**Exercises, March 27, 2025**

1) Let $k$ be a field of characteristic 3 and consider the $k$–group $G = \mu_{3,k} \rtimes (\mathbb{Z}/2\mathbb{Z})_k$. where $(\mathbb{Z}/2\mathbb{Z})_k$ acts by inversion on $\mu_{3,k}$.
(i) Show that $G(k)$ is the group with two elements.
(ii) Show that $G(k)$ is non central in $G(S)$ for some suitable $k$–algebra $S$.

2) Let $R$ be a commutative ring.
(i) Let $f \in R$. Show that $R_f$ is flat over $R$.
(ii) Let $S$ be a multiplicative subset of $R$. Show that $R_S$ is flat over $R$.

3) Let $R$ be a commutative ring and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Assume that $M''$ is flat over $R$.
(i) Let $N$ be an $R$-module. Show that the sequence

$$0 \to M' \otimes N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

of $R$-modules is exact. [*Hint: use an exact sequence of* $R$*-modules* $0 \to K \to L \to N \to 0$ *where* $L$ *is a free module*].
(ii) Show that $M$ is flat if and only if $M'$ is flat.

4) Let $R$ be a DVR and let $\pi$ be a uniformizer parameter and put $K = R[\pi^{-1}]$. What is the schematic closure in $\mathbb{A}^2_R$ of the following closed $K$-subchemes of $\mathbb{A}^2_K$? (i) $x^2 = \pi y^2$; (ii) $x^2 = \pi^{-1} y^2$; (iii) $x^2 = \pi^{-1} y^2 + 1$.

5) Let $R$ be a DVR with fraction field $K$ and residue field $k$. Let $\mathfrak{X}$ be an affine $R$–scheme and let $\mathfrak{X}'$ be the schematic closure of $\mathfrak{X}_K = X \times_R K$ in $\mathfrak{X}$.
(i) Show that $\mathfrak{X}'$ is an affine flat $R$–scheme of generic fiber $\mathfrak{X}_K$.
(ii) Show that $\mathfrak{X}'(R) = \mathfrak{X}(R)$.
(iii) Show that $\mathfrak{X}'(B) = \mathfrak{X}(B)$ for each flat $R$–ring $B$.