Affine group schemes I

We shall work over a base ring R (commutative and unital).

2. Sorites

2.1. **R-Functors.** We denote by $\mathcal{A}ff_R$ the category of affine R-schemes. We are interested in R–functors, i.e. covariant functors from $\mathcal{A}ff_R$ to the category of sets. If $\mathfrak{X}$ an R-scheme, it defines a covariant R–functor

$$h_{\mathfrak{X}}: \ \mathcal{A}ff_R \ \to \ Sets, \ S \mapsto \mathfrak{X}(S).$$

Given a map $f : \mathfrak{Y} \to \mathfrak{X}$ of R-schemes, there is a natural morphism of functors $f_* : h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ of R-functors.

We recall now Yoneda's lemma in our setting. Let F be an R-functor. If $\mathfrak{X} = \mathrm{Spec}(R[\mathfrak{X}])$ is an affine R–scheme and $\zeta \in F(R[\mathfrak{X}])$, we define a morphism of R-functors

$$\phi(\zeta) : h_{\mathfrak{X}} \to F$$

by $\phi(\zeta)(S) : h_{\mathfrak{X}}(S) = \mathrm{Hom}_R(R[\mathfrak{X}], S) \to F(S)$, $x \mapsto F(f_x)(\zeta)$ for each R-ring S where $f_x \in \mathrm{Hom}_R(R[\mathfrak{X}], S)$ is the evaluation function at x.

2.1.1. **Lemma.** *(Yoneda lemma)*

(1) The assignment $\zeta \to \phi(\zeta)$ induces a bijection

$$F(R[\mathfrak{X}]) \xrightarrow{\sim} \mathrm{Hom}_{R-func}(h_{\mathfrak{X}}, F).$$

(2) Let $\mathfrak{Y}$ be an R–scheme. Then we have

$$\mathrm{Hom}_{R-sch}(\mathfrak{X}, \mathfrak{Y}) = h_{\mathfrak{Y}}(R[\mathfrak{X}]) \xrightarrow{\sim} \mathrm{Hom}_{R-func}(h_{\mathfrak{X}}, h_{\mathfrak{Y}}).$$

Proof. (1) The strategy is to construct the inverse map. We are given $\alpha \in \mathrm{Hom}_{R-func}(h_{\mathfrak{X}}, F)$, it gives rise to a map $\alpha_{R[\mathfrak{X}]} : h_{\mathfrak{X}}(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$ so that the universal point $x^{univ} \in h_{\mathfrak{X}}(R[\mathfrak{X}]) = \mathrm{Hom}_R(R[\mathfrak{X}], R[\mathfrak{X}])$ defines an element $\psi(\alpha) = \alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) \in F(R[\mathfrak{X}])$ or for short $\alpha(id_{R[\mathfrak{X}]}$.

Step 1: $\psi \circ \phi = id_{F(R[\mathfrak{X}])}$. Let $\zeta \in F(R[\mathfrak{X}])$. We apply $\phi(\zeta)_{R[\mathfrak{X}]} : h_{\mathfrak{X}}(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$ to $R[\mathfrak{X}]$ and obtain $\psi(\phi(\zeta)) = F(id_{R[\mathfrak{X}]})(\zeta) = \zeta$.

Step 2: $\phi \circ \psi = id_{\mathrm{Hom}_{R-func}(h_{\mathfrak{X}}, F)}$. Let $\alpha \in \mathrm{Hom}_{R-func}(h_{\mathfrak{X}}, F)$. Then $\psi(\alpha) = \alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) \in F(R[\mathfrak{X}])$ and we consider the element $\eta = \phi(\psi(\alpha)) \in \mathrm{Hom}_R(h_{\mathfrak{X}}, F)$ defined as follows. For each $f_x \in \mathrm{Hom}_R(R[\mathfrak{X}], S)$, $\eta(S) : h_X(S) \to F(S)$ applies f_x to

$$F(f_x)\big(\psi(\alpha)\big) = F(f_x)\big(\alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]})\big) = \alpha(f_x \circ id_{R[\mathfrak{X}]}) = \alpha(f_x)$$

where we used the functorial property in the second equality. Thus $\phi \circ \psi = id_{\mathrm{Hom}_{R-func}(h_{\mathfrak{X}}, F)}$.

(2) We apply (1) to $F = h_{\mathfrak{Y}}$. ☐

2.1.2. **Remarks.** (a) The formula $F(f_x)\big(\psi(\alpha)\big) = \alpha(f_x)$ arising in the proof expresses the fact that an R–functor $h_X \to F$ is determined by its value on the universal point of X.

(b) For more on the Yoneda lemma, see [Wa, §1.2], [GW, §4.2] or [Vi, §2.1]. Part (2) holds then for general R-schemes.

An R-functor F is representable by an R scheme (resp. an affine R–scheme) if there exists an R-scheme $\mathfrak{X}$ (resp. an affine R–scheme $\mathfrak{X}$) together with an isomorphism of functors $h_X \to F$. We say that $\mathfrak{X}$ represents F.

If $\mathfrak{X}$ is affine, the isomorphism $h_X \to F$ comes from an element $\zeta \in F(R[\mathfrak{X}])$ which is called the universal element of $F(R[\mathfrak{X}])$. The pair $(\mathfrak{X}, \zeta)$ satisfies the following universal property:

For each affine R-scheme $\mathfrak{T}$ and for each $\eta \in F(R[\mathfrak{T}])$, there exists a unique morphism $u : \mathfrak{T} \to \mathfrak{X}$ such that $F(u^*)(\zeta) = \eta$.

Given a morphism of rings $j : R \to R'$, an R–functor F defines by restriction an R'–functor denoted by j_*F or $F_{R'}$. If $F = h_{\mathfrak{X}}$ for an affine R-scheme $\mathfrak{X}$, we have $F_{R'} = h_{\mathfrak{X} \times_R R'}$.

2.1.3. **Examples.** We will see later more non representable R-functors.

(a) The empty R–functor is not representable by an affine R-scheme (and not actually by any R-scheme). Denote by F the empty functor and assume that $h_{\mathfrak{X}} \cong F$ for an R-scheme $\mathfrak{X}$. Then $id_{\mathfrak{X}} \in h_{\mathfrak{X}}(R[\mathfrak{X}])$ contradicting the fact that F is the empty R-functor.

(b) We consider the R-functor $F(S) = S^{(\mathbb{N})}$ and claim that it not representable by an affine R–scheme. Assume that $h_{\mathfrak{X}} \cong F$ so that $\mathrm{Hom}_R(R[\mathfrak{X}], R[\mathfrak{X}]) \cong R[\mathfrak{X}]^{(\mathbb{N})}$. Then the image of $id_{R[\mathfrak{X}]}$ has bounded support d so that $F(S) \subset S^d \subset S^{(\mathbb{N})}$ for each R–ring S. This is a contradiction.

2.1.4. **Remark.** We denote by $F_0(S) = \{\bullet\}$ for each R–ring S. Let F be an R-functor. Then there is a canonical map $F \to F_0$; in other words F_0 is a terminal object of the category of R–functors.

2.2. **Monomorphisms.** The fibered product of R-functors is defined as follows. For $\alpha_1 : F_1 \to E$ and $\alpha_2 : F_1 \to E$ two morphisms of R-functors, we set $(F_1 \times_E F_2)(S) = F_1(S) \times_{E(S)} F_2(S)$ for each R–ring S.

2.2.1. **Lemma.** *Let $\alpha : F \to E$ be a morphism of R-functors. The following conditions are equivalent:*

(i) α is a monomorphism;

(ii) the diagonal $\Delta : F \to F \times_E F$ is an isomorphism;

(iii) $F(S) \to E(S)$ is injective for each R–ring S.

Proof. $(i) \Longrightarrow (ii)$. We consider the projections $p_i : F \times_E F \to F$ for $i = 1, 2$. Since $\alpha \circ p_1 = \alpha \circ p_2$, we obtain that $p_1 = p_2$. Thus p_1 is an isomorphism and so is Δ.

$(ii) \Longrightarrow (i)$. We are given $\beta_1, \beta_2 : G \to F$ be morphisms of R–functors such that $\alpha \circ \beta_1 = \alpha \circ \beta_2$. This defines a map $\beta : G \to F \times_E F \xleftarrow{\sim} F$, so that $\beta_1 = \beta_2$.

$(iii) \Longrightarrow (ii)$. For each R-ring S, we have $F(S) \xrightarrow{\sim} F(S) \times_{E(S)} F(S)$ so that Δ is an isomorphism of R-functors.

$(ii) \Longrightarrow (iii)$. Obvious.

□

We consider now the case of schemes.

2.2.2. **Lemma.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of R-schemes. The following conditions are equivalent:*

(i) f is a monomorphism;

(i') The R–functor $h_f : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ is a monomorphism;

(ii) the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is an isomorphism;

(iii) $F(S) \to E(S)$ is injective for each R–ring S.

Proof. The proof of the implications $(i) \Longleftrightarrow (ii) \Longrightarrow (iii)$ is similar with the previous lemma. The implication $(iii) \Longrightarrow (ii)$ Lemma 2.2.1, $(iii) \Longrightarrow (i)$ yields the implication $(iii) \Longrightarrow (i')$.

It remains to establish the implication $(i') \Longrightarrow (ii)$. Lemma 2.2.1, $(i) \Longrightarrow (ii)$ shows that the diagonal $h_{\mathfrak{X}} \to h_{\mathfrak{X}} \times_{h_{\mathfrak{Y}}} h_{\mathfrak{X}}$ is a an isomorphism of R–functors. Let $\mathfrak{Z}$ be an R-scheme, we need to establish that the diagonal map $\mathfrak{X}(\mathfrak{Z}) \to \mathfrak{X}(\mathfrak{Z}) \times_{\mathfrak{Y}(\mathfrak{Z})} \mathfrak{X}(\mathfrak{Z})$ is an isomorphism. If $\mathfrak{Z}$ is affine over R it is true. Let $g, h \in \mathfrak{X}(\mathfrak{Z})$ mapping to the same element of $\mathfrak{Y}(\mathfrak{Z})$.

We consider then an affine cover $(\mathfrak{U}_i)_{\imath \in I}$ of $\mathfrak{Z}$ so that the restrictions $g_i : \mathfrak{U}_i \subset \mathfrak{Z} \to \mathfrak{X}$ $h_i : \mathfrak{U}_i \subset \mathfrak{Z} \to \mathfrak{X}$ define an unique element $f_i \in \mathfrak{X}(\mathfrak{U}_i)$. Since the diagonal is split by the first projection, f_i and f_j agree on $\mathfrak{U}_i \cap \mathfrak{U}_j$ so that define $f : \mathfrak{Z} \to \mathfrak{X}$. Then $f = g = h$ and we are done.

□

2.2.3. **Remark.** The equivalence $(i) \Longleftrightarrow (ii)$ in (1) holds in any category with fiber products, see [St, Tag 01L3].

We consider now the epimorphisms of R–functors. If $\alpha : F \to E$ satisfies that $F(S) \to E(S)$ is surjective for each R–ring S, we claim that α is an epimorphism.

Let $\gamma_1, \gamma_2 : E \to D$ be morphisms of R–functors such that $\gamma_1 \circ \alpha = \gamma_2 \circ \alpha$. Then $\gamma_1 : E(S) \to D(S)$ agrees with $\gamma_2 : E(S) \to D(S)$ for each R–ring S so that $\beta_1 = \beta_2$. Thus α is an epimorphism.

It can be shown by using coproducts that the epimorphisms are all of that shape, see [KS, §2, Ex. 2.4, 2.23] or [SGA3, §I.1.4]; those references put also the monomorphism case in a much wider setting.

In the category of R–schemes, we have to pay attention that there are epimorphisms whose associated functor is not surjective, see [GW, Ex. 8.2.(d)] for the construction of a bunch of epimorphisms. A concrete example is with $k = \mathbb{R}$ and the morphism $u : \mathfrak{X} = \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R}) = \mathfrak{Y}$.

Let $\mathfrak{Z}$ be an $\mathbb{R}$-scheme and let $f_1, f_2 : \mathfrak{Y} \to \mathfrak{Z}$ such that $f_1 \circ u = f_2 \circ u$. In other words we have two points $z_1, z_2 \in \mathfrak{Z}(\mathbb{R})$ which coincide as complex points. Since $\mathfrak{Z}(\mathbb{R})$ injects in $\mathfrak{Z}(\mathfrak{C})$, it follows that $z_1 = z_2$ so that u is an epimorphism. The fact that $\mathfrak{Z}(\mathbb{R})$ injects in $\mathfrak{Z}(\mathbb{C})$ reduces to an affine scheme $\mathrm{Spec}(A)$ for which we have $\mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R}) \subset \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(A, \mathbb{C})$.

2.3. **Zariski sheaves.** We say that an R–functor F is a Zariski sheaf if it satisfies the following requirements:

(A) for each R–ring S and each decomposition $1 = f_1 + \cdots + f_n$ in S, then

$$F(S) \xrightarrow{\sim} \Big\{ (\alpha_i) \in \prod_{i=1,..,n} F(S_{f_i}) \mid (\alpha_i)_{S_{f_i f_j}} = (\alpha_j)_{S_{f_i f_j}} \text{ for } \ i, j = 1, ..., n \Big\}.$$

(B) $F(0) = \{\bullet\}$.

2.3.1. **Lemma.** *Let F be an R–functor F which a Zariski sheaf. Then F is additive, i.e. the map $F(S_1 \times S_2) \to F(S_1) \times F(S_2)$ is bijective for each pair (S_1, S_2) of R–algebras.*

Proof. We are given an R–ring $S = S_1 \times S_2$; we write it $S = S_1 \times S_2 = Se_1 + Se_2$ where e_1, e_2 are idempotents satisfying $e_1 + e_2 = 1$, we have $S_1 = S_{e_1}$, $S_2 = S_{e_2}$ and $S_{e_1 e_2} = 0$ [St, Tag 00ED]. Then

$$F(S) \xrightarrow{\sim} \big\{ (\alpha_1, \alpha_2) \in F(S_1) \times F(S_2) \mid \alpha_{1,0} = \alpha_{2,0} \in F(0) \big\}.$$

Since $F(0) = \{\bullet\}$, we conclude that $F(S) = F(S_1) \times F(S_2)$. □

Representable R-functors are clearly Zariski sheaves. In particular, to be a Zariski sheaf is a necessary condition for an R–functor to be representable.

2.3.2. **Lemma.** *Let $1 = f_1 + \cdots + f_n$. Let F be an R–functor which is a Zariski sheaf and such that $F_{R_{f_i}}$ is representable by an affine R_{f_i}-scheme for $i = 1, ..., n$. Then F is representable by an affine R–scheme.*

Proof. Let $\mathfrak{X}_i$ be an R_{f_i}-scheme together with an isomorphism $\zeta_i : h_{\mathfrak{X}_i} \xrightarrow{\sim} F_{R_{f_i}}$ of R_{f_i}–functors for $i = 1, .., n$. Then for $i \neq j$, $F_{R_{f_i f_j}}$ is represented by $\mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j}$ and $\mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}$. More precisely, the isomorphism $\zeta^{-1}_{j, R_{f_i f_j}} \circ \zeta_{i, R_{f_i f_j}} : h_{\mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j}} \xrightarrow{\sim} h_{\mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}}$ defines an isomorphism $u_{i,j} : \mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j} \xrightarrow{\sim} \mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}$ and we have compatiblities $u_{i,j} \circ u_{j,k} = u_{i,k}$ once restricted to $R_{f_i f_j f_k}$. It follows that the $\mathfrak{X}_i$'s glue in an affine R–scheme $\mathfrak{X}$. Also the map ζ_i^{-1} glue in an R–map $F \to h_{\mathfrak{X}}$. Since F is a Zariski sheaf, we conclude that $F \xrightarrow{\sim} h_{\mathfrak{X}}$. □

2.4. **Functors in groups.**

2.5. **Definition.** An R–group scheme $\mathfrak{G}$ is a group object in the category of R-schemes. It means that $\mathfrak{G}/R$ is an affine scheme equipped with a section $\epsilon : \mathrm{Spec}(R) \to \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \to \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{tikzcd}
(\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} \arrow[r, "m \times id"] \arrow[dd, "can", "\wr"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[dr, "m"] & \\
& & \mathfrak{G} \\
\mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) \arrow[r, "id \times m"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[ur, "m"] &
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "id \times \epsilon"] \arrow[dr, "can", "\cong"'] & \mathfrak{G} \times_R \mathfrak{G} & \mathrm{Spec}(R) \times_R \mathfrak{G} \arrow[l, "\epsilon \times id"'] \arrow[dl, "can"', "\cong"] \\
& \mathfrak{G} &
\end{tikzcd}$$

Symmetry:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathfrak{G} \arrow[r, "id \times \sigma"] \arrow[d] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"] \\
\mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathfrak{G}.
\end{tikzcd}$$

We say that $\mathfrak{G}$ is commutative if furthermore the following diagram commutes

$$\begin{tikzcd}
& \mathfrak{G} \times_R \mathfrak{G} \arrow[dl, "m"'] \arrow[dd, "switch"] \\
\mathfrak{G} & \\
& \mathfrak{G} \times_R \mathfrak{G} \arrow[ul, "m"]
\end{tikzcd}$$

We will mostly work with affine R–group schemes, that is, when $\mathfrak{G}$ is an affine R–group scheme.

Let $R[\mathfrak{G}]$ be the coordinate ring of $\mathfrak{G}$. We call $\epsilon^* : R[\mathfrak{G}] \to \mathfrak{G}$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \to R[G]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. By means of the dictionary affine schemes/rings, they satisfy the following commutativity rules:

Co-associativity:

$$\begin{tikzcd}
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[r, "id\otimes\Delta"] & R[\mathfrak{G}] \otimes_R (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \arrow[dd, "\wr"] \\
R[\mathfrak{G}] \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[r, "\Delta\otimes id"] & (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

Counit: The following composite maps are $id_{R[\mathfrak{G}]}$

$$R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \xrightarrow{id\otimes\epsilon} R[\mathfrak{G}] \otimes_R R \xrightarrow{\sim} R[\mathfrak{G}]$$

$$R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \xrightarrow{\epsilon\otimes id} R[\mathfrak{G}] \otimes_R R \xrightarrow{\sim} R[\mathfrak{G}].$$

Cosymmetry:

$$\begin{tikzcd}
 & R[\mathfrak{G}] \arrow[r, "\Delta"] \arrow[dl, "\epsilon^*"] & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[dd, "id\times\sigma^*"] \\
R \arrow[dr] & & \\
 & R[\mathfrak{G}] & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[l, "product"']
\end{tikzcd}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R–algebra[1]. Given an affine R–scheme $\mathfrak{X}$, there is then a one to one correspondence between group structures on $\mathfrak{X}$ and commutative R–algebra structures on $R[\mathfrak{X}]$.

