Exercises, March 20, 2025

1) Let k be a field. Let $\mathfrak{S} := \mathrm{Spec}(k[X,Y]/(X^2+Y^2-1))$ be the unit sphere group with group structure given by $(x,y)\dot{(}x',y') = (xx'-yy', xy'+x'y)$.

(i) Show that if $k=\mathbb{C}$, then the map $\mathfrak{S} \to \mathbb{G}_{m,\mathbb{C}}$ given by $t \mapsto X+iY$ is an isomorphism of group schemes.

(ii) Show that if $k=\mathbb{R}$ then the ideal $I := (X, Y-1)$ is not a principal ideal of $\mathbb{R}[X,Y]/(X^2+Y^2-1)$ (*Hint: consider a generator of* $I \otimes_{\mathbb{R}} \mathbb{C}$.).

(iii) Show that $\mathbb{R}[T,T^{-1}]$ is a principal ideal domain.

(iv) Show that if $k=\mathbb{R}$ then $\mathfrak{S}$ is not isomorphic to $\mathbb{G}_m$ as schemes.

2) Let k be a field. Show that an algebraic action of the affine k–group $\mathbb{G}_a$ on $\mathbb{A}^1_k \setminus \{0\}$ is trivial.

3) Let R be a ring and let G be an affine R–group scheme.

(i) Suppose that the elements in the various $G(S)$ (for S running over the R–algebras) do not have uniformly bounded orders, i.e. for each n there is an S for which $g \mapsto g^n$ is nontrivial on $G(S)$. Show that some $G(S)$ contains an element of infinite order.

(ii) Let p be a prime number and let H be the R–functor of p-power roots of unity, i.e. for each R–ring S we have

$$H(S) = \big\{ x \in S^\times \mid x^{p^n} = 1 \quad \text{for some}\, n \big\}.$$

If R is non zero, show that H is not representable by an affine R–scheme.

(iii) Show that H is an inductive limit of representable R–functors.

4) Let R be a ring and let $f_* : N_1 \to N_2$ be a morphism of R-modules where N_1, N_2 are finitely generated projective. We consider the morphism of R–group schemes $f : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$.

(i) Show that there is a largest open subcheme $U \subset \mathrm{Spec}(R)$ such that the restriction $\mathfrak{W}(N_1) \times_{\mathrm{Spec}(R)} U \to \mathfrak{W}(N_2) \times_{\mathrm{Spec}(R)} U$ is a closed immersion (*Hint*: use Tag 01QO or Remark 3.47 in Goertz-Wedhorn).

(ii) Show that a point $x \in \mathrm{Spec}(R)$ belongs to U if and only if $N_1 \otimes_R \kappa(x) \to N_2 \otimes_R \kappa(x)$ is injective (where $\kappa(x)$ is the residue field of the point x).

(iii) If $N_1 = R^{n_1}$, $N_2 = R^{n_2}$, provide an explicit description of U.