Exercises, May 8, 2025

1) We put $R = \mathbb{R}[[t]]$ and consider its extensions $S_1 = \mathbb{C}[[t]]$ and $S_2 = R[\sqrt{t}]$.
(i) For $i = 1, 2$, show that S_i is a free R–module of rank 2 and construct an R–group homomorphism

$$f_i : \prod_{S_i/R} (\mathbb{G}_{m,S_i}) \to \mathbb{G}_{m,R}$$

such that its precomposite with the diagonal map $\mathbb{G}_{m,R} \to \prod_{S_i/R}(\mathbb{G}_{m,S_i})$ is $t \mapsto t^2$.
(ii) Describe $\prod\limits_{S_i/R} (\mathbb{G}_{m,S_i}) \times_R \mathbb{R}$ for $i = 1, 2$ and the associated morphisms $g_i = f_i \times_R \mathbb{R}$.
(iii) Show that the character g_2 is a square (that is $g_2 = 2\,g'$ for some character g') but not f_2.

2) Let R be a ring and let S be a finite locally free extension of R of constant degree $d \geq 1$ and denote by $j : R \to S$.
(a) Construct a natural linear faithful representation $\prod\limits_{S/R} \mathbb{G}_{m,S} \to \mathrm{GL}(j_* S)$.
(b) Assume that R is connected. Show that $\prod\limits_{S/R} \mathbb{G}_{m,S}$ is a diagonalizable R–group scheme if and only if $S \cong R \times \cdots \times R$ (d times). [*Hint:* Use (a) and rank considerations (the rank of an abelian group is the minimal cardinal of a set of generators)].

3) (Connection with the definition of smoothness in the book *Néron models*).
Let $S \cong R[T_1, \dots, T_n]/(f_1, \dots, f_c)$ with $0 \leq c \leq n$.
(a) Assume that S is standard smooth. For each $s \in \mathrm{Spec}(S)$, that show that $df_1(s), \dots, df_c(s)$ is a free family of the $\kappa(s)$–vector space $\Omega^1_{R[T_1,\dots,T_n]/R} \otimes_{R[T_1,\dots,T_n]} \kappa(s)$.
(b) If $df_1(s), \dots, df_c(s)$ is a free family of the $\kappa(s)$–vector space $\Omega^1_{R[T_1,\dots,T_n]/R} \otimes_{R[T_1,\dots,T_n]} \kappa(s)$ for each $s \in \mathrm{Spec}(S)$, show that S is a smooth R–algebra (with the definition of the lectures).