Also $\mathfrak{G}$ is commuative if and only if the following diagram commutes

$$\begin{tikzcd}
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[dd, "switch"] \\
R[\mathfrak{G}] \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\
 & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]
\end{tikzcd}$$

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[1] This is Waterhouse definition [Wa, §I.4], other people talk about cocommutative coassociative Hopf algebra.

If $\mathfrak{G}/R$ is an (affine) R–group scheme, then for each R–algebra S the abstract group $\mathfrak{G}(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \to \mathfrak{G}(S)$. It means that the functor $h_{\mathfrak{G}}$ is actually a group functor.

2.5.1. **Lemma.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the Yoneda lemma induces a one to one correspondence between group structures on $\mathfrak{X}$ and group structures on $h_{\mathfrak{X}}$.*

In other words, defining a group law on $\mathfrak{X}$ is the same that to define compatible group laws on each $\mathfrak{G}(S)$ for S running over the R-algebras.

Proof. This is an immediate consequence of Yoneda's lemma. We assume that the R-functor h_X is equipped with a group structure. The Yoneda lemma shows that this group structure arises in an unique way of an affine R-group scheme structure. ☐

2.5.2. **Remark.** We shall encounter certain non-affine group R-schemes. A group scheme $\mathfrak{G}/R$ is a group object in the category of R-schemes. More generally the previous lemma holds for a non affine R–group scheme.

3. EXAMPLES

3.1. **Constant group schemes.** Let I be a set and consider theconsider the R–scheme $I_R = \bigsqcup_{\gamma \in I} \mathrm{Spec}(R) = \bigsqcup_{\gamma \in I} U_i$. We claim that its functor of points h_{I_R} identifies with

$$\Big\{ \text{locally constant functions}\, \mathrm{Spec}(S)_{top} \to I \Big\}.$$

To see this let S be an R–ring and let $f \in h_{I_R}(S) = \mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spec}(S), I_R)$. By pulling back the open cover (U_i) of I_R, we obtain a decomposition $S = \bigsqcup_{\gamma \in I} S_i$ in open subschemes of R. This defines a locally constant function $\mathrm{Spec}(S)_{top} \to I$ having the value i on each S_i (for more details see [GW, Ex. 4.43] or [St, Tag 03YW]).

Next let Γ be an abstract group. We consider the R–scheme $\Gamma_R = \bigsqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)$. Its functor of points h_{Γ_R} identifies with

$$\Big\{ \text{locally constant functions}\, \mathrm{Spec}(S)_{top} \to \Gamma \Big\}.$$

The group structure on Γ induces an R-group scheme structure on Γ_R. If R is non zero, this group scheme is affine and only if Γ is finite.

3.2. **Vector groups.** Let N be an R–module. We consider the commutative group functors

$$V_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto \mathrm{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee,$$

$$W_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto N \otimes_R S.$$

3.2.1. **Lemma.** *The R–group functor V_N is representable by the affine R–scheme $\mathfrak{V}(N) = \mathrm{Spec}(S^*(N))$ which is then a commutative R–group scheme. Furthermore if the R–module N is of finite presentation then the R–scheme $\mathfrak{V}(N)$ is of finite presentation.*

Proof. It follows readily of the universal property of the symmetric algebra

$$\mathrm{Hom}_{R'-mod}(N \otimes_R R', R') \xleftarrow{\sim} \mathrm{Hom}_{R-mod}(N, R') \xrightarrow{\sim} \mathrm{Hom}_{R-alg}(S^*(N), R')$$

for each R-algebra R'.

We assume that the R–module N is finitely presented, that is, there exists an exact sequence $0 \to M \to R^n \to N \to 0$ where M is a finitely generated R–module. According to [St, Tag 00DO] the kernel I of the surjective map $S^*(R^n) \to S^*(N)$ is generated by M (seen in degree one) so is a finitely generated $S^*(R^n)$-module. Since $S^*(R^n) = R[t_1, \dots, t_n]$, we conclude that the R–algebra $S^*(N)$ is of finite presentation.

□

3.2.2. **Remark.** The converse of the last assertion holds as well by using the limit characterizations of the finite presentation property, see [St, Tags 0G8P, 00QO].

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N. We note that $N = \mathfrak{V}(N)(R)$. In the special case $N = R^d$, this is nothing but the affine space $\mathbf{A}^d_R$ of relative dimension d.

Its group law on the R–group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \to S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The cosymmetry is $\sigma^* : S^*(N) \to S^*(N)$, $X \mapsto -X$ and the counit is the augmentation map $S^*(N) \to R$.

If $N = R$, we get the affine line over R. Given a map $f : N \to N'$ of R–modules, there is a natural map $f^* : \mathfrak{V}(N') \to \mathfrak{V}(N)$ of R–group schemes.

3.2.3. **Lemma.** *The assignement $N \to \mathfrak{V}(N)$ is a faithful contravariant (essentially surjective) functor from the category of R-modules and that of vector group R-schemes.*

Proof. Since this functor is essentially surjective, it is enough to show that it is faithful. Given two R–modules N, N' we want to show that the morphism

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_{R-gp}\big(\mathfrak{V}(N'), \mathfrak{V}(N)\big), \quad f \mapsto f^*$$

is injective. This is clear since $f_* : S^*(N) \to S^*(N')$ is a graded morphism and applies N to N' by f. □

3.2.4. **Remark.** Let k be a field of characteristic $p > 0$ and consider the Frobenius morphism $\mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$, $x \mapsto x^p$. It is a k–group homomorphism but is linear. This shows that the functor above is not fully faithful and then not an anti-equivalence of categories. For obtaining an anti-equivalence of categories, we need to restrict the morphisms to linear morphisms, see [SGA3, I.4.6.2].

We consider also the R-functor $W(N)$ defined by $W(N)(S) = N \otimes_R S$. The assignement $N \to W(N)$ is an equivalence of categories from the category of R-modules and that of functors W with linear maps. Together with Lemma 3.2.3, it follows that there is an anti-equivalence of categories between the category of functors W with linear maps and the category of vector R–group schemes.

If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that the R–functor $W(N)$ is representable by an affine group scheme. In this case we denote by $\mathfrak{W}(N)$ the associated R–group scheme.

3.2.5. **Theorem.** *The R–functor $W(N)$ is representable if and only if N is projective and finitely generated.*

If R is noetherian, this is due to [Ni04]. The general case has been handled by Romagny [Ro, Thm. 5.4.5]. Note that it is coherent with the example 2.1.3.(b).

3.3. **Group of invertible elements, linear groups.** Let A/R be an algebra (unital, associative). We consider the R-functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times .$$

3.3.1. **Lemma.** *If A/R is finitely generated projective, then $\mathrm{GL}_1(A)$ is representable by an affine group scheme. Furthermore, $\mathrm{GL}_1(A)$ is of finite presentation.*

Proof. Up to localize for the Zariski topology (Lemma 2.3.2), we can assume that A is a free R–module of rank d.

We shall use the norm map $N : A \to R$ defined by $a \mapsto \det(L_a)$ where $L_a : A \to A$ is the R-endomorphism of A defined by the left translation by A. We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a. More precisely, let $P_a(X) = X^d - \mathrm{Tr}(L_a)X^{d-1} + \cdots + (-1)^{d-1}c_{d-1}(L_a)X + (-1)^d \det(L_a) \in R[X]$ be the characteristic polynomial of L_a; according to the Cayley-Hamilton theorem we have $P_a(L_a) = 0$ [Bbk1, III, §11] so that $L_{P_a(a)} = 0$ and $P_a(a) = 0$. If $\det(L_a) \in R^\times$, it follows that

$$a\Big(a^{d-1} - \mathrm{Tr}(L_a)a^{d-2} + \cdots + (-1)^{d-1}c_{d-1}(L_a)a\Big) = (-1)^{d+1}\det(A)$$

so that $ab = ba = 1$ with $b = (-1)^{d+1}\det(A)^{-1}\Big(a^{d-1} - \mathrm{Tr}(L_a)a^{d-2} + (-1)^{d-1}c_{d-1}(L_a)a\Big)$.

The same is true after tensoring by S, so that

$$\mathrm{GL}_1(A)(S) = \Big\{a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \quad | \quad N(a) \in S^\times\Big\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{tikzcd}
\mathfrak{G} \arrow[r] \arrow[d] & \mathfrak{W}(A) \arrow[d, "N"'] \\
\mathbb{G}_{m,R} \arrow[r] & \mathfrak{W}(R).
\end{tikzcd}$$

□

Given an R–module N, we consider the R–group functor

$$S \mapsto \mathrm{GL}(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S) = \mathrm{End}_S(N \otimes_R S)^\times .$$

So if N is finitely generated projective. then $\mathrm{GL}(N)$ is representable by an affine R–group scheme. Furthermore $\mathrm{GL}(N)$ is of finite presentation.

3.3.2. **Remark.** If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

3.4. **Diagonalizable group schemes.** Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R–algebra of A. As R-module, we have

$$R[A] = \bigoplus_{a \in A} R\, e_a$$

and the multiplication is given by $e_a\, e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R. We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R-algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \to R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$,

Antipode: $\sigma^* : R[A] \to R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \to R$, $\epsilon\big(\sum_{a \in A} r_a\, e_a\big) = r_0$.

We can check easily that it satisfies the axioms of affine commutative group schemes. One important example is that of $A = \mathbb{Z}$. In this case, we find $\mathbb{G}_{m,R} = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. Another one is $A = \mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$ for which we have $\mu_{n,R} = \mathrm{Spec}(R[T]/(T^n - 1)$ called the R–scheme of n–roots of unity.

3.4.1. **Definition.** *We denote by* $\mathfrak{D}(A)/R$ *(or* $\widehat{A}$*) the affine commutative group scheme* $\mathrm{Spec}(R[A])$. *It is called the diagonalizable R–group scheme of base A. An affine R–group scheme is diagonalizable if it is isomorphic to some* $\mathfrak{D}(B)$.

We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$.

If $f : B \to A$ is a morphism of abelian groups, it induces a group homomorphism $f^* : \mathfrak{D}(A) \to \mathfrak{D}(B)$. In particular, when taking $B = \mathbb{Z}$, we have a natural mapping

$$\eta_A : A \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathbb{G}_m).$$

3.4.2. **Remark.** For $a \in A$, put $\chi_a = \eta_A(a) : \mathfrak{D}(A) \to \mathbb{G}_m$. The map $\chi_a^* : R[t, t^{-1}] \to R[A]$ applies t to e_a. Using the commutative diagram

$$\begin{tikzcd}
\mathfrak{D}(A)(R[A]) \arrow[r, "\chi_a"] \arrow[d, "\wr"] & \mathbb{G}_m(R[A]) \arrow[d, "\wr"] \\
\mathrm{Hom}_R(R[A], R[A]) \arrow[r] & \mathrm{Hom}_R(R[t, t^{-1}], R[A]) = R[A]^\times,
\end{tikzcd}$$

we see that the universal element of $\mathfrak{D}(A)$ maps to χ_a^* which corresponds to e_a.

3.4.3. **Lemma.** *If R is connected, η_A is bijective.*

Proof. We establish first the injectivity. If $\eta_A(a) = 0$, it means that the map $R[T, T^{-1}] \to R[A]$, $T \mapsto e_a$ factorises by the augmentation $R[T, T^{-1}] \to R$ hence $a = 0$.

For the surjectivity, let $f : \mathfrak{D}(A) \to \mathbb{G}_m$ be a morphism of R–group schemes. Equivalently it is given by the map $f^* : R[T, T^{-1}] \to R[A]$ of Hopf algebra which satisfies in particular the following compatibility

$$\begin{tikzcd}
R[T, T^{-1}] \arrow[r, "f^*"] \arrow[d, "\Delta"] & R[A] \arrow[d, "\Delta_A"] \\
R[T, T^{-1}] \otimes_R R[T, T^{-1}] \arrow[r, "f^* \otimes f^*"] & R[A] \otimes_R R[A].
\end{tikzcd}$$

In other words, it is determined by the function $X = f^*(T) \in R[A]^\times$ satisfying $\Delta(X) = X \otimes X$. Writing $X = \sum_{a \in A} r_a e_a$, we have

$$\sum_{a \in A} r_a \, e_a \otimes e_a = \sum_{a, a' \in A} r_a \, r_{a'} \, e_a \otimes e_{a'}.$$

It follows that $r_a \, r_b = 0$ if $a \neq b$ and $r_a \, r_a = r_a$. Since the ring is connected, 0 and 1 are the only idempotents so that $r_a = 0$ or $r_a = 1$. Then there exists a unique a such that $r_a = 1$ and $r_b = 0$ for $b \neq a$. This shows that the map η_A is surjective. We conclude that η_A is bijective. ☐

3.4.4. **Proposition.** *(Cartier duality) Assume that R is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable R–group schemes.*

Proof. It is enough to contruct the inverse map $\mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{D}(B)) \to \mathrm{Hom}(A, B)$ for abelian groups A, B. We are given a group homomorphism $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R-gp}(\mathfrak{D}(B), \mathbb{G}_m) \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathbb{G}_m),$$

hence a map $B \to A$. It is routine to check that the two functors are inverse of each other. ☐

3.4.5. **Lemma.** *Assume that R is connected. The following are equivalent:*

(i) A is finitely generated;

(ii) $\mathfrak{D}(A)/R$ is of finite presentation;

(iii) $\mathfrak{D}(A)/R$ is of finite type.

Proof. $(i) \Longrightarrow (ii)$. We use the structure theorem of abelian groups $A \cong: \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \cdots \times \mathbb{Z}/n_c\mathbb{Z}$. Using the compatibility with products we are reduced to the case of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ which correspond to $\mathbb{G}_{m,R}$ and $\mu_{n,R}$. Both are finitely presented over R.

$(ii) \Longrightarrow (iii)$. Obvious.

$(iii) \Longrightarrow (i)$. We assume that $R[A]$ is a finitely generated R–ring. We write $A = \varinjlim_i A_i$ as the inductive limit of finitely generated subgroups. We have $R[A] = \varinjlim_i R[A_i]$. Since the ring $R[A]$ is finitely generated over R, the identity $\mathbb{Z}[A] \to \mathbb{Z}[A]$ factorizes through $\mathbb{Z}[A_i]$ for some i. It implies that $\mathbb{Z}[A_i] \xrightarrow{\sim} \mathbb{Z}[A]$. Cartier duality shows that $A_i \xrightarrow{\sim} A$. Thus A is finitely generated. □

There are other notable properties of Cartier duality, see [SGA3, VIII.2.1]. In practice we will work with finiteness assumptions, however it is remarkable that the theory holds for arbitrary abelian groups.

3.5. **Monomorphisms of group schemes.** We recall that a morphism of R–functors $f : F \to F'$ is a monomorphism if $f(S) : F(S) \to F'(S)$ is injective for each R–algebra S/R (§2.2). If F and F' are functors in groups and f respects the group structure, the kernel of f is the R–group functor defined by $\ker(f)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S.

We recall that a morphism $f : \mathfrak{G} \to \mathfrak{H}$ of affine R-group schemes is a monomorphism if h_f is a monomorphism (Lemma 2.2.2).

3.5.1. **Lemma.** *Let $f : \mathfrak{G} \to \mathfrak{H}$ be a morphism of R–group schemes. Then the R–functor $\ker(f)$ is representable by a closed subgroup scheme of $\mathfrak{G}$.*

Proof. Indeed the carthesian product

$$\begin{tikzcd}
\mathfrak{N} \arrow[r] \arrow[d] & \mathfrak{G} \arrow[d, "f"'] \\
\mathrm{Spec}(R) \arrow[r, "\epsilon'"] & \mathfrak{H}
\end{tikzcd}$$

does the job. □

Summarizing $f : \mathfrak{G} \to \mathfrak{H}$ is a monomorphism if and only if the kernel R-group scheme $\ker(f)$ is the trivial group scheme.

Over a field F, we know that a monomorphism of algebraic groups is a closed immersion [SGA3, VI_B.1.4.2].

Over a DVR, it is not true in general that an open immersion (and a fortiori a monomorphism as seen in the exercise session) of group schemes of finite type is a closed immersion. We consider the following example

[SGA3, VIII.7]. Assume that R is a DVR and consider the constant group scheme $\mathfrak{H} = (\mathbb{Z}/2\mathbb{Z})_R$. Now let $\mathfrak{G}$ be the open subgroup scheme of $\mathfrak{H}$ which is the complement of the closed point 1 in the closed fiber. By construction $\mathfrak{G}$ is dense in $\mathfrak{H}$, so that the immersion $\mathfrak{G} \to \mathfrak{H}$ is not closed. Raynaud constructed a more elaborated example where $\mathfrak{H}$ and $\mathfrak{G}$ are both affine over $\mathbf{F}_2[[t]]$ and a monomorphism which is not an immersion [SGA3, XVI.1.1.c].

However diagonalizable groups have a wonderful behaviour with that respect by using Cartier duality (Proposition 3.4.4).

3.5.2. **Proposition.** *Assume that R is connected. Let $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ be a group homomorphism of diagonalizable R–group schemes. Then the following are equivalent:*

(i) $f^* : A \to B$ is onto;

(ii) f is a closed immersion;

(iii) f is a monomorphism.

Proof. $(i) \Longrightarrow (ii)$: Then $R[B]$ is a quotient of $R[A]$ so that $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is a closed immersion.

$(ii) \Longrightarrow (iii)$: obvious.

$(iii) \Longrightarrow (i)$: We denote by $B_0 \subset B$ the image of $f^* : A \to B$. We consider the compositum

$$\begin{tikzcd}
\mathfrak{D}(B/B_0) \arrow[r] \arrow[rr, bend right, "v"'] & \mathfrak{D}(B) \arrow[r, "v"] \arrow[rr, bend left, "f"] & \mathfrak{D}(B_0) \arrow[r, "w"] & \mathfrak{D}(A).
\end{tikzcd}$$

We observe that it is the trivial morphism (v is trivial) and is a monomorphism as compositum of the monomorphisms u and f. It follows that $\mathfrak{D}(B/B_0) = \mathrm{Spec}(R)$ and we conclude that $B_0 = B$ by Cartier duality. □

Of the same flavour, the kernel of a map $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is isomorphic to $\mathfrak{D}(f(A))$. The case of vector groups is more subtle.

