Exercises, April 10, 2025

Let R be a ring.

1) Assume that R is a DVR of fraction field K and consider the R–algebra

$$A = \big\{ P \in R[T] \mid P(0) \in A \big\}.$$

(i) Show that A is flat and that the map $A \to K[T]$ induces an isomorphism $A \otimes_R K \xrightarrow{\sim} K[T]$.

(ii) Show that there exists a unique structure of R–group scheme on $\mathrm{Spec}(A)$ such that its extension to K identifies (with the map of (1)) with the additive K–group.

(iii) Let $M = Rm_1 \oplus Rm_2$ be a free R-module of rank 2 and consider the endomorphism u defined by $u(m_1) = m_2$, $u(m_2) = 0$. Show that

$$c : M \to M \otimes_R A$$

defined by

$$c(m) = m \otimes 1 + u(m) \otimes T$$

provides a $\mathfrak{G}$–module where $\mathfrak{G} = \mathrm{Spec}(A)$ is the R–group scheme of (ii).

(iv) Show that $R\,m_1$ is the intersection of the $\mathfrak{G}$-submodules containing m_1 and show that $R\,m_1$ is not a $\mathfrak{G}$-submodule of M.

2) Let $\mathfrak{G}$ be an affine R–group scheme. Let M be a $\mathfrak{G}$-module with coaction $c : M \to M \otimes_R R[\mathfrak{G}]$ and associated representation ρ. We assume that M is a free R-module.

(a) Show that there exists a smallest ideal I of $R[\mathfrak{G}]$ such that $c(m) = m \otimes 1$ mod $M \otimes I$ for all $m \in M$ and show that its formation commutes with arbitrary change of rings.

(b) If $g \in \ker(\rho)(R)$, show that $\epsilon_g(I) = 0$.

(c) Show that the R–functor $\ker(\rho)$ is represented by the closed R–subscheme $\mathfrak{H} = \mathrm{Spec}(R[\mathfrak{G}]/I)$ of $\mathfrak{G}$.

3) (a) Let G be a group R-functor. Its center $Z(G)$ is defined by letting $h \in G(R)$ be in $Z(G)(R)$ if for every R-ring S and every $g \in G(S)$, we have $hg = gh$. Show that $Z(G)$ is an R-subfunctor of G and show that it is normal in G (that is, $Z(G)(S)$ is a normal subgroup of $G(S)$ for each R-ring S).

(b) Suppose that G is an affine R-group scheme and put $A = R[G]$. Consider the representation $\rho : h_G \to \mathrm{GL}(A)$ defined by

$$\rho(h)(f) = (L_{h^{-1}})^* \big((R_h^*)(f) \big).$$

Show that $\ker(\rho) = Z(G)$.

(c) Suppose also that $R[\mathfrak{G}]$ is a free R-module. Show that $Z(G)$ is represented by $\mathrm{Spec}(A/I)$, where I is the smallest ideal of A such that $\varphi(f) = f \otimes 1$ mod $A \otimes I$ for all $f \in A$. [*Hint.* Use the preceding exercise].

(d) Discuss the case of the semi-direct product $\mu_3 \rtimes (\mathbb{Z}/2\mathbb{Z})$ over a field of characteristic 3.

4) What is the center of the affine R–group scheme GL_n for $n \geq 2$?