Exercises, April 3, 2025

Let R be a ring.

1) Let M be a finitely generated faithfully projective R–module, that is, $M \otimes_R R/\mathfrak{m} \neq 0$ for each maximal ideal $\mathfrak{m} \subset R$. Show that that $R.1$ is a direct summand of $\mathrm{End}_R(M)$. [*Hint:* dualize the map $R \to \mathrm{End}_R(M)$].

2) Let G be an affine R-group scheme over a ring R.

(i) Given two G-modules M, N, construct a natural structure of G-module on $M \otimes_R N$.

(ii) Furthermore assume that the R–module N is f.g. faithfully projective. If M is faithful, show that $M \otimes_R N$ is faithful [*Hint:* use the first exercise]. What about the converse?

3) Let A be an abelian group and let k be a field. Identify the simple objects of the category of $\mathfrak{D}(A)$–modules and show that any $\mathfrak{D}(A)$–module is a direct sum of simple ones.

4) Let Γ be a finite group. Consider the R–algebra $A = R^{(\Gamma)} = \bigoplus_{\sigma \in \Gamma} R e_\sigma$ with $e_\sigma^2 = e_\sigma$, $e_\sigma e_\tau = 0$ for $\sigma \neq \tau$ and $1 = \sum_{\sigma \in \Gamma} e_\sigma$. Consider the following triple $(\Delta, \epsilon, anti)$ defined by

(a) $\Delta : A \to A \otimes_R A$, $e_\sigma \mapsto \sum_{\sigma_1 \sigma_2 = \sigma} e_{\sigma_1} \otimes e_{\sigma_2}$;

(b) $\epsilon : A \to R$, $e_\sigma \mapsto \delta_{\sigma,1}$;

(c) $anti : A \to A$, $e_\sigma \mapsto e_{\sigma^{-1}}$.

(i) Show that $(A, \Delta, \epsilon, anti)$ is the Hopf algebra sttucture attached to the constant group scheme Γ_R.

(ii) Let M be an R–module. Given a representation $\Gamma \to \mathrm{Aut}(M)$, show that it gives rise to a representation $\mathfrak{G} \to \mathrm{Aut}(V(M))$ and explicit its coaction.

(iii) Show that any representation $\mathfrak{G} \to \mathrm{Aut}(V(M))$ arises in the way of (ii).