3.5.3. **Proposition.** *Let $f : N_1 \to N_2$ be a morphism of finitely generated projective R-modules. Then the morphism of functors $f_* : W(N_1) \to W(N_2)$ is a monomorphism if and only if f identifies N_1 as a direct summand of N_2. If it the case, $f_* : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$ is a closed immersion.*

Proof. If N_1 is a direct summand of N_2, the morphism $f_* : W(N_1) = V(N_1^\vee) \to W(N_2^\vee)$ is a closed immersion and a fortiori a monomorphism. Conversely we assume that $f_* : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$ is a monomorphism.

Conversely suppose that f_* is a monomorphism. Since $W(N_1)(R)$ injects in $W(N_2)(R)$, we have that $f : N_1 \to N_2$ is injective. We put $N_3 = N_2/f(N_1)$. To show that N_1 is a direct summand of N_2 it is enough to show that N_3 is (finitely generated projective). This is our plan. Since

N_2 and N_3 are f.g. projective R–modules, the R–module N_3 is of finite presentation. In view of the characterization of f.g. projective modules [Bbk2, II.5.2], it is enough to show that $N_3 \otimes R_{\mathfrak{m}}$ is free for each maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{m}$ be a maximal ideal of R.

Applying the criterion of Lemma 2.2.1 to the residue field $S = R/\mathfrak{m}$ we have that the map

$$f_*(R/\mathfrak{m}) : N_1 \otimes_R R/\mathfrak{m} \to N_2 \otimes_R R/\mathfrak{m}$$

is injective. It follows that there exists an $R/\mathfrak{m}$-base $(\overline{w}_1, \dots, \overline{w}_r, \overline{w}_{r+1}, \dots, \overline{w}_n)$ of $N_2 \otimes_R R/\mathfrak{m}$ such that $(\overline{w}_1, ..., \overline{w}_r)$ is a base of $f(N_1 \otimes_R R/\mathfrak{m})$. We have $\overline{w}_i = f(\overline{v}_i)$ for $i = 1, .., r$. We lift the $\overline{v}_i$'s in an arbitrary way in $N_1 \otimes_R R_{\mathfrak{m}}$ and the $\overline{w}_{r+1}, \dots, \overline{w}_n$ in $N_2 \otimes_R R_{\mathfrak{m}}$. Then $(v_1, \dots, v_r)$ is an $R_{\mathfrak{m}}$-base of $N_1 \otimes_R R_{\mathfrak{m}}$ and $(f(v_1), \dots, f(v_r), w_{r+1}, \dots, w_n)$ is an $R_{\mathfrak{m}}$–base of $N_2 \otimes_R R_{\mathfrak{m}}$. Thus $N_3 \otimes_R R_{\mathfrak{m}}$ is free.

We conclude that f identifies N_1 as a direct summand of N_2. □

4. SEQUENCES OF GROUP FUNCTORS

4.1. **Exactness.** We say that a sequence of R–group functors

$$1 \to F_1 \overset{u}{\to} F_2 \overset{v}{\to} F_3 \to 1$$

is exact if for each R–algebra S, the sequence of abstract groups

$$1 \to F_1(S) \overset{u(S)}{\to} F_2(S) \overset{v(S)}{\to} F_3(S) \to 1$$

is exact. Similarly we can define the exactness of a sequence $1 \to F_1 \to \cdots \to F_n \to 1$. If $w : F \to F'$ is a map of R–group functors, recall the definition of the R–group functor $\ker(w)$ by $\ker(w)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S. Also the cokernel $\operatorname{coker}(w)(S) = \operatorname{coker}(F(S) \to F'(S))$ is an R-functor (but not necessarily an R-functor in groups).

4.1.1. **Example.** We consider an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of finitely generated modules with N_1, N_2 projective. We claim that it induces an exact sequence of R–functors in groups

$$0 \to W(N_1) \to W(N_2) \to W(N_3) \to 0$$

if and only if the starting sequence is split (equivalently N_3 is projective). The converse implication is obvious. If the sequence above of R–functors in groups is exact, then $W(N_1) \to W(N_2)$ is a monomorphism so that Proposition 3.5.3 shows that N_1 is a direct summand of N_2.

We can define also the cokernel of a morphism R–group schemes. But it is very rarely representable. The simplest example is the Kummer morphism $f_n : \mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$, $x \mapsto x^n$ for $n \geq 2$ for $R = \mathbb{C}$, the field of complex numbers. Assume that there exists an affine $\mathbb{C}$–group scheme $\mathfrak{G}$ such that there is a four terms exact sequence of $\mathbb{C}$–functors

$$1 \to h_{\mu_n} \to h_{\mathbb{G}_m} \overset{h_{f_n}}{\to} h_{\mathbb{G}_m} \to h_{\mathfrak{G}} \to 1$$

We denote by T' the parameter for the first $\mathbb{G}_m$ and by $T = (T')^n$ the parameter of the second one. Then $T \in \mathbb{G}_m(R[T, T^{-1}])$ defines a non trivial element of $\mathfrak{G}(R[T, T^{-1}])$ which is trivial in $\mathfrak{G}(R[T', T'^{-1}])$ It is a contradiction.

We provide a criterion.

4.1.2. **Lemma.** *Let*

$$1 \to \mathfrak{G}_1 \overset{u}{\to} \mathfrak{G}_2 \overset{v}{\to} \mathfrak{G}_3 \to 1$$

be a sequence of affine R–group schemes. Then the sequence of R–functors

$$1 \to h_{\mathfrak{G}_1} \to h_{\mathfrak{G}_2} \to h_{\mathfrak{G}_3} \to 1$$

is exact if and only if the following conditions are satisfied:

(i) $u : \mathfrak{G}_1 \to \ker(v)$ is an isomorphism;

(ii) $v : \mathfrak{G}_2 \to \mathfrak{G}_3$ admits a splitting $f : \mathfrak{G}_3 \to \mathfrak{G}_2$ as R–schemes.

4.1.3. **Remark.** Note that if (ii) holds, we have $\mathfrak{G}_2(S) = u(\mathfrak{G}_1(S))f(\mathfrak{G}_3(S))$ for each R-algebra S. Let S be an R–algebra and let $g_2 \in \mathfrak{G}_2(S)$. Since $\mathfrak{G}_1(S) \to \mathfrak{G}_2(S) \to \mathfrak{G}_3(S)$ is exact, $g_2\, f(v(g_2))^{-1} \in \mathfrak{G}_1(S)$. We conclude that $\mathfrak{G}_2(S) = u(\mathfrak{G}_1(S))f(\mathfrak{G}_3(S))$.

We proceed to the proof of Lemma 4.1.2.

Proof. We assume that the sequence of R-functors $1 \to h_{\mathfrak{G}_1} \to h_{\mathfrak{G}_2} \to h_{\mathfrak{G}_3} \to 1$ is exact. We have seen that $\mathfrak{G}_1$ is the kernel of v. This shows (i). The assertion (ii) is an avatar of Yoneda's lemma. We consider the surjective map $\mathfrak{G}_2(R[\mathfrak{G}_3]) \to \mathfrak{G}_3(R[\mathfrak{G}_3])$ and lift the identity of $\mathfrak{G}_3$ to a map $t : \mathfrak{G}_2(R[\mathfrak{G}_3]) = \mathrm{Hom}_{R-sch}(\mathfrak{G}_3, \mathfrak{G}_2)$. Then t is an R–scheme splitting of $v : \mathfrak{G}_2 \to \mathfrak{G}_3$.

Conversely we assume (i) and (ii). Clearly $h_{\mathfrak{G}_1} \to h_{\mathfrak{G}_2}$ is a monomorphism and $h_{\mathfrak{G}_2} \to h_{\mathfrak{G}_3}$ is a epimorphism (see §2.2). We only have to check the exactness of $\mathfrak{G}_1(S) \to \mathfrak{G}_2(S) \to \mathfrak{G}_3(S)$ for each S/R but it follows from (ii). □

4.1.4. **Examples.** (a) It is not obvious to construct examples of exact sequences of group functors which are not split as R–group functors. An example is the exact sequence of Witt vectors groups over $\mathbb{F}_p$ $0 \to W_1 \to W_2 \to W_1 \to 0$. It provides a non split exact sequence of commutative affine $\mathbb{F}_p$–group schemes $0 \to \mathbb{G}_a \to W_2 \to \mathbb{G}_a \to 0$. For other examples see [DG, III.6]. (b) Also it is natural question to ask whether the existence of sections of the map $\mathfrak{G}_2 \to \mathfrak{G}_3$ locally over $\mathfrak{G}_3$ is enough. It is not the case and an example of this phenomenon is by using the $\mathbb{R}$–group scheme G_2 defined as the unit group scheme of the $\mathbb{R}$-algebra $\mathbb{C}$; recall that its functor of points is $G_2(S) = (S \otimes_{\mathbb{R}} \mathbb{C})^\times$ (§3.3). It comes with a norm morphism $N : G_2(S) \to \mathbb{G}_{m,\mathbb{R}}$ and we consider the kernel $G_3 = \ker(N)$. Note that G_2

comes with an involution σ given by the complex conjugation. We consider the sequence of $\mathbb{R}$–group schemes

$$1 \to \mathbb{G}_m \to G_2 \xrightarrow{\sigma - id} G_3 \to 1.$$

The associated sequence for real points is $1 \to \mathbb{R}^\times \to \mathbb{C}^\times \to S^1 \to 1$, where the last map is $z \mapsto \overline{z}/z$. For topological reasons[2], there is no continuous section of the map $\mathbb{C}^\times \to S^1$. A fortiori, there is no algebraic section of the map $G_2 \xrightarrow{\sigma - id} G_3$. On the other hand this map admits local splittings, let us explain how it works for example on $\mathfrak{G}_3 \setminus \{(-1,0)\}$. We map $t \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) = (\sigma - 1).(1+ti)$ induces an isomorphism $\mathbb{R}[\mathfrak{G}_3]_{(-1,0)} \xrightarrow{\sim} \mathbb{R}\bigl[t, \frac{1}{t^2+1}\bigr]$ and defines a section of $\sigma - id$ on $\mathfrak{G}_3 \setminus \{(-1,0)\}$. The sequence above is not exact in the category of $\mathbb{R}$–functors.

4.2. **Semi-direct product.** Let $\mathfrak{G}/R$ be an affine group scheme acting on another affine group scheme $\mathfrak{H}/R$, that is we are given a morphism of R–functors

$$\theta : h_{\mathfrak{G}} \to \mathrm{Aut}(h_{\mathfrak{H}}).$$

The semi-direct product $h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ is well defined as R–functor.

4.2.1. **Lemma.** *$h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ is representable by an affine R-scheme denote by $\mathfrak{H} \rtimes^\theta \mathfrak{G}$. Furthermore we have an exact sequence of affine R-group schemes*

$$1 \to \mathfrak{H} \to \mathfrak{H} \rtimes^\theta \mathfrak{G} \to \mathfrak{G} \to 1.$$

Proof. We consider the affine R-scheme $\mathfrak{X} = \mathfrak{H} \times_R \mathfrak{G}$. Then $h_{\mathfrak{X}} = h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ has a group structure so defines a group scheme structure on $\mathfrak{X}$. The sequence holds in view ot the criterion provided by Lemma 4.1.2. □

A nice example of this construction is the "affine group" of a finitely generated. projective R–module N. The R-group scheme $\mathrm{GL}(N)$ acts on the vector R–group $\mathfrak{W}_N$ so that we can form the R–group scheme $\mathfrak{W}_N \rtimes \mathrm{GL}(N)$ of affine transformations of N.

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[2] The induced map $\mathbb{Z} = \pi(\mathbb{C}^\times, 1) \to \mathbb{Z} = \pi_1(S^1, 1)$ is the multipliczation by 2.

Affine group schemes II

5. Flatness

We will explain in this section why flatness is a somehow a minimal reasonable assumption when studying affine group schemes. This includes a nice behaviour of the dimension of geometric fibers, see Thm. 5.3.1 below.

5.1. Examples of flat affine group schemes.

5.1.1. **Lemma.** *Let $\mathfrak{G}$ be an affine R-group scheme. Then $\mathfrak{G}$ is flat if and only if $\mathfrak{G}$ is faithfully flat.*

Proof. Faithfully flat means that the structural morphism $\mathfrak{G} \to \mathrm{Spec}(R)$ is flat and surjective. Since $\mathfrak{G} \to \mathrm{Spec}(R)$ admits the unit section, the structural morphism is surjective. This explains the equivalence between flatness and faithfully flatness in our setting. □

All examples we have seen so far were flat. Constant group schemes are obviously flat. If A is an abelian group, the diagonalizable R–group scheme $\mathfrak{D}(A)$ is R–flat since $R[A]$ is a free R-module.

If N is a finitely generated projective R-module, the affine group schemes $\mathfrak{V}(N)$ and $\mathfrak{W}(N)$ are flat. Indeed, flatness is a local property for the Zariski topology on $\mathrm{Spec}(R)$ [St, Tag 00HJ] so that it reduces to the case of the affine space $\mathbb{A}^d_R$ which is clear since the R-module $R[t_1, \dots, t_d]$ is free. A more complicated fact is the following.

5.1.2. **Lemma.** *Let M be an R-module. Then M is flat if and only if $\mathfrak{V}(M)$ is a flat R-scheme.*

Proof. By definition the R-scheme $\mathfrak{V}(M)$ is flat if and only is the symmetric algebra $S^*(M)$ is a flat R–module. Since M is a direct summand of $S^*(M)$ as R-module, the flatness of $S^*(M)$ implies that M is flat.

For the converse we use Lazard's theorem stating that M is isomorphic to a direct limit $\varinjlim_{i \in I} M_i$ of f.g. free R-modules [St, Tag 058G]. Since $S^*(M) = \varinjlim_{i \in I} S^*(M_i)$ and each $S^*(M_i)$ is a free R-module (so a fortiori flat), it follows that $S^*(M)$ is a flat R–algebra in view of [St, Tag 05UU] (use the case $R_i = R$ for all i).

□

Finally the group scheme of invertible elements $U(A)$ of an algebra A/R f.g. projective is flat. We have seen that $U(A)$ is principal open in $\mathfrak{W}(A)$ so that $R[U(A)]$ is flat over $R[\mathfrak{W}(A)]$ [St, Tag 00HT]. Since flatness behaves well for composition [GW, prop. 14.3], we conclude that the affine R-scheme $U(A)$ is flat.

5.2. **The DVR case.** Assume that R is a DVR with uniformizing parameter π and denote by K its field of fractions. We recall the following well-known fact.

5.2.1. **Lemma.** *Let M be an R–module. Then the following are equivalent:*

(i) M is flat;

(ii) M is torsion free, that is $\times\pi : M \to M$ is injective;

(iii) $M \to M \otimes_R K$ is injective.

Furthermore, if M is finitely generated, this is equivalent to $M \cong R^n$.

Proof. $(i) \implies (ii)$. It means that the functor $\otimes_R M$ is exact. Since $\pi : R \to R$ is injective, it follows that $\times\pi : M \to M$.

$(ii) \implies (i)$. The R module M is the filtered inductive limit of its finitely generated submodules. Also, submodules of torsionfree modules are torsionfree, and inductive limits of flat modules are flat [St, Tag 05UU]. This is why it suffices to prove that finitely generated torsionfree R-modules are flat, or even free. We assume then that M is a finitely generated R-module. Choose $m_1, \dots, m_n \in M$ such that $\overline{m_1}, \dots, \overline{m_n}$ is a k-basis of the k–vector space $M \otimes_R k$. By Nakayama's Lemma, $m_1, \dots, m_n$ is a generating set of M; in other words we have a surjective R–map $f : R^n \to M$. Consider a non zero relation $f(r_1, \dots, r_n) = \sum_{i=1}^n r_i \, m_i = 0$. Since M is torsionfree, dividing the r_i' by the largest possible power π^c occuring so that we get a non-trivial relation $\sum_{i=1}^n \overline{r_i} \, \overline{m}_i = 0$. This is a contradiction.

$(ii) \implies (iii)$. Once again this reduces to the finitely generated case which is free. Since $R^n \to K^n$ is injective, we are done.

$(iii) \implies (ii)$. Obvious.

□

Note that there are generalization to Dedekind domains and valuation rings [St, Tags 0AUW, 0539]. From the lemma, we know that an affine scheme $\mathfrak{X}/R$ is flat, that is, $R[\mathfrak{X}]$ is torsionfree or equivalently that $R[\mathfrak{X}]$ embeds in $K[\mathfrak{X}]$.

5.2.2. **Proposition.** [EGA4, 2.8.1] *(see also* [GW, §14.3]*)*

Let $\mathfrak{X}/R$ be a flat affine R-group scheme. There is a one to one correspondence between the flat closed R-subschemes of $\mathfrak{X}$ and the closed K–subschemes of the generic fiber $\mathfrak{X}_K$.

Furthermore this correspondence commutes with fibered products over R and is functorial with respect to R-morphisms $\mathfrak{X} \to \mathfrak{X}'$ of flat R–schemes.

The correspondence goes as follows. In one way we take the generic fiber and in the way around we take the schematic closure (in the sense of the scheme theoretic image of the immersion map $Y \subset \mathfrak{X}_K \hookrightarrow \mathfrak{X}$ [St, Tag 01R7]). The schematic closure $\mathfrak{Y}$ of Y in $\mathfrak{X}$ is the smallest closed subscheme $\mathfrak{X}$ such that $Y \subset \mathfrak{X}_K \hookrightarrow \mathfrak{X}$ factorizes through $\mathfrak{Y}$. Let us explain its construction in terms of rings. If Y/K is a closed K–subscheme of X/K, it is defined by the ideal $I(Y) = \mathrm{Ker}(K[\mathfrak{X}] \to K[Y])$ of $K[\mathfrak{X}]$. Similarly we deal with the ideal $I(\mathfrak{Y}) = \mathrm{Ker}(R[\mathfrak{X}] \to R[\mathfrak{Y}])$ of $R[\mathfrak{X}]$. This fits in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I(\mathfrak{Y}) \arrow[r] \arrow[d] & R[\mathfrak{X}] \arrow[r] \arrow[d, hook] & R[\mathfrak{Y}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I(Y) \arrow[r] & K[\mathfrak{X}] \arrow[r] & K[Y] \arrow[r] & 0
\end{tikzcd}$$

The ideal $I(\mathfrak{Y})$ of $R[\mathfrak{X}]$ is the smallest ideal which maps in $I(Y)$, it follows that $I(\mathfrak{Y}) = I \cap R[\mathfrak{X}]$. Since $I(\mathfrak{Y}) \otimes_R K = I(Y)$, we have $R[\mathfrak{Y}] \otimes_R K = K[Y]$, that is, $\mathfrak{Y} \times_R K = Y_K$. Also the map $R[\mathfrak{Y}] \to K[Y]$ is injective, i.e. $\mathfrak{Y}$ is a flat affine R–scheme. It remains to show that the other composite is the identity and also the functorial properties. We proceed then to the end of the proof of Proposition 5.2.2.

Proof. Given $\mathfrak{Y} \subset \mathfrak{X}$ a flat closed R–subcheme, we consider the ideal $I(\mathfrak{Y}) = \mathrm{Ker}(R[\mathfrak{X}] \to R[\mathfrak{Y}])$. We denote by $\mathfrak{Y}' \subset \mathfrak{X}$ the schematic closure of $\mathfrak{Y}_K \subset \mathfrak{X}$. We have $I(\mathfrak{Y}') = I(\mathfrak{Y}_K) \cap R[\mathfrak{X}]$. We consider the commutative diagram of exact sequences of R-modules

$$\begin{tikzcd}
0 \arrow[r] & I(\mathfrak{Y}) \arrow[r] \arrow[d, hook] & R[\mathfrak{X}] \arrow[r] \arrow[d, hook] & R[\mathfrak{Y}] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I(\mathfrak{Y}_K) \arrow[r] & K[\mathfrak{X}] \arrow[r] & K[\mathfrak{Y}] \arrow[r] & 0
\end{tikzcd}$$

where the two vertical maps on the right express flatness of $\mathfrak{X}$ and $\mathfrak{Y}$. By diagram chase we have $I(\mathfrak{Y}) = I(\mathfrak{Y}')$.

We examine now the behaviour for fibered products, We are given two affine flat R–schemes $\mathfrak{X}_1$, $\mathfrak{X}_2$ with closed flat R-subschemes $\mathfrak{Y}_1 \subset \mathfrak{X}_1$ and $\mathfrak{Y}_2 \subset \mathfrak{X}_2$. Then $\mathfrak{Y}_1 \times_R \mathfrak{Y}_2$ is a flat closed R–subscheme (using that flatness behaves well with tensor products, see [Bbk2, §I.7]) of $\mathfrak{X}_1 \times_R \mathfrak{X}_2$ and of generic fiber $\mathfrak{Y}_{1,K} \times_K \mathfrak{Y}_{2,K}$ so is the schematic closure of $\mathfrak{Y}_{1,K} \times_K \mathfrak{Y}_{2,K}$ in $\mathfrak{X}_1 \times_R \mathfrak{X}_2$.

Next we deal with a morphism $f : \mathfrak{X} \to \mathfrak{X}'$ of affine flat R–schemes. For an affine flat closed R–subcheme $\mathfrak{Y} \subset \mathfrak{X}$ (resp. $\mathfrak{Y}' \subset \mathfrak{X}'$), if f induces a morphism $\mathfrak{Y} \to \mathfrak{Y}'$ then f_K induces a map $\mathfrak{Y}_K \to \mathfrak{Y}'_K$. Conversely assume that f_K induces a map $f_K : Y' \to Y$ where $Y \subset \mathfrak{X}_K$ (resp. $Y' \subset \mathfrak{X}'_K$) and denote by $\mathfrak{Y} \subset \mathfrak{X}$ (resp. $\mathfrak{Y}' \subset \mathfrak{X}'$) the schematic adherence of Y. We need to check that f induces a map $\mathfrak{Y} \to \mathfrak{Y}'$. We consider the diagram

$$\begin{tikzcd}
R[\mathfrak{X}'] \arrow[r, "f^*"] \arrow[d, hook] & R[\mathfrak{X}] \arrow[d, hook] \\
K[\mathfrak{X}'] \arrow[r, "f_K^*"] & K[\mathfrak{X}] \\
I(Y') \arrow[r] \arrow[u, phantom, "\bigcup"] & I(Y) \arrow[u, phantom, "\bigcup"]
\end{tikzcd}$$

It shows that $f^*(R[\mathfrak{X}'] \cap I(Y')) \subseteq R[\mathfrak{X}] \cap I(Y)$ whence $f^*(R[\mathfrak{Y}']) \subseteq R[\mathfrak{Y}]$ as desired. □

In particular, if $\mathfrak{G}/R$ is a flat group scheme, it induces a one to one correspondence between flat closed R-subgroup schemes of $\mathfrak{G}$ and closed K–subgroup schemes of $\mathfrak{G}_K$[3].

5.2.3. **Example.** We consider the centralizer closed subgroup scheme of $\mathrm{GL}_{2,R}$

$$\mathfrak{Z} = \Big\{ g \in \mathrm{GL}_{2,R} \ \mid \ g\,A = A\,g \Big\}$$

of the element $A = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$. Then $\mathfrak{Z} \times_R R/\pi R \xrightarrow{\sim} \mathrm{GL}_{2,R}$ and

$$\mathfrak{Z} \times_R K = \mathbb{G}_{m,K} \times_K \mathbb{G}_{a,K} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\}$$

Then the closure of $\mathfrak{Z}_K$ in $\mathrm{GL}_{2,R}$ is $\mathbb{G}_{m,R} \times_R \mathbb{G}_{a,R}$, so does not contain the special fiber of $\mathfrak{Z}$. We conclude that $\mathfrak{Z}$ is not flat.

5.3. **A necessary condition.** In the above example, the geometrical fibers were of dimension 4 and 2 respectively. It illustrates then the following general result.

5.3.1. **Theorem.** [SGA3, VI_B.4.3] *Let R be a ring and let $\mathfrak{G}/R$ be a flat group scheme of finite presentation. Then the dimension of the geometrical fibers is locally constant.*

It means that the dimension of the fibers cannot jump by specialization.

6. REPRESENTATIONS

Let $\mathfrak{G}/R$ be an a affine group scheme.

6.0.1. **Definition.** *A (left) $R - \mathfrak{G}$-module (or $\mathfrak{G}$-module for short) is an R–module M equipped with a morphism of group functors*

$$\rho : h_{\mathfrak{G}} \to \mathrm{Aut}_{lin}(W(M)).$$

We say that the $\mathfrak{G}$–module M is faithful is ρ is a monomorphism.

---

[3] Warning: the fact that the schematic closure of a group scheme is a group scheme is specific to Dedekind rings.

Here $\mathrm{Aut}_{lin}(W(M))$ stands for for linear automorphisms of the functor $W(M)$, that is, $\mathrm{Aut}_{lin}(W(M))(S) = \mathrm{End}_S(M \otimes_R S)^\times$ for each R–algebra S. We denote by $\mathrm{GL}(M)$ and we bear in mind that is not necessarily representable.

If $M = R^n$, then $\mathrm{GL}(M)$ is representable by $\mathrm{GL}_{n,R}$ so that it corresponds to an R–group homomorphism $\mathfrak{G} \to \mathrm{GL}_{n,R}$ and faithfulness corresponds to the triviality of the kernel.

Coming back to the general setting, it means that for each algebra S/R, we are given an action of $\mathfrak{G}(S)$ on $W(M)(S) = M \otimes_R S$. We use again Yoneda lemma. The mapping ρ is defined by the image of the universal point $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ provides an element called the coaction

$$c_\rho \in \mathrm{Hom}_R\Big(M, M \otimes_R R[\mathfrak{G}]\Big) \xrightarrow{\sim} \mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big).$$

Yoneda lemma implies that c_ρ determines ρ. We denote $\widetilde{c}_\rho$ its image in $\mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big)$. For $g \in \mathfrak{G}(R)$, we use the evaluation $\epsilon_g : R[\mathfrak{G}] \to \mathfrak{R}$ and have by functoriality the bottom of the following commutative diagram

(6.0.2)

$$\begin{tikzcd}
M \arrow[d, "id"] \arrow[rd, "c_\rho"] \arrow[dd, bend right=70, "id"'] & \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "\widetilde{c}_\rho"] \arrow[d, "id\otimes\epsilon_g"] & M \otimes_R R[G] \arrow[d, "id\otimes\epsilon_g"] \\
M \arrow[r, "\rho(g)"] & M.
\end{tikzcd}$$

In other words we have

$$\rho(g).m = \epsilon_g(c_\rho.m) \qquad (g \in \mathfrak{G}(R), m \in M). \tag{6.0.3}$$

6.0.4. **Remark.** For the trivial representation, we have that $\tilde{c}_{triv} = id_{M \otimes_R R[\mathfrak{G}]}$ so that $c_{triv}(m) = m \otimes 1$.

6.0.5. **Proposition.** *(1) Both diagrams*

(6.0.6)

$$\begin{tikzcd}
M \arrow[r, "c_\rho"] \arrow[d, "c_\rho"'] & M \otimes_R R[\mathfrak{G}] \arrow[d, "id\otimes\Delta_{\mathfrak{G}}"] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "c_\rho\otimes id"] & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}],
\end{tikzcd}$$

(6.0.7)

$$\begin{tikzcd}
M \arrow[r, "c_\rho"] \arrow[d, "id"'] & M \otimes_R R[\mathfrak{G}] \arrow[ld, "id\times\epsilon^*"] \\
M &
\end{tikzcd}$$

commute.

(2) Conversely, if an R*–map* $c : M \to M \otimes_R R[\mathfrak{G}]$ *satisfying the two rules above, there is a unique representation* $\rho_c : h_{\mathfrak{G}} \to \mathrm{GL}(W(M))$ *such that* $c_{\rho_c} = c$*.*

A module M equipped with an R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above is called a $\mathfrak{G}$-module (and also a comodule over the Hopf algebra $R[\mathfrak{G}]$). The proposition shows that it is the same to talk about representations of $\mathfrak{G}$ or about $\mathfrak{G}$-modules (or also $R - \mathfrak{G}$-modules).

6.0.8. **Remark.** There is of course a compatibility with the inverse map but it follows from the other rules.

In particular, the comultiplication $R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ defines a $\mathfrak{G}$-structure on the R–module $R[\mathfrak{G}]$. It is called the regular representation and is studied more closely in Example 6.0.9. We proceed to the proof of Proposition 6.0.5.

Proof. (1) We double the notation by putting $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$. We consider the following commutative diagram

$$\begin{tikzcd}
\mathfrak{G}(R[\mathfrak{G}_1]) \times \mathfrak{G}(R[\mathfrak{G}_2]) \arrow[r, "\rho\times\rho"] \arrow[d] & \mathrm{GL}(M)(R[\mathfrak{G}_1]) \times \mathrm{GL}(M)(R[\mathfrak{G}_2]) \arrow[d] \\
\mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \arrow[r, "\rho\times\rho"] \arrow[d, "m"'] & \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \arrow[d, "m"'] \\
\mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \arrow[r, "\rho"] & \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2])
\end{tikzcd}$$

and consider the image $\eta \in \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2])$ of the couple (ζ_1, ζ_2) of universal elements by the left vertical map. Then η is defined by the ring homomorphism $\eta^* : R[\mathfrak{G}] \overset{\Delta_{\mathfrak{G}}}{\to} R[\mathfrak{G} \times \mathfrak{G}] \xrightarrow{\sim} R[\mathfrak{G}_1 \times \mathfrak{G}_2]$ so that $\rho(\eta)$ is defined by the following commutative diagram (in view of the compatibility (6.0.2))

$$\begin{tikzcd}
M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \arrow[r, "\rho(\eta)"] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "\tilde{c}_\rho"] \arrow[u, "id_M\otimes\Delta"] & M \otimes_R R[\mathfrak{G}] \arrow[u, "id_M\otimes\Delta"]
\end{tikzcd}$$

On the other hand we have that $\rho(\eta) = \widetilde{c}_{\rho,2} \circ \widetilde{c}_{\rho,1}$ where we did not write the extensions to $R[\mathfrak{G}_1 \times \mathfrak{G}_2]$. Reporting that fact in the diagram above provides the commutative diagram

$$\begin{tikzcd}
M \arrow[r] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \arrow[r, "\tilde{c}_{\rho_2}\circ\tilde{c}_{\rho_1}"] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
M \arrow[r] \arrow[u, "id"] & M \otimes_R R[\mathfrak{G}] \arrow[r, "\tilde{c}_\rho"] \arrow[u, "id_M\otimes\Delta"] & M \otimes_R R[\mathfrak{G}]. \arrow[u, "id_M\otimes\Delta"]
\end{tikzcd}$$

By restricting to M, we get the commutative square

$$\begin{tikzcd}
M \otimes_R R[\mathfrak{G}_1] \arrow[r, "c_{\rho_2} \otimes id_{R[\mathfrak{G}_1]}"] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
M \arrow[u, "c_{\rho,1}"] \arrow[r, "c_\rho"] & M \otimes_R R[\mathfrak{G}] \arrow[u, "\Delta"']
\end{tikzcd}$$

as desired. The other rule comes from the fact that $1 \in G(R)$ acts trivially on M and is a special case of the diagram (6.0.2).

(2) We are given $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules. We define first a morphism of R–functors $h_{\mathfrak{G}} \to W(End_R(M))$. According to Yoneda lemma 2.1.1, we have

$$\operatorname{Hom}_{R-func}\big(h_{\mathfrak{G}}, W(End_R(M))\big) = W(End_R(M))(R[\mathfrak{G}])$$

$$= \operatorname{Hom}_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]) \xleftarrow{\sim} \operatorname{Hom}_R(M, M \otimes_R R[\mathfrak{G}]).$$

It follows that c defines a (unique) morphism of R–functors $\rho_c : h_{\mathfrak{G}} \to W(End_R(M))$ such that the universal element of $\mathfrak{G}$ is applied to $\tilde{c}$. The first rule insures the multiplicativity (check it) and the second rule says that the unit element $1 \in \mathfrak{G}(R)$ is applied to id_M. It follows that ρ_c factorizes through the subfunctor $\mathrm{GL}(M)$ of $W(End_R(M))$ and induces a homomorphism of R–group functors $h_{\mathfrak{G}} \to \mathrm{GL}(M)$. □

6.0.9. **Example.** We claim that the regular representation is nothing but the right translation on $R[\mathfrak{G}]$ and that it is faithful. We consider the $\mathfrak{G}$–module $A = R[\mathfrak{G}]$ defined by the comultiplication $\Delta : A \to A \otimes_R R[\mathfrak{G}]$. It defines the regular representation $\rho : \mathfrak{G} \to \mathrm{GL}(A)$. Given $g \in \mathfrak{G}(R)$, we consider the following diagram (special case of the diagram (6.0.2))

(6.0.10)

$$\begin{tikzcd}
A \arrow[d, "id"] \arrow[dr, "\Delta"] & \\
A \otimes_R R[\mathfrak{G}] \arrow[r, "\widetilde{\Delta}"] \arrow[d, "id\otimes\epsilon_g"] & A \otimes_R R[\mathfrak{G}] \arrow[d, "id\otimes\epsilon_g"] \\
A \arrow[r, "\rho(g)"] & A.
\end{tikzcd}$$

where ϵ_g is the evaluation at g and where the bottom is the compatibility (6.0.2). In terms of schemes, the map below is $\mathfrak{G} = \mathfrak{G} \times_R \operatorname{Spec}(R) \xrightarrow{id_{\mathfrak{G}} \times g} \mathfrak{G} \times_R \mathfrak{G} \xrightarrow{product} \mathfrak{G}$. It follows that $\rho(g).a = a \circ R_g = R_g^*(a)$ for each $a \in A = R[\mathfrak{G}]$ where $R_g : \mathfrak{G} \to \mathfrak{G}$ is the right translation by g, $x \mapsto xg$.

Let us show that the regular representation $R[\mathfrak{G}]$ is faithful. Let S be an R–ring and let $g \in \mathfrak{G}(S)$ acting trivially on $S[\mathfrak{G}]$. It means that $f \circ R_g = f$ for each $f \in S[\mathfrak{G}]$ hence $f(g) = f(1)$ for each $f \in S[\mathfrak{G}]$. But

$\mathfrak{G}(S) = \mathrm{Hom}_S(S[\mathfrak{G}], S)$, so that $g = 1$. This shows that the regular representation is faithful.

A morphism of $\mathfrak{G}$-modules is an R–morphism $f : M \to M'$ such that $f(S) \circ \rho(g) = \rho'(g) \circ f(S) \in \mathrm{Hom}_S(M \otimes_R S, M' \otimes_R S)$ for each S/R and for $g \in \mathfrak{G}(S)$. Equivalenty, this is to require the commutativty of the following diagram

(6.0.11)
$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "c_\rho"] & M' \arrow[d, "c_{\rho'}"] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "f \otimes id"] & M' \otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

It is clear that the R–module $\mathrm{coker}(f)$ is equipped then with a natural structure of $\mathfrak{G}$-module. For the kernel $\ker(f)$, we cannot proceed similarly because the mapping $\ker(f) \otimes_R S \to \ker(M \otimes_R S \overset{f(S)}{\to} M' \otimes_R S)$ is not necessarily injective. One tries to use the module viewpoint by considering the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] & M \arrow[r, "f"] \arrow[d, "c_\rho"'] & M' \arrow[d, "c_{\rho'}"'] \\
 & \ker(f) \otimes_R R[\mathfrak{G}] \arrow[r] & M \otimes_R R[\mathfrak{G}] \arrow[r, "f \otimes id"] & M' \otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

If $\mathfrak{G}$ is flat, then the left bottom map is injective, and the diagram defines a map $c : \ker(f) \to \ker(f) \otimes_R R[\mathfrak{G}]$. This map c satisfies the two compatibilities and define then a $\mathfrak{G}$-module structure on $\ker(f)$. We have proven the important fact.

6.0.12. **Proposition.** *Assume that $\mathfrak{G}/R$ is flat. Then the category of $\mathfrak{G}$-modules is an abelian category.*

6.0.13. **Remark.** It is actually more than an abelian category since it carries tensor products, see below.

6.1. **Tensor products.** Given two homomorphisms $\rho_1 : h_{\mathfrak{G}} \to \mathrm{GL}(M_1)$, $\rho_2 : h_{\mathfrak{G}} \to \mathrm{GL}(M_2)$ we can form the tensor product

$$\rho_1 \otimes \rho_2 : h_{\mathfrak{G}} \to \mathrm{GL}(M_1 \otimes_R M_2)$$

by means on the homomorphism

$$h_{\mathfrak{G}} \xrightarrow{\rho_1 \times \rho_2} \mathrm{GL}(M_1) \times \mathrm{GL}(M_2) \xrightarrow{tensor\ representation} \mathrm{GL}(M_1 \otimes_R M_2)$$

6.1.1. **Lemma.** *Let $c_i : M_i \to M_i \otimes_R R[\mathfrak{G}]$ be the coaction for $i = 1, 2$ and let $c : M_1 \otimes_R M_2 \to (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}]$ be the coaction of the tensor* *representation. Then the following diagram commutes*

$$\begin{tikzcd}
M_1 \otimes_R M_2 \arrow[r, "c_1\otimes c_2"] \arrow[rrd, "c"'] & (M_1 \otimes_R R[\mathfrak{G}]) \otimes_R (M_2 \otimes_R R[\mathfrak{G}]) \arrow[r, "\sim"] & (M_1 \otimes_R M_2) \otimes_R (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \arrow[d, "id\otimes mult"] \\
 & & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

Proof. We need to identify the coaction c of $M_1 \otimes_R M_2$ starting from

$$\widetilde{c} = \widetilde{c}_1 \otimes \widetilde{c}_2 \in \mathrm{End}_{R[\mathfrak{G}]}\Big((M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}])\Big) \xrightarrow{\sim} \mathrm{End}_{R[\mathfrak{G}]}\Big((M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}]\Big)$$

where the isomorphism arises from the identification

$$\begin{tikzcd}
(M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) \arrow[r, "\alpha", "\sim"'] & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}] \\
(m_1 \otimes a_1) \otimes (m_2 \otimes a_2) \arrow[r, mapsto] & (m_1 \otimes m_2) \otimes (a_1\, a_2).
\end{tikzcd}$$

We consider then the following commutative diagram

$$\begin{tikzcd}
(M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) \arrow[r, "\alpha", "\sim"'] \arrow[d, "\tilde{c}_1\otimes\tilde{c}_2"] & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}] \arrow[d, "\tilde{c}"] & M_1 \otimes_R M_2 \arrow[l] \arrow[ld, "c"] \\
(M_1 \otimes_R R[\mathfrak{G}]) \otimes_{R[\mathfrak{G}]} (M_2 \otimes_R R[\mathfrak{G}]) \arrow[r, "\alpha", "\sim"'] & (M_1 \otimes_R M_2) \otimes_R R[\mathfrak{G}] &
\end{tikzcd}$$

It follows that $c(m_1 \otimes m_2) = \alpha(c_1(m_1) \otimes c(m_2))$ whence the desired statement.

□

In particular, if M_1 is a trivial $\mathfrak{G}$-module, we have $c(m_1 \otimes m_2) = m_1 \otimes c_2(m_2)$ so that $c = id_{M_1} \otimes c_2$. The coaction has then an interpretation with tensor products.

6.1.2. **Lemma.** *Let M be a $\mathfrak{G}$-module with coaction c and denote by $\widetilde{c} \in \mathrm{End}_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}])$ the action of the universal element of $\mathfrak{G}$. Let M_{tr} be the underlying trivial $\mathfrak{G}$–module and consider the tensor structure on the R–modules $M_{tr} \otimes_R R[\mathfrak{G}]$ and $M \otimes_R R[\mathfrak{G}]$.*

(1) $c : M \to M_{tr} \otimes_R R[\mathfrak{G}]$ is a $\mathfrak{G}$-morphism.

(2) $\widetilde{c} : M \otimes_R R[\mathfrak{G}] \to M_{tr} \otimes_R R[\mathfrak{G}]$ is a $\mathfrak{G}$-isomorphism.

Proof. (1) The coaction of $M_{tr} \otimes_R R[\mathfrak{G}]$ is $id_M \otimes \Delta$ so that the first rule

(6.1.3)
$$\begin{tikzcd}
M \arrow[r, "c"] \arrow[d, "c"'] & M \otimes_R R[\mathfrak{G}] \arrow[d, "id\otimes\Delta_{\mathfrak{G}}"'] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "c\otimes id"] & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}],
\end{tikzcd}$$

so that the top horizontal map provides a $\mathfrak{G}$-morphism $M \to M_{tr} \otimes_R R[\mathfrak{G}]$.

(2) By using the viewpoint of representations $\widetilde{c} : M \otimes_R R[\mathfrak{G}] \to M \otimes_R R[\mathfrak{G}]$ (which is defined by $\widetilde{c}(m \otimes a) = c(m)a)$ is a $\mathfrak{G}$-morphism. It is invertible as we have seen in the beginning of §6. □

6.2. **Representations of diagonalizable group schemes.** Let $\mathfrak{G} = \mathfrak{D}(A)/R$ be a diagonalizable group scheme. For each $a \in A$, we can attach a character $\chi_a = \eta_A(a) : \mathfrak{D}(A) \to \mathbb{G}_m = \mathrm{GL}_1(R)$. It defines then a $\mathfrak{G}$–structure on the R–module R.

To identify the relevant coaction, we use again Yoneda's technique by considering the homomorphism $\chi_{a,*} = \mathfrak{D}(A)(R[A]) \to \mathbb{G}_m(R[A]) = R[A]^\times$ and the image of the universal element which is e_a in view of Remark 3.4.2. It follows that the coaction is defined by $\widetilde{c}_a : R[A] \xrightarrow{\sim} R[A]$, $u \mapsto e_a\, u$ so that we have $c_a(r) = r \otimes e_a \in R \otimes_R R[A] = R[A]$.

If $M = \oplus_{a \in A} M_a$ is an A-graded R–module, the group scheme $\mathfrak{D}(A)$ acts diagonally on it by χ_a on each piece M_a.

We have constructed a covariant functor from the category of graded A–modules to the category of representations of $\mathfrak{D}(A)$.

6.2.1. **Proposition.** *The functor above is an equivalence of abelian categories from the category of A–graded R-modules to the category of $R-\mathfrak{D}(A)$-modules.*

Proof. Step 1: full faithfulness. Let $M_\bullet$ and $N_\bullet$ be A–graded modules. We have maps

$$\prod_{a \in A} \mathrm{Hom}_R(M_a, N_a) \to \mathrm{Hom}_{\mathfrak{D}(A)-mod}(M_\bullet, N_\bullet) \hookrightarrow \prod_{a,b \in A} \mathrm{Hom}_R(M_a, N_b).$$

It is then enough to show that $\mathrm{Hom}_R(M_a, N_b) = 0$ if $a \neq b$. For $a \neq b$, let $f : M_a \to N_b$ be a morphism of $\mathfrak{D}(A)$-modules. Then for $l \in M_a$, we have $c_{N_b}(f(m)) = f(c_{M_a}(m))$ so that $f(m) \otimes e_b = f(m \otimes e_a) = f(m) \otimes e_a \in N_b \otimes R[A]$. Since $R[A] = \bigoplus_{a \in A} Re_a$, we conclude that $f(m) = 0$. We conclude that $\mathrm{Hom}_R(M_a, N_b) = 0$ if $a \neq b$.

Step 2: Essential surjectivity. Let M be an $R - \mathfrak{D}(A)$-module and consider the underlying map $c : M \to M \otimes_R R[A]$. We write $c(m) = \sum\limits_{a \in A} \varphi_a(m) \otimes e_a$. We apply the first rule (6.0.6), that is, the commutativity of

(6.2.2)
$$\begin{tikzcd}
M \arrow[r, "c"] \arrow[d, "c"'] & M \otimes_R R[A] \arrow[d, "id\otimes\Delta"] \\
M \otimes_R R[A] \arrow[r, "c\otimes id"] & M \otimes_R R[A] \otimes_R R[A].
\end{tikzcd}$$

We have then

$$(c \otimes id)(c(m)) = (c \otimes id)\Big(\sum_{a \in A} \varphi_a(m) \otimes e_a\Big) = \sum_{b \in A} \sum_{a \in A} \varphi_b(\varphi_a(m))\, e_b \otimes e_a.$$

On the other hand we have

$$(id\otimes\Delta)(c(m))=(id\otimes\Delta)\Big(\sum_{a\in A}\varphi_a(m)\otimes e_a\Big)=\sum_{a\in A}\varphi_a(m)\,e_a\otimes e_a.$$

It follows that

$$\varphi_b\circ\varphi_a=\delta_{a,b}\,\varphi_a\qquad(a,b\in A)$$

We consider also the other compatibility (6.0.7)

(6.2.3)
$$\begin{tikzcd}
M \arrow[r, "c"] \arrow[d, "id"'] & M\otimes_R R[A] \arrow[dl, "id\times\epsilon^*"] \\
M &
\end{tikzcd}$$

It implies that

$$m=id\times\epsilon^*\Big(\sum_{a\in A}\varphi_a(m)\,e_a\Big)=\sum_{a\in A}\varphi_a(m).$$

We obtain that

$$\sum_{a\in A}\varphi_a=id_M.$$

Hence the φ_a's are pairwise orthogonal projectors whose sum is the identity. Thus $M=\bigoplus_{a\in A}\varphi_a(M)$ which decomposes a direct summand of eigenspaces as desired. □

6.2.4. **Corollary.** *Let* $0\to M_1\to M_2\to M_3\to 0$ *be an exact sequence of* $R-\mathfrak{D}(A)$*–modules.*

(1) For each $a\in A$*, it induces an exact sequence* $0\to(M_1)_a\to(M_2)_a\to(M_3)_a\to 0$.

(2) The sequence $0\to M_1\to M_2\to M_3\to 0$ *splits as sequence of* $R-\mathfrak{D}(A)$*–modules if and only if it splits as sequence of* R*–modules.*

Proof. (1) It readily follows of the equivalence of categories stated in Proposition 6.2.1.

(2) The direct sense is obvious. Conversely, let $s:M_3\to M_2$ be a splitting. Then for each $a\in A$, the composite $(M_3)_a\to M_3\overset{s}{\to}M_2\overset{\varphi_a}{\to}(M_2)_a$ provides the splitting of $(M_2)_a\to(M_3)_a$. □

We record also the following property.

6.2.5. **Corollary.** *Let* M *be an* $R-\mathfrak{G}$*-module. Then for each* S/R *and for each* $a\in A$*, we have* $M_a\otimes_R S=(M_a\otimes_R S)_a$.

6.2.6. **Corollary.** *Assume that* R *is a field. Then the category of representations of* $\mathfrak{D}(A)$ *is semisimple abelian category, that is, all short exact sequences split* [KS, 8.3.16].

Proof. Since the category of k–vector spaces is semisimple so is the category of A-graded vector spaces. Proposition 6.2.1 shows that the category of representations of $\mathfrak{D}(A)$ is semisimple. □

It is also of interest to know kernels of representations.

6.2.7. **Lemma.** *Let $A^\sharp$ be a finite subset of A and denote by A_0 the subgroup generated by $A^\sharp$. We consider the representation $M = \oplus_{a\in A^\sharp} R^{n_a}$ of $\mathfrak{G} = \mathfrak{D}(A)$, with $n_a \geq 1$. Then the representation $\rho : \mathfrak{G} \to \mathrm{GL}(M)$ factorizes as*

$$\mathfrak{G} = \mathfrak{D}(A) \to \mathfrak{D}(A_0) \overset{\rho_0}{\to} \mathrm{GL}(M)$$

where ρ_0 is a closed immersion. Furthermore $\ker(\rho) = \mathfrak{D}(A/A_0)$ *is a closed subgroup scheme.*

Proof. First case: $A = A_0$. Then the map $\mathfrak{G} \to \mathrm{GL}(M)$ factorizes by the closed subgroup scheme $T = \prod\limits_{a\in A^\sharp} \mathbb{G}_{m,R}^{n_a}$. Since the map $\widehat{T} \to A_0 = A$ is onto, the map $\mathfrak{G} \to \prod\limits_{a\in A^\sharp} \mathbb{G}_{m,R}^{n_a}$ is a closed immersion (Proposition 3.5.2). A composite of closed immersions being a closed immersion, ρ is a closed immersion.

General case. The representation $\rho : \mathfrak{G} \to \mathrm{GL}(M)$ factorizes as

$$\mathfrak{G} = \mathfrak{D}(A) \overset{q}{\to} \mathfrak{D}(A_0) \overset{\rho_0}{\to} \mathrm{GL}(M)$$

where ρ_0 is a closed immersion. It follows that $\ker(\rho) = \ker(q)$. This kernel $\ker(q)$ is $\mathfrak{D}(A/A_0)$ and is a closed subgroup scheme of $\mathfrak{G}$ (*ibid*). □

6.2.8. **Remark.** If R is a field, all finite dimensional representations of $\mathfrak{D}(A)$ are of this shape, so one knows the kernel of each finite dimensional representation.

6.3. **Existence of faithful finite dimensional representations (field case).** Let k be a field and let $\mathfrak{G}$ be an affine k-group.

6.3.1. **Theorem.** *Let V be a $k - \mathfrak{G}$-representation. Then $V = \varinjlim_{i\in I} V_i$ where V_i runs over the f.d. subrepresentations of V.*

Proof. We write $c : V \to V \otimes_k k[\mathfrak{G}]$ for the coaction. A sum of f.d. subrepresentations of $\mathfrak{G}$ is again one, so it is enough to show that each $v \in V$ belongs in some finite-dimensional subrepresentation. Let $(a_i)_{i\in I}$ be a basis of the k–vector space A. We write $c(v) = \sum\limits_{i\in I} v_i \otimes a_i$, where all but finitely many v_i's are zero. Next we have $\Delta(a_i) = \sum\limits_{j,l\in I} r_{i,j,l}\, a_j \otimes_k a_l$. Using the first rule (6.0.6) of comodules we have

$$\sum_{i\in I} c(v_i) \otimes a_i = (c \otimes id)(c(v)) = (id \otimes \Delta)c(v) = \sum_{i,j,l} r_{i,j,l}\ v_i \otimes a_j \otimes a_l.$$

Comparing the coefficients, we get $c(v_l) = \sum\limits_{i,j\in I} r_{i,j,l}\ v_i \otimes a_j$. Hence the subspace W spanned by v and the v_i's is a subrepresentation. □

6.3.2. **Theorem.** *Assume that $\mathfrak{G}$ is algebraic, that is, the k–algebra $k[\mathfrak{G}]$ is of finite type. Then $\mathfrak{G}$ admits a finite dimensional faithful k–representation V.*

Proof. We start with the regular representation V of G which is faithful in view of Example 6.0.9. We write $V = \varinjlim_{i\in I} V_i$ as in the previous theorem where the V_i's are finite dimensional. We put $\mathfrak{H}_i = \ker(\mathfrak{G} \to \mathrm{GL}(V_i))$, this is a closed k-subgroup of G. For each k–algebra S, we have

$$\bigcap_i \mathfrak{H}_i(S) = \ker\Big(\mathfrak{G}(S) \to \mathrm{GL}(V)(S)\Big) = 1.$$

We put $\mathfrak{H} = \bigcap_i \mathfrak{H}_i$, this is a closed k–subgroup of $\mathfrak{G}$ with trivial functor of points so that $H = 1$. We write $k[\mathfrak{H}_i] = k[\mathfrak{G}]/J_i$. Then

$$\ker(R[\mathfrak{G}] \to R) = \mathop{\Huge +}_{i\in I} J_i.$$

Since the ring $k[\mathfrak{G}]$ is a noetherian ring, its ideals are finitely generated so that there exists $i_1, ..., i_c \in I$ such that $\ker(R[\mathfrak{G}] \to R) = J_{i_1} + \cdots + J_{i_c}$. We consider the index $i \in I$ defined by $V_i = V_{i_1} + \cdots + V_{i_c}$. We have $\mathfrak{H}_i = \mathfrak{H}_{i_1} \cap \cdots \cap \mathfrak{H}_{i_c}$ so that $J_i = J_{i_1} + \cdots + J_{i_c} = \ker(R[\mathfrak{G}] \to R)$. Thus $\mathfrak{H}_i = 1$ and V_i is a faithful representation of $\mathfrak{G}$. $\square$

6.3.3. **Remark.** We will see later that a monomorphism of affine algebraic k-group is a closed immersion, see also [DG, §III.7.2] or [Mi2, thm. 3.34]. An easier thing ro do is to upgrade Theorem 6.3.2 by requiring that the homomorphism is a closed immersion, see [Wa, Thm. 3.4].

6.4. **Existence of faithful finite rank representations.** This question is rather delicate for general groups and general rings, see [SGA3, VI_B.13] and the paper [Th] by Thomason. Over a field or a Dedekind ring, faithful representations occur.

6.4.1. **Theorem.** *Assume that R or a Dedekind ring (e.g. DVR). Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a faithful $\mathfrak{G}$-module M which is f.g. free as R-module.*

The key thing is the following fact due to Serre [Se4, §1.5, prop. 2].

6.4.2. **Proposition.** *Assume that R is noetherian and let $\mathfrak{G}/R$ be an affine flat group scheme. Let M be a $\mathfrak{G}$-module. Let N be an R-submodule of M of finite type. Then there exists an $R-\mathfrak{G}$-submodule $\widetilde{N}$ of M which contains N and is f.g. as R-module.*

We can now proceed to the proof of Theorem 6.4.1. We take $M = R[\mathfrak{G}]$ seen as the regular representation, it is faithful (Example 6.0.9). The proposition shows that M is the direct limit of the family of $\mathfrak{G}$-submodules $(M_i)_{i\in I}$ which are f.g. as R-modules. The M_i's are torsion–free so are flat. Hence the M_i's are projective (in view of Lemma 5.2.1).

We look at the kernel $\mathfrak{N}_i/R$ of the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$. The regular representation is faithful and its kernel is the intersection of the $\mathfrak{N}_i$. Since $\mathfrak{G}$ is a noetherian scheme, there is an index i such that $\mathfrak{N}_i = 1$ (argument of the proof of Theorem 6.3.2). In other words, the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$ is faithful. Now M_i is a direct summand of a free module R^n, i.e. $R^n = M_i \oplus M_i'$. It provides a representation $\mathfrak{G} \to \mathrm{GL}(M_i) \to \mathrm{GL}(M_i \oplus M_i')$ which is faithful and such that the underlying module is free.

An alternative proof is §1.4.5 of [BT2] which shows that the provided representation $\mathfrak{G} \to \mathrm{GL}(M)$ is actually a closed immersion. This occurs as special case of the following result.

6.4.3. **Theorem.** *(Raynaud-Gabber* [SGA3, VI$_B$.13.2]*) Assume that R is a regular noetherian ring of dimension ≤ 2. Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a $\mathfrak{G}$-module M isomorphic to R^n as R–module such that $\rho_M : \mathfrak{G} \to \mathrm{GL}_n(R)$ is a closed immersion.*

Finally there are examples due to Grothendieck of rank two tori over the local ring of a nodal curve which do not admit a faithful representation [SGA3, X.1.6], see also [G2, §3].

6.5. **Hochschild cohomology.** We assume that $\mathfrak{G}$ is flat. If M is a $\mathfrak{G}$–module, we consider the R–module of invariants $M^{\mathfrak{G}}$ defined by

$$M^{\mathfrak{G}} = \Big\{ m \in M \ \mid m \otimes 1 = c(m) \in M \otimes_R R[\mathfrak{G}] \Big\}.$$

It is the largest trivial $\mathfrak{G}$-submodule of M and we have also $M^{\mathfrak{G}} = \mathrm{Hom}_{\mathfrak{G}}(R, M)$ and is denoted by $H^0(\mathfrak{G}, M)$.

6.5.1. **Example.** For an R-module N, we consider the tensor product $N \otimes_R R[\mathfrak{G}]$. We claim that the map $N \to N \otimes_R R[\mathfrak{G}]$ induces an isomorphism

$$N \xrightarrow{\sim} H^0(\mathfrak{G}, N \otimes_R R[\mathfrak{G}]).$$

Clearly the above map is injective. Conversely let $\sum_i n_i \otimes a_i \in H^0(\mathfrak{G}, N \otimes_R R[\mathfrak{G}])$. Then we have

$$\sum_i n_i \otimes a_i \otimes 1 = c\big(\sum_i n_i \otimes a_i\big) = \sum_i n_i \otimes \Delta(a_i) \in N \otimes_R R[\mathfrak{G}] \otimes R[\mathfrak{G}].$$

By applying $id \otimes \epsilon \otimes id$, we get $\sum_i n_i \otimes \epsilon(a_i) = \sum_i n_i \otimes a_i$ so that $\sum_i n_i \otimes a_i$ belongs to N.

We can then mimick the theory of cohomology of groups.

6.5.2. **Lemma.** *The category of $R - \mathfrak{G}$-modules has enough injective.*

We shall use the following extrem case of induction, see [J, §2, 3] for the general theory.

6.5.3. **Lemma.** *(Frobenius reciprocity) Let N be an R–module. Then for each $\mathfrak{G}$-module M the mapping*

$$\psi : \mathrm{Hom}_{\mathfrak{G}}(M, N \otimes_R R[\mathfrak{G}]) \to \mathrm{Hom}_R(M, N),$$

given by taking the composition with the augmentation map, is an isomorphism.

Proof. We define first the converse map. We are given an R-map $f_0 : M \to N$ and consider the following map of $\mathfrak{G}$-modules

$$M \xrightarrow{c_M} M_{tr} \otimes_R R[\mathfrak{G}] \xrightarrow{f_0 \otimes id} N \otimes_R R[\mathfrak{G}]$$

where we use again Lemma 6.1.2.(1). By construction we have $\psi(f) = f_0$. In the way around we are given a $\mathfrak{G}$–map $h : M \to N \otimes_R R[\mathfrak{G}]$ and denote by $h_0 : M \to N \otimes_R R[\mathfrak{G}] \xrightarrow{id \otimes \epsilon} N \to 0$. We consider the following commutative diagram

$$\begin{tikzcd}
M \arrow[r, "h"] \arrow[d, "c_M"'] & N \otimes_R R[\mathfrak{G}] \arrow[d, "id \times \Delta_{\mathfrak{G}}"'] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "h \otimes id"] \arrow[d, "id \times \epsilon^*"'] & N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[d, "id \times \epsilon^*"'] \\
M \arrow[r, "h \otimes id"] & N \otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

The composite $N \times_R R[\mathfrak{G}] \xrightarrow{id \times \Delta} N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \xrightarrow{id_N \times \epsilon^* \times id} N \otimes_R R[\mathfrak{G}]$ is the same than the left vertical map and is equal to $h_0 \otimes id$. Thus $h = h_0 \otimes id$ as desired. □

6.5.4. **Remark.** Now if N is a $\mathfrak{G}$-module, we remind the canonical isomorphism $N \otimes_R R[\mathfrak{G}] \xrightarrow{\sim} N_{tr} \otimes_R R[\mathfrak{G}]$ of $\mathfrak{G}$-modules where N_{tr} denotes the underlying R-module seen as trivial $R - \mathfrak{G}$–module (Lemma 6.1.2). Lemma 6.5.3 gives rise then to an isomorphism

$$\mathrm{Hom}_{\mathfrak{G}}(M, N \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} \mathrm{Hom}_R(M, N),$$

for any $\mathfrak{G}$-module M.

We can proceed to the proof of Lemma 6.5.2.

Proof. The argument is similar as Godement's one in the case of sheaves. Let M be a $\mathfrak{G}$–module and let us embed the R–module M_{tr} in some injective R–module I (this exists, see [We, Exercise 2.3.5]). Then we consider the following injective $\mathfrak{G}$-map

$$M \xrightarrow{c_M} M_{tr} \otimes_R R[\mathfrak{G}] \to I \otimes_R R[\mathfrak{G}]$$

where we use Lemma 6.1.2. We claim that $I \otimes_R R[\mathfrak{G}]$ is an injective $\mathfrak{G}$-module. We consider a diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "f"'] & N' \\
& I \otimes_R R[\mathfrak{G}]. &
\end{tikzcd}$$

From Frobenius reciprocity (i.e. Lemma 6.5.3), we have the following

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{G}}(N', I \otimes_R R[\mathfrak{G}]) \arrow[r, "i^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathfrak{G}}(N, I \otimes_R R[\mathfrak{G}]) \arrow[d, "\cong"'] \\
\mathrm{Hom}_R(N', I) \arrow[r, "i^*"] & \mathrm{Hom}_{\mathfrak{G}}(N, I).
\end{tikzcd}$$

Since I is an injective R–module, the bottom map is onto. Thus f extends to a $\mathfrak{G}$-map $N' \to I \otimes_R R[\mathfrak{G}]$. $\square$

We can then take the right derived functors of the left exact functor $R - \mathfrak{G} - mod \to R - Mod$, $M \to M^{\mathfrak{G}} = H_0^0(\mathfrak{G}, M)$, see [We, §2.5]. It defines the Hochschild cohomology groups $H_0^i(\mathfrak{G}, M)$. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of $\mathfrak{G}$–modules, we have the long exact sequence

$$\cdots \to H_0^i(\mathfrak{G}, M_1) \to H_0^i(\mathfrak{G}, M_2) \to H_0^i(\mathfrak{G}, M_3) \overset{\delta_i}{\to} H_0^{i+1}(\mathfrak{G}, M_1) \to \dots$$

6.5.5. **Lemma.** *Let M be an $R[\mathfrak{G}]$-module. Then $M \otimes_R R[\mathfrak{G}]$ is acyclic, i.e. satisfies*

$$H_0^i(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0 \qquad \forall i \geq 1.$$

Proof. We embed the M in an injective R-module I and put $Q = I/M$. The sequence of $\mathbf{G}$-modules

$$0 \to M \otimes_R R[\mathfrak{G}] \to I \otimes_R R[\mathfrak{G}] \to Q \otimes_R R[\mathfrak{G}] \to 0$$

is exact. We have seen that $I \otimes_R R[\mathfrak{G}]$ is injective, so that $H_0^i(\mathfrak{G}, I \otimes_R R[\mathfrak{G}]) = 0$ for each $i > 0$. The long exact sequence induces an exact sequence

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{G}}(R, I \otimes_R R[\mathfrak{G}]) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathfrak{G}}(R, Q \otimes_R R[\mathfrak{G}]) \arrow[r] \arrow[d, "\cong"'] & H_0^1(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) \to 0 \\
I \arrow[r] & Q \arrow[r] & 0.
\end{tikzcd}$$

Therefore $H_0^1(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$. The isomorphisms

$$H_0^i(\mathfrak{G}, Q \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}])$$

permits to use the standard shifting argument to conclude that $H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$ for each $i \geq 0$. $\square$

As in the usual group cohomology, these groups can be computed by means of cocycles. This provides a description of $H_0^i(\mathfrak{G}, M)$ in terms of Hochschild cocycles, see [DG, II.3] or [J, §4.16] for details. A n–cocycle (resp. a boundary) in this setting is the data of a n-cocycle (resp. a boundary) $c(S) \in Z^n(\mathfrak{G}(S), M \otimes_R S)$ in the usual sense and which agree with base changes.

6.5.6. **Remark.** In particular, there is a natural map $Z^n(\mathfrak{G}, M) \to Z^n(\mathfrak{G}(R), M)$. If $\mathfrak{G} = \Gamma_R$ is finite constant, this map induces an isomorphism $H_0^*(\Gamma, M) \xrightarrow{\sim} H_0^*(\Gamma, M)$ with the usual group cohomology (see [DG, II.3.4]).

We can state an important vanishing statement.

6.5.7. **Theorem.** *(Grothendieck) Let* $\mathfrak{G} = \mathfrak{D}(A)$ *be a diagonalizable group scheme. Then for each* $\mathfrak{G}$*-module* M*, we have* $H^i(\mathfrak{G}, M) = 0$ *for each* $i \geq 1$.

Proof. Once again we embed M in $M_{tr} \otimes_R R[A]$, it is a direct summand as R–module. According to Corollary 6.2.4, the $\mathfrak{G}$-module M is a direct summand of the flasque $\mathfrak{G}$–module $M \otimes_R R[\mathfrak{G}]$ (see Lemma 6.5.5). Hence M is flasque as well and has trivial cohomology (for $i \geq 1$). $\square$

6.6. **First Hochschild cohomology group.** We just focus on H^1 and H^2. Then

$$H^1_0(\mathfrak{G}, M) = Z^1_0(\mathfrak{G}, M)/B^1_0(\mathfrak{G}, M)$$

are given by equivalence of Hochschild 1-cocycles. More precisely, a 1-cocycle (or crossed homomorphism) is an R–functor

$$z : h_{\mathfrak{G}} \to W(M)$$

which satisfies the following rule for each algebra S/R

$$z(g_1 g_2) = z(g_1) + g_1 \,.\, z(g_2) \quad \forall \;\, g_1, g_2 \in \mathfrak{G}(S).$$

Note that $z(1_S 1_S) = z(1_S) + z(1_S)$ so that $z(1_S) = 0$. The coboundaries are of the shape $g \,.\, m \otimes 1 \,-\, m \otimes 1$ for $m \in M$. As in the classical case, we can attach to $z \in Z^1_0(\mathfrak{G}, M)$ an R-map

$$s_z \in \mathrm{Hom}_{R-func}(h_{\mathfrak{G}}, W(M) \rtimes h_{\mathfrak{G}})$$

defined by

$$s_z(g) = (z(g), g) \in (M \otimes_R S) \rtimes \mathfrak{G}(S)$$

for each R-algebra S and each $g \in \mathfrak{G}(S)$. We have the following dictionnary.

6.6.1. **Lemma.** *(1) The assignment*

$$Z^1_0(\mathfrak{G}, M) \to \mathrm{Hom}_{R-func}(h_{\mathfrak{G}}, W(M) \rtimes h_{\mathfrak{G}}), \quad z \mapsto s_z,$$

is a bijection between $Z^1_0(\mathfrak{G}, M)$ *and the homomorphic sections of the homomorphism of* R*-group functors* $W(M) \rtimes h_{\mathfrak{G}} \to h_{\mathfrak{G}}$.

(2) Furthermore it induces a bijection between $Z^1_0(\mathfrak{G}, M)$ *and the set of* M*-conjugacy classes of those sections.*

Proof. (1) Let us check first that s_z is a homomorphic section of the map $W(M) \rtimes h_{\mathfrak{G}} \to h_{\mathfrak{G}}$. Let S be an R–algebra and let $g_1, g_2 \in \mathfrak{G}(S)$. We have

$$s_z(g_1)\, s_z(g_2) = (z(g_1), g_1)\,(z(g_2), g_2) = \big(z(g_1) + g_1.z(g_2), g_1\, g_2\big) = (z(g_1\, g_2), g_1\, g_2) = s_z(g_1\, g_2)$$

by using the cocycle condition. Since $s_z(1_S) = (0, 1_S)$, s_z is an homomorphic section of $(M \otimes_R S) \rtimes \mathfrak{G}(S) \to \mathfrak{G}(S)$.

Conversely we are given a homomorphic section s of $W(M) \rtimes h_{\mathfrak{G}} \to h_{\mathfrak{G}}$. For each R–algebra S, it is of the shape $s(g) = (a(g), g)$ for each $g \in G(S)$ with $a(g) \in M \otimes_R S$. The above computation shows that $a : \mathfrak{G}(S) \to M \otimes_R S$ satisfies the cocycle relation. The functoriality in S enables us to conclude that a is an Hochschild 1-cocycle.

(2) For an homomorphic section s and $m \in M$, we consider the homomorphic section ${}^m s$ defined by ${}^m s : \mathfrak{G}(S) \to (M \otimes_R S) \rtimes \mathfrak{G}(S)$; i.e. by $({}^m s)(g) = m\, s(g)\, m^{-1}$. We have

$$({}^m s)(g) = (m, 1_S)\,(a(g), g)\,(-m, 1_S) = (m{+}a(g), g)\,(-m, 1_S) = (m{+}a(g){-}g.m, g).$$

The dictionnary tells us that s_z and $s_{z'}$ are M–conjugated if and only if z and z' are cohomologous. □

6.7. H^2 **and group extensions.** A 2-cocycle for $\mathfrak{G}$ and M is the data for each S/R of a 2-cocycle $f(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to M \otimes_R S$ in a compatible way. It satisfies the rule

$$g_1 \,.\, f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0$$

for each S/R and each $g_1, g_2, g_3 \in \mathfrak{G}(S)$. The 2-cocycle c is normalized if it satisfies furthermore the rule

$$f(g, 1) = f(1, g) = 0.$$

each S/R and each $g \in \mathfrak{G}(S)$. Up to add a coboundary, we can always deal with normalized cocycles. The link in the usual theory between normalized classes and group extensions [We, §6.6] extends mecanically. Given a normalized Hochschild cocycle $c \in Z^2(\mathfrak{G}, M)$, we can define the following group law on the R–functor $W(M) \times \mathfrak{G}$ by

$$(m_1, g_1) \,.\, m_2, g_2) = \Big(m_1 + g_1 \,.\, m_2 + c(g_1, g_2),\; g_1\, g_2 \Big)$$

for each S/R and each $m \in M \otimes_R S$ and $g \in \mathfrak{G}(S)$. In other words, we defined a group extension E_f of R-functors in groups of $h_{\mathfrak{G}}$ by $W(M)$.

In the way around, we are given an extension

$$0 \to W(M) \to E \to h_{\mathfrak{G}} \to 1$$

of R–functors in groups. Since $E \to h_{\mathfrak{G}}$ is an epimorphism, the universal point $g^{univ} \in \mathfrak{G}(R[\mathfrak{G}])$ lifts to an element $e \in E(R[\mathfrak{G}])$ (see §2.2). In other words we have a section $s : h_{\mathfrak{G}} \to E$ and we will associate a 2-cocycle which measures how far s is a homomorphism. As in abstract group case [We, th. 6.6.3], for each R-ring S, we set

$$c_s(g_1, g_2) = s(g_1)\, s(g_2)\, s(g_1\, g_2)^{-1} \qquad (g_1, g_2 \in \mathfrak{G}(S)).$$

We can check that c_s is a normalized 2-cocycle and that two normalized cocycles c, c' are cohomologous if and only if the extensions E_c and $E_{c'}$ are isomorphic. Now we denote by $\mathrm{Ext}_{R-functor}(\mathfrak{G}, W(M))$ the abelian group of classes of extensions (equipped with the Baer sum as defined in the classical setting in [Bn, IV, exercise 1]) of R–group functors of $h_{\mathfrak{G}}$ by $W(M)$ with the given action $h_{\mathfrak{G}} \to \mathrm{GL}(M)$.

The 0 is the class of the semi-direct product $W(M) \rtimes h_{\mathfrak{G}}$. As in the classical case, it provides a nice description of the H^2.

6.7.1. **Theorem.** [DG, II.3.2] *The construction above induces a group isomorphism* $H^2_0(\mathfrak{G}, M) \xrightarrow{\sim} \mathrm{Ext}_{R-functor}(\mathfrak{G}, W(M))$.

As consequence of the vanishing theorem 6.5.7, we get the following

6.7.2. **Corollary.** *Let A be an abelian group and let M be a $\mathfrak{D}(A)$–module. Let $0 \to W(M) \to F \to \mathfrak{D}(A) \to 1$ be a group R-functor extension. Then F is the semi-direct product of $\mathfrak{D}(A)$ by $W(M)$ and all sections of $F \to \mathfrak{D}(A)$ are M-conjugated.*

6.8. **Linearly reductive algebraic groups.** Let k be a field and let G/k be an affine algebraic group. Recall that a $k - G$–module V is simple if 0 and V are its only G-submodules. Note that simple $k - G$–module are finite dimensional according to Proposition 6.4.2. A $k - G$–module is semisimple if it is a direct sum of its simple submodules.

6.8.1. **Definition.** *The k–group $\mathfrak{G}$ is linearly reductive if each finite dimensional representation of $\mathfrak{G}$ is semisimple.*

We have seen (in an exercise) that diagonalizable groups are linearly reductive. An important point is that this notion is stable by base change and is geometrical, namely G is linearly reductive and only if $G \times_k \overline{k}$ is linearly reductive (see [Mg, prop. 3.2]). Exactly as in the case of diagonalizable groups, we have the following vanishing statement.

6.8.2. **Theorem.** *Assume that the affine algebraic group G/k is linearly reductive. Then the category of G-modules is semisimple and for each representation V of G, we have $H^i_0(G, V) = 0$ for each $i > 0$.*

Proof. We have to show that each short exact sequence $0 \to V' \to V \xrightarrow{p} V'' \to 0$ of G-modules split.
Step 1: V is finite dimensional. This is clear by decomposing it in a direct sum of simple representations.
Step 2: V'' is finite dimensional. We write $V = \varinjlim_{i \in I} V_i$ of its f.g. G-submodules (Thm. 6.3.1). Then the above sequence induces sequences of $\mathfrak{G}$–modules

$$0 \to V_i' \to V_i \to V_i'' \to 0$$

which are split. For i large enough, we have $V_i'' = V''$ so the sequence is split.
Step 3: General case. We consider the set $\mathcal{E}$ of the pairs (W, s) where W is a G-submodule of V'' and $s : W \to V''$ is a G–hommorphism such that $p \circ s : W \to V''$ is the inclusion map. This set is partially ordered, we say that $(W_1, s_1) \leq (W_2, s_2)$ is $W_1 \subseteq W_2$ and $s_{2,|W_1} = s_1$. Clearly $\mathcal{E}$ admits upper bounds for every chain so Zorn's lemma provides a maximal element (W, s) of $\mathcal{E}$. Assume that $W \subsetneq V''$ and pick $x \in V'' \setminus W$. Then x belongs to a finite dimensional G-submodule V_x'' in view of Theorem 6.3.1; at least one of the simple G-submodule V_0'' of V_x'' is not included in W. Since V_0 is simple, we have $W \cap V_0'' = 0$ hence a direct sum $W \oplus V_0'' \subseteq W$. By the step 2, there exists a section $s_0 : V_0'' \to V$ so that $s \oplus s_0$ extends s. This contradicts the maximality of W. Thus $W = V''$ and we are done.

The argument for the vanishing of Hochschild cohomology is then the same than for diagonalizable groups. We embed a representation V in $V \otimes_k k[G]$ so that V is a direct summand of $V \otimes_k k[G]$. But $V \otimes_k k[G]$ is flasque (see Lemma 6.5.5), so that $H^i(G, V) = 0$ for all $i \geq 1$. $\square$

6.8.3. **Corollary.** *Under the assumptions of Theorem 6.8.2, each extension of group functors of G by a vector algebraic group $W(M)$ (M finite dimensional representation of G) splits. Furthermore M acts transitively on the sections of $W(M) \rtimes G \to G$.*

Proof. This follows of the interpretation of $0 = H^2_0(G, V)$ in terms of group extensions (Thm. 6.7.1) and $0 = H^1_0(G, V)$ in terms of sections (Lemma 6.6.1.(2)). $\square$

The smooth connected linearly reductive groups are the reductive groups in characteristic zero and only the tori in positive characteristic (Nagata, see [DG, IV.3.3.6]).

For example, GL_n (for $n \geq 2$) is reductive in characteristic zero but not over a field of positive characteristic.

6.8.4. **Example.** Let k be a field. The additive k–group $\mathbb{G}_a$ is not linearly reductive. We consider the representation $\rho : \mathbb{G}_a \to \mathrm{GL}_2$,

$$x \mapsto \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}$$

Then the second projection $p_2 : k^2 \to k$ is a $\mathbb{G}_a$-homomorphism with k the trivial representation. We have $H^0(\mathbb{G}_a, k \oplus k) = k.e_1$ and it does surject by p_2 on k. The exact sequence $0 \to k \to k \oplus k \xrightarrow{p_2} k \to 0$ is then not split. Furthermore it induces a sequence

$$0 \to k \to H^0(\mathbb{G}_a, k \oplus k) \xrightarrow{p_{2,*}} k \to H^1(\mathbb{G}_a, k) \to \dots$$

so that $H^1(\mathbb{G}_a, k)$ is non zero.

For more on the topic, see [Mg] and [Wn]. By using a similar method (involving sheaves) in the non-abelian setting, Demarche gave a proof of the following classical result [De].

6.8.5. **Theorem.** *(Mostow* [Mo]*) Assume that* $\mathrm{char}(k) = 0$ *and let G/k be a linearly reductive group and let U/k be a split unipotent k-group. Then each extension of algebraic groups of G by U is split and the sections are conjugated under $U(k)$.*

Lie algebras, lifting tori

7. WEIL RESTRICTION

We are given the following equation $z^2 = 1 + 2i$ in $\mathbb{C}$. A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine schemes and for functors.

We are given a ring extension S/R or $j : R \to S$. Since a S-algebra is an R–algebra, an R-functor F defines an S-functor denoted by F_S and called the scalar extension of F to S. For each S–algebra S', we have $F_S(S') = F(S')$. If X is an R-scheme, we have $(h_X)_S = h_{X \times_R S}$.

Now we consider a S–functor E and define its Weil restriction to S/R denoted by $\prod\limits_{S/R} E$ by

$$\Big(\prod_{S/R} E\Big)(R') = E(R' \otimes_R S)$$

for each R–algebra R'. We note the two following functorial facts.

(I) For an R-map or rings $u : S \to T$, we have a natural map

$$u_* : \prod_{S/R} E \to \prod_{T/R} E_T. \tag{7.0.1}$$

(II) For each R'/R, there is natural isomorphism of R'–functors

$$\Big(\prod_{S/R} E\Big)_{R'} \xrightarrow{\sim} \prod_{S \otimes_R R'/R'} E_{S \otimes_R R'}. \tag{7.0.2}$$

For other functorial properties, see appendix A.5 of [CGP], for example the construction.

At this stage, it is of interest to discuss the example of vector group functors. Let N be an R–module. We denote by j_*N the scalar restriction of N from S to R [Bbk1, §II.1.13]. The module j_*N is N equipped with the R–module structure induced by the map $j : R \to S$. It satisfies the adjunction property $\mathrm{Hom}_R(M, j_*N) \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N)$ (*ibid*, §III.5.2).

7.0.3. **Lemma.** *(1) We have a canonical isomorphism* $\prod\limits_{S/R} W(N) \xrightarrow{\sim} W(j_*N)$.

(2) If N is f.g. projective and S/R is finite and locally free, then the R–module j_*N is f.g. projective and $\prod\limits_{S/R} W(N)$ is representable by the vector group scheme $\mathfrak{W}(j_*N)$.

For a more general statement, see [SGA3, I.6.6].

Proof. (1) For each R-algebra R', we have

$$\Big(\prod_{S/R} W(N)\Big)(R') = W(N)(R'\otimes_R S) = N\otimes_S(R'\otimes_R S) = j_*N\otimes_R R' = W(j_*N)(R').$$

(2) We write $N\oplus N' = S^n$ so that $j_*N\oplus j_*N' = (j_*S)^n$. Since the R–module S is f.g. projective, $(j_*S)^n$ is f.g. projective and so is j_*N. Hence $W(j_*N)$ is representable by the vector R–group scheme $\mathfrak{W}(j_*N)$. $\square$

7.0.4. **Example.** We have $h_{\mathrm{Spec}(R)} = \prod_{S/R} h_{\mathrm{Spec}(S)}$. This is the case $N = 0$ of Lemma 7.0.3.(1).

If F is an R-functor, we have for each R'/R a natural map

$$\eta_F(R') : F(R') \to F(R'\otimes_R S) = F_S(R'\otimes_R S) = \Big(\prod_{S/R} F_S\Big)(R');$$

it defines a natural mapping of R–functor $\eta_F : F \to \prod\limits_{S/R} F_S$ called often the diagonal map. For each S–functor E, it permits to defines a map

$$\phi : \mathrm{Hom}_{S-functor}(F_S, E) \to \mathrm{Hom}_{R-functor}\big(F, \prod_{S/R} E\big)$$

by applying a S–functor map $g : F_S \to E$ to the composition

$$F \overset{\eta_F}{\to} \prod_{S/R} F_S \xrightarrow{\prod_{S/R} g} \prod_{S/R} E.$$

7.0.5. **Lemma.** *The map ϕ is bijective.*

Proof. We apply the compatibility (7.0.2) with $R' = S_2 = S$. The map $S \to S\otimes_R S_2$ is split by the codiagonal map $\nabla : S\otimes_R S_2 \to S, s_1\otimes s_2 \to s_1 s_2$. Then we can consider the map

$$\theta_E : \Big(\prod_{S/R} E\Big)_{S_2} \xrightarrow{\sim} \prod_{S\otimes_R S_2/S_2} E_{S\otimes_R S_2} \overset{\nabla_*}{\to} \prod_{S/S} E = E.$$

This map permits to construct the inverse map ψ of ϕ as follows

$$\psi(h) : F_S \overset{l_S}{\to} \Big(\prod_{S/R} E\Big)_{S_2} \overset{\theta_E}{\to} E$$

for each $l \in \mathrm{Hom}_{R-functor}\big(F, \prod\limits_{S/R} E\big)$. By construction, the maps ϕ and ψ are inverse of each other. $\square$

In conclusion, the Weil restriction from S to R is then right adjoint to the functor of scalar extension from R to S.

7.0.6. **Proposition.** *Assume that S is finite locally free over R. Let $\mathfrak{Y}/S$ be an affine scheme of finite type (resp. of finite presentation). Then the functor $\prod_{S/R} h_{\mathfrak{Y}}$ is representable by an affine scheme of finite type (resp. of finite presentation).*

Again, it is a special case of a much more general statement, see [BLR, §7.6]. We denote by $\prod_{S/R} \mathfrak{Y}$ the R-scheme representing $\prod_{S/R} h_{\mathfrak{Y}}$.

Proof. The R–functor $\prod_{S/R} h_{\mathfrak{Y}}$ is a Zariski sheaf. According to Lemma 2.3.2, up to localize for the Zariski topology, we can assume that S is free over R, namely $S = \oplus_{i=1,\dots,d} R\,\omega_i$. We see $\mathfrak{Y}$ as a closed subscheme of some affine space $\mathbb{A}^n_S$, that is given by a system of equations $(P_\alpha)_{\alpha\in I}$ with $P_\alpha \in S[t_1, \dots, t_n]$. Then $\prod\limits_{S/R} h_{\mathfrak{Y}}$ is a subfunctor of $\prod\limits_{S/R} W(S^n) \xrightarrow{\sim} W(j_*(S^n)) \xrightarrow{\sim} W(R^{nd})$ by Lemma 7.0.3. For each I, we write

$$P_\alpha\Big(\sum_{i=1,..,d} y_{1,i}\omega_i, \sum_{i=1,..,d} y_{2,i}\omega_i, \dots, \sum_{i=1,..,d} y_{n,i}\Big) = Q_{\alpha,1}\,\omega_1 + \cdots + Q_{\alpha,r}\,\omega_r$$

with $Q_{\alpha,i} \in R\big[y_{k,i}; i = 1, .., d; k = 1, ..., n\big]$. Then for each R'/R, $\Big(\prod\limits_{S/R} h_{\mathfrak{Y}}\Big)(R')$ inside ${R'}^{nd}$ is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine R-scheme $\mathfrak{X}$ of finite type. Furthermore, if $\mathfrak{Y}$ is of finite presentation, we can take I finite so that $\mathfrak{X}$ is of finite presentation too. □

In particular, if $\mathfrak{H}/S$ is an affine group scheme of finite type, then the R–group functor $\prod\limits_{S/R} h_{\mathfrak{H}}$ is representable by an R-affine group scheme of finite type. There are nice functoriality issues, for example for open (resp. closed) immersions appendix A.5 of [CGP]. There are two basic examples of Weil restrictions.

(a) The case of a finite separable field extension k'/k (or more generally an étale k-algebra). Given an affine algebraic k'-group G'/k', we associate the affine algebraic k–group $G = \prod\limits_{k'/k} G'$ which is often denoted by $R_{k'/k}(G)$, see [Vo, §3. 12]. In that case, $R_{k'/k}(G) \times_k k_s \xrightarrow{\sim} (G'_{k_s})^d$. In particular, the dimension of G is $[k' : k] \dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(b) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For example the quotient k-group $\prod\limits_{k[\epsilon]/k} (\mathbb{G}_m)/\mathbb{G}_m$ is the additive k–group. Also if $p = \mathrm{char}(k) > 0$, $\prod\limits_{k[\epsilon]/k} \mu_{p,k[\epsilon]}$ is of dimension 1.

A side statement is the following.

7.0.7. **Lemma.** *Assume that S is locally free over R of constant rank $d \geq 1$. Let $\mathfrak{X}$ be an affine R–scheme and consider the diagonal map $\eta_{\mathfrak{X}} : \mathfrak{X} \to \prod_{S/R}(\mathfrak{X} \times_R S)$. Then $\eta_{\mathfrak{X}}$ is a closed immersion.*

Proof. Without loss of generality we may assume that R is non zero and so is S. Let $i : \mathfrak{X} \to \mathbb{A}^n_R = \mathfrak{W}(R^n)$ be a closed immersion. We consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\eta"] \arrow[d, "i"] & \prod_{S/R}(\mathfrak{X} \times_R S) \arrow[d, "\prod_{S/R}(i_S)"] \\
\mathfrak{W}(R^n) \arrow[r, "\eta"] & \prod\limits_{S/R} \mathfrak{W}(S^n) = \mathfrak{W}(j_* S)^n.
\end{tikzcd}$$

Since the two vertical maps are closed immersions, we are reduced to the case of $\mathfrak{W}(R^n)$ and even to the case of $\mathfrak{W}(R)$. As R-module, we claim that R is a direct summand of S, this implies that $\mathfrak{W}(R) \to \mathfrak{W}(j_* S)$ is a (split) monomorphism hence a closed immersion in view of Proposition 3.5.3. To establish the claim we embed S as direct summand in R^l. The vector $j(1) = (r_1, \dots, r_l)$ is unimodular, that is, $\sum r_i R = R$ [4]. Thus R is a direct summand of the R–module S and the claim is proven. □

Let us give an application of Weil restriction.

7.0.8. **Proposition.** *Let $\mathfrak{G}/R$ be an affine group scheme. Assume that there exists a finite locally free extension S/R of degree $d \geq 1$ such that $\mathfrak{G} \times_R S$ admits a faithful representation N f.g. locally free as S–module. Then $\mathfrak{G}$ admits a faithful representation M which is f.g. locally free as R–module.*

Proof. Let $\rho : \mathfrak{G} \times_R S \to \mathrm{GL}(N)$ be a faithful S–representation and denote by M/R the restriction of N from S to R. We have seen that M is f.g. projectivein Lemma 7.0.3.(2).

We have a natural embedding $\mathrm{End}_S(N) \subset \mathrm{End}_R(M)$ of R-algebras. Given an R-algebra R', we can map

$$\prod_{S/R} \big(W(\mathrm{End}_S(N))\big)(R') = \mathrm{End}_S(N) \otimes_S (S \otimes R') = \mathrm{End}_S(N) \otimes_R R'.$$

in $\mathrm{End}_R(M) \otimes_R R' = W(\mathrm{End}_R(M))(R')$. We have then a morphism of R–functors

$$\prod_{S/R} \big(W(\mathrm{End}_S(N))\big) \to W(\mathrm{End}_R(M))$$

and we claim that is a monomorphism. The S–module $\mathrm{End}_S(N)$ is finite locally free so that $\mathrm{End}_S(N) \otimes_S (S \otimes R') = \mathrm{End}_{S \otimes_R R'}\big(N \otimes_S (S \otimes R')\big)$ [Bbk1, II.5.3, prop. 7]. This embeds in $\big(W(\mathrm{End}_R(M))\big)(R') = \mathrm{End}_R(M) \otimes_R R' = \mathrm{End}_{R'}\big(M \otimes_R R'\big) = \mathrm{End}_{R'}\big(N \otimes_S (S \otimes_R R')\big)$ so that the claim is established. We have then a monomorphism of R–schemes $\prod_{S/R}\big(\mathfrak{W}(\mathrm{End}_S(N))\big) \to \mathfrak{W}(\mathrm{End}_R(M))$. We obtain then a monomorphism of R–group schemes $\prod\limits_{S/R} \mathrm{GL}(N) \to \mathrm{GL}(M)$

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[4] This is a standard argument. If not $r_1, \dots, r_l$ belong to a maximal proper ideal $\mathfrak{m}$ of R, contradicting the fact that $1_S \otimes R/\mathfrak{m}$ is non zero.

of R–group schemes. We consider then the R–map

$$\mathfrak{G} \xrightarrow{\eta_{\mathfrak{G}}} \prod_{S/R} \mathfrak{G} \times_R S \xrightarrow{\prod\limits_{S/R} \rho} \prod_{S/R} \mathrm{GL}(N) \to \mathrm{GL}(M)$$

Lemma 7.0.7 states that the left handside map is a closed immersion. The map in the diagram is a composite of monomorphisms, hence a monomorphism. ☐

7.0.9. **Remark.** If ρ is a closed immersion, we claim that so is the constructed map $\mathfrak{G} \to \mathrm{GL}(M)$. Since $\prod\limits_{S/R} \rho$ is a closed immersion, it is enough to check that $\prod\limits_{S/R} \mathrm{GL}(N) \to \mathrm{GL}(M)$ is a closed immersion. We claim that we have a cartesian diagram

$$\begin{tikzcd}
\prod\limits_{S/R} \mathrm{GL}(N) \arrow[rr] \arrow[d] & & \mathrm{GL}(M) \arrow[d] \\
\prod\limits_{S/R} \mathfrak{W}(\mathrm{End}_S(N)) = \mathfrak{W}(j_* \mathrm{End}_S(N)) \arrow[rr] & & \mathfrak{W}(\mathrm{End}_R(M))
\end{tikzcd}$$

where the bottom horizontal map is a closed immersion in view of Proposition 3.5.3. The cartesianity follows from $\mathrm{End}_S(N)^\times = \mathrm{End}_R(M)^\times \cap \mathrm{End}_S(N)$ and similarly after any change of rings R'/R.

7.0.10. **Remark.** It is natural to ask whether the functor of scalar extension from R to S admits a left adjoint. It is the case and we denote by $\bigsqcup\limits_{S/R} E$ this left adjoint functor, see [DG, §I.1.6]. It is called the Grothendieck restriction.

If $\rho : S \to R$ is an R–ring section of j, it defines a structure R^ρ of S–ring. We have $\bigsqcup\limits_{S/R} E = \bigsqcup\limits_{\rho: S \to R} E(R^\rho)$. If $E = h_{\mathfrak{Y}}$ for a S–scheme $\mathfrak{Y}$, $\bigsqcup\limits_{S/R} \mathfrak{Y}$ is representable by the R–scheme $\mathfrak{Y}$. This is simply the following R-scheme $\mathfrak{Y} \to \mathrm{Spec}(S) \overset{j^*}{\to} \mathrm{Spec}(R)$.

8. Tangent spaces and Lie algebras

8.1. **Derivations.** Let S be an R–ring and let M be an S–module. An R-derivation on M is an R-module homomorphism $d : S \to M$ to the S-module M satisfying the Leibniz rule

$$d(f\, g) = f\, d(g) + g\, d(f) \quad (f, g \in S).$$

We have $d(1) = d(1.1) = 1\, d(1) + 1\, d(1)$ so that $d(1) = 0$ and $d(R) = 0$. We denote by $\mathrm{Der}_R(S, M)$ the R–module of R-derivations on S to M.

We define the S–module of Kähler differentials $\Omega^1_{S/R}$ as the quotient of the free S–module $S^{(S)} = \bigoplus\limits_{s \in S} S\, ds$ by the S-module of relations generated by

(a) dr, $r \in R$;

(b) $d(s+t) = ds + dt$, $s, t \in S$;

(c) $d(st) = s\,dt + t\,ds$, $s, t \in S$.

The map $d : S \to \Omega^1_{S/R}$, $s \to ds$, is then a derivation (note that R-linearity follows from (a)). Next let $f : \Omega^1_{S/R} \to M$ be a morphism of S-modules. We define $d_f(s) = f(ds)$, then $d_f(s\,t) = f(d(st)) = f(s\,dt + t\,ds) = s\,f(dt) + t\,f(ds)$ so it is a derivation. The derivation d is actually universal in the sense of the following statement.

8.1.1. **Theorem.** [St, Tag 00RO] *For each S–module M, the map*

$$\mathrm{Hom}_S\big(\Omega^1_{S/R}, M\big) \to \mathrm{Der}_R(S, M), \quad f \mapsto d \circ f$$

is an isomorphism.

8.1.2. **Example.** (see [St, 00RX]) If $S = R[T_1, \dots, T_n]$, we claim that we have

$$\Omega^1_{S/R} = S\,dT_1 \oplus \cdots \oplus S\,dT_n \cong R^n.$$

Since S is generated as R–algebra by $T_1, ..., T_n$, the map

$$f : S\,dT_1 \oplus \cdots \oplus S\,dT_n \to \Omega^1_{S/R}, (P_1, ..., P_n) \mapsto P_1\,dT_1 + \cdots + P_n\,dT_n,$$

is onto. Next consider the R-derivation $\partial/\partial T_i : S \to S$. By the universal property this corresponds to an S-module map $l_i : \Omega^1_{S/R} \to S$ which maps dT_i to 1 and dT_j to 0 for $j \neq i$. Thus it is clear that there are no S-linear relations among the elements $dT_1, ..., dT_n$.

In particular for $M = R$ with S–structure $P(T_1, \dots, T_n).r = P(0, ..., 0).r$, we have $\mathrm{Der}_R(S, R^{(0)}) = \mathrm{Hom}_S(S^n, R^{(0)}) = R^n$ with generators $D_1, \dots, D_n$ defined by $D_i(P) = (\partial P/\partial T_i)(0)$.

8.2. **Tangent spaces.** We are given an affine R–scheme $\mathfrak{X} = \mathrm{Spec}(A)$. Given a point $x \in \mathfrak{X}(R)$, it defines an ideal $I(x) = \ker(A \overset{s_x}{\to} R)$ and defines an A–structure on R denoted R^x. We denote by $R[\epsilon] = R[t]/t^2$ the ring of R-dual numbers. We claim that we have a natural exact sequence of pointed set

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Der}_A(A, R^x) \arrow[r, "i_x"] & \mathfrak{X}(R[\epsilon]) \arrow[r] \arrow[d, no head, equal] & \mathfrak{X}(R) \arrow[r] & 1 \\
& & \mathrm{Hom}_R\big(A, R[\epsilon]\big). & &
\end{tikzcd}$$

where the base points are $x \in \mathfrak{X}(R) \subset \mathfrak{X}(R[\epsilon])$. The map i_x applies a derivation D to the map $f \mapsto s_x(f) + \epsilon\,D(f)$. It is a ring homomorphism since for $f, g \in A$ we have

$$\begin{aligned}
i_x(fg) &= s_x(fg) + \epsilon\,D(fg) \\
&= s_x(f)\,s_x(g) + \epsilon\,D(f)\,s_x(g) + \epsilon s_x(f)\,D(g) \qquad \text{[derivation rule]} \\
&= (s_x(f) + \epsilon\,D(f))\,.\,(s_x(g) + \epsilon\,D(g)) \qquad [\epsilon^2 = 0].
\end{aligned}$$

Conversely, one sees that a map $u : A \to R[\epsilon]$, $f \mapsto u(f) = s_x(f) + \epsilon \, v(f)$ is a ring homomorphism and only if $v \in \mathrm{Der}_A(A, R^x)$.

8.2.1. **Remark.** The geometric interpretation of $\mathrm{Der}_A(A, R^x)$ is the tangent space at x of the scheme $\mathfrak{X}/R$ (see [Sp, 4.1.3]).

We have a natural A-map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \; \to \; \mathrm{Der}_A(A, R^x);$$

it applies an A–map $l : I(x)/I^2(x) \to R$ to the derivation $D_l : A \to R$, $f \mapsto D_l(f) = l(f - f(x))$. This map is clearly injective but is split by mapping a derivation $D \in \mathrm{Der}_A(A, R^x)$ to its restriction on $I(x)$. Hence the map above is an isomorphism. Furthermore $I(x)/I^2(x)$ is an R^x–module hence the forgetful map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \xrightarrow{\sim} \mathrm{Hom}_{R-mod}(I(x)/I^2(x), R)$$

is an isomorphism. We conclude that we have the fundamental exact sequence of pointed sets

$$1 \longrightarrow \big(I(x)/I^2(x)\big)^\vee \xrightarrow{i_x} \mathfrak{X}(R[\epsilon]) \longrightarrow \mathfrak{X}(R) \to 1.$$

We record that the R–module structure on $I(x)/I(x)^2$ is also induced by the change of variable $\epsilon \mapsto \lambda \, \epsilon$. This construction behaves well with fibered products.

8.2.2. **Lemma.** *Let $\mathfrak{Y} = \mathrm{Spec}(B)$ be an affine R–scheme and let $y \in \mathfrak{Y}(R)$. The dual of the R–module map $v : I(x)/I^2(x) \oplus I(y)/I^2(y) \to I(x,y)/I^2(x,y)$ is an isomorphism and fits in the following commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \big(I(x)/I^2(x)\big)^\vee \oplus \big(I(y)/I^2(y)\big)^\vee \arrow[r, "i_x \times i_y"] & \mathfrak{X}(R[\epsilon]) \times \mathfrak{Y}(R[\epsilon]) \arrow[r] & \mathfrak{X}(R) \times \mathfrak{Y}(R) \arrow[r] & 1 \\
1 \arrow[r] & \big(I(x,y)/I^2(x,y)\big)^\vee \arrow[r, "i_{(x,y)}"] \arrow[u, "v^\vee", "\cong"'] & (\mathfrak{X} \times_R \mathfrak{Y})(R[\epsilon]) \arrow[r] \arrow[u, "\cong"'] & (\mathfrak{X} \times_R \mathfrak{Y})(R) \arrow[r] \arrow[u, "\cong"'] & 1.
\end{tikzcd}$$

commutes.

Proof. We write the two sequences and the map between them is provided by the fact that the map $(\mathfrak{X} \times_R \mathfrak{Y})(R[\epsilon]) \xrightarrow{\sim} \mathfrak{X}(R[\epsilon]) \times \mathfrak{Y}(R[\epsilon])$ is bijective. $\square$

We note that an R-module, $I(x)$ is a direct summand of $R[\mathfrak{X}]$. If we consider an R-ring S, il follows that $I(x) \otimes_R S$ is the kernel of $R[\mathfrak{X}] \overset{s_x \otimes id}{\to} S$. In conclusion, we have then defined a (split) exact sequence of pointed R–functors

$$1 \longrightarrow \mathfrak{V}(I(x)/I(x)^2) \xrightarrow{i_x} \prod_{R[\epsilon]/R} \mathfrak{X}_{R[\epsilon]} \longrightarrow \mathfrak{X} \longrightarrow 1.$$

8.3. **Smoothness.** There are several equivalent definitions for expressing that an R–algebra S is smooth. We have chosen to follow a variant of [GW2, §10.18] provided by the Stacks Project [St, Tags 00T6, 00T7].

8.3.1. **Definition.** *(1) An R–algebra S is* standard smooth *if*

$$S \cong R[T_1, \dots, T_n]/(f_1, \dots, f_c)$$

with $0 \leq c \leq n$ *such that*

$$\det\Big((\partial f_i / \partial T_j)_{i,j=1,\dots,c} \Big) \in S^\times.$$

(2) An R–algebra S is smooth *if it is of finite presentation and if for each geometric point $y \in \mathrm{Spec}(S)$ of image $x \in \mathrm{Spec}(R)$, there exists $f \in R$ and $g \in S$ such that $x \in \mathrm{Spec}(R_f)$, $y \in \mathrm{Spec}(S_g)$, and $R \to S$ induces a map $R_f \to S_g$ which is standard smooth.*

8.3.2. **Remarks.** (a) If the R–algebra S is *standard smooth* with the above presentation, it follows that the non-empty geometric fibers are of dimension $n - c$ in view of [GW2, Thm. 18.56.(v)]. In particular is S is non zero, the relative dimension d is well-defined. We will see later another way to understand that, see Remark 8.3.7.
(b) Etale means smooth of relative dimension 0. We have to pay attention that the notion of *standard étale* is different [St, Tag 00UB], this is $S \cong R[T]_Q/P(T)$ where P is a monic polynomial such that $P'(T) \in S^\times$.

The two notions are stable by arbitrary base change on R.

8.3.3. **Examples.** (a) A localization R_f is a smooth R–algebra.
(b) The polynomial R–algebra $R[T_1, \dots, T_d]$ is smooth.

The advantage of this definition is to be close of the intuition coming from differential geometry but it is not intrinsecal. However a good point is that it behaves well under composition [St, Tags 00T9, 00TD]. It turns out that smooth R–algebras are flat [GW, thm. 14.24], we refer to [GW2, §18.10] for the equivalence with other definitions. The most important result is that smoothness can be characterized on the functor of points.

8.3.4. **Theorem.** *(see* [GW2, Cor. 18.57], [St, Tag 00TN, 00UR]*) Let $\mathfrak{X} = \mathrm{Spec}(A)$ be an affine R–scheme which is finitely presented.*

(1) The R–scheme $\mathfrak{X}$ is smooth (i.e. A is an R–smooth algebra) if and only if $\mathfrak{X}$ is formally smooth, that is: for each R–ring B and each ideal $I \subset B$ satisfying $I^2 = 0$, the map $\mathfrak{X}(B) \to \mathfrak{X}(B/I)$ is onto.

(2) The R-scheme $\mathfrak{X}$ is étale (i.e. S is an R-étale algebra) if and only if $\mathfrak{X}$ is formally étale, that is: for each R–ring B and each ideal $I \subset B$ satisfying $I^2 = 0$, the map $\mathfrak{X}(B) \to \mathfrak{X}(B/I)$ is bijective.

We make now the connection with tangent spaces.

8.3.5. **Lemma.** *Let* $S = R[T_1, \dots, T_n]/(f_1, \dots, f_c) = R[T_1, \dots, T_n]/I$ *be a standard smooth algebra with* $0 \leq c \leq n$ *and* $\det\Big((\partial f_i / \partial T_j)_{i,j=1,\dots,c} \Big) \in S^\times$.

(1) The S*–module* I/I^2 *is free of base* $f_1, \dots, f_c$.

(2) The S*–module* $\Omega^1_{S/R}$ *is free of base the images of* dT_{c+1}, …, dT_n.

Proof. (1) and (2) We put $B = R[T_1, \dots, T_n]$ and denote by $p : B \to S = B/I$ the quotient map. According to [St, Tag 00RU], we have an exact sequence of S–modules

$$I/I^2 \xrightarrow{d \otimes id} \Omega^1_{B/R} \otimes_B S \to \Omega^1_{S/R} \to 0.$$

By taking into account Example 8.1.2, we have $\Omega^1_{B/R} = B^n$, this sequence becomes

$$I/I^2 \xrightarrow{d \otimes id} S^n \to \Omega^1_{S/R} \to 0.$$

We precompose by the surjective map $S^c \to I/I^2$, $(s_1, \dots, s_c) \mapsto \sum\limits_{j=1}^{c} s_j \, f_j$.

The matrix of $S^c \to S^n$, $(s_1, \dots, s_c) \mapsto \Big(\sum\limits_{j=1}^{c} s_j \, \partial f_j / \partial T_i \Big)_{i=1,\dots,n}$ is

$$\Big(\partial f_j / \partial T_i \Big)_{j=1,..,c, i=1,\dots,n}$$

which admits an invertible minor. It follows that the S-linear map $S^c \to S^n$ admits a left inverse and that $S^c \xrightarrow{\sim} I/I^2$. Thus I/I^2 is a free S-module of rank c.

We conclude also that $\Omega^1_{S/R}$ is a free S–module of rank $n - c$.

□

8.3.6. **Lemma.** *Let* $\mathfrak{X} = \mathrm{Spec}(A)$ *be affine* R*–scheme* $\mathfrak{X}/R$ *which is smooth of relative dimension* d.

(1) The $R[\mathfrak{X}]$*-module* $\Omega^1_{R[\mathfrak{X}]/R}$ *is locally free of rank* d.

(2) Let $x \in \mathfrak{X}(R)$ *and consider the ideal* $I(x) = \mathrm{Ker}\big(R[\mathfrak{X}] \xrightarrow{ev_x} R\big)$. *Then the* R*–module* $(I(x)/I(x)^2)^\vee$ *is locally free of rank* d.

Proof. (1) We are allowed to localize on $R[\mathfrak{X}]$ (using [St, Tag 00RT, (2)], so that the statement boils down to the composite $R \to R_f \to S$ for some $f \in R$ where S is a standard smooth R_f–algebra. In view of [St, Tag 00RT, (1)], we have an isomorphism of S–modules $\Omega^1_{S/R} \xrightarrow{\sim} \Omega^1_{S/R_f}$ so that we are reduced to the case when S is a standard smooth R–algebra. That case is treated by Lemma 8.3.5.(2) so we are done.

(2) By (1), the $R[\mathfrak{X}]$-module $\Omega^1_{R[\mathfrak{X}]/R}$ is locally free of rank d. We write $\Omega^1_{R[\mathfrak{X}]/R} \oplus N = R[\mathfrak{X}]^n$, so that the R-module $\mathrm{Hom}_{R[\mathfrak{X}]}(\Omega^1_{R[\mathfrak{X}]/R}, R^x)$ is a direct summand of $\mathrm{Hom}_{R[\mathfrak{X}]}(R[\mathfrak{X}]^n, R^x) = R^n$ so that is locally free. Thus the R–module $(I(x)/I(x)^2)^\vee$ is locally free. To check this is of rank d, we can localize on $\mathfrak{X}$. □

8.3.7. **Remark.** It provides another way to see that the relative dimension d of $\mathfrak{X}$ is well-defined by taking a non-empty geometric fiber of $\mathfrak{X} \to \mathrm{Spec}(R)$.

8.4. **Lie algebras.** Now let $\mathfrak{G}/R$ be an affine group scheme. We denote by $\mathrm{Lie}(\mathfrak{G})(R)$ the tangent space at the origin $1 \in \mathfrak{G}(R)$. This is the dual of I/I^2 where $I \subset R[\mathfrak{G}]$ is the kernel of the augmentation ideal. We define the "Lie algebra of $\mathfrak{G}$" vector R–group scheme by

$$\mathrm{Lie}(\mathfrak{G}) = \mathfrak{V}(I/I^2)$$

and we shall define later the Lie algebra structure. We recall that it fits in the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Lie}(\mathfrak{G})(R) \arrow[r] & \mathfrak{G}(R[\epsilon]) \arrow[r] & \mathfrak{G}(R) \arrow[r] & 1 \\
& X \arrow[r, mapsto] & \exp(\epsilon X) & &
\end{tikzcd}$$

which is a split exact of abstract groups where $\mathrm{Lie}(\mathfrak{G})(R)$ is equipped with the induced group law.

8.4.1. **Lemma.** *That induced group law is the additive law on* $\mathrm{Lie}(\mathfrak{G})(R)$, *namely* $\exp(\epsilon X + \epsilon Y) = \exp(\epsilon X) \,.\, \exp(\epsilon Y)$ *for each* $X, Y \in \mathrm{Lie}(\mathfrak{G})(R)$.

Proof. We apply Lemma 8.2.2 and use the product map $m : \mathfrak{G} \times_R \mathfrak{G} \to \mathfrak{G}$ to construct the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \big(I/I^2\big)^\vee \oplus \big(I/I^2\big)^\vee \arrow[r, "\exp \times \exp"] & \mathfrak{G}(R[\epsilon]) \times \mathfrak{G}(R[\epsilon]) \arrow[r] & \mathfrak{G}(R) \times \mathfrak{G}(R) \arrow[r] & 1 \\
1 \arrow[r] & \big(I(\mathfrak{G} \times_R \mathfrak{G})/I(\mathfrak{G} \times_R \mathfrak{G})^2\big)^\vee \arrow[u, "v^\vee", "\cong"'] \arrow[r, "\exp_{\mathfrak{G} \times_R \mathfrak{G}}"] \arrow[d, "m_*"'] & (\mathfrak{G} \times_R \mathfrak{G})(R[\epsilon]) \arrow[u, equal] \arrow[r] \arrow[d, "m"'] & (\mathfrak{G} \times_R \mathfrak{G})(R) \arrow[u, equal] \arrow[r] \arrow[d, "m"'] & 1. \\
1 \arrow[r] & \big(I/I^2\big)^\vee \arrow[r, "\exp"] & \mathfrak{G}(R[\epsilon]) \arrow[r] & \mathfrak{G}(R) \arrow[r] & 1.
\end{tikzcd}$$

Since the composite $\mathfrak{G} \overset{id \times \epsilon}{\to} \mathfrak{G} \times_R \mathfrak{G} \overset{m}{\to} \mathfrak{G}$ is the identity, the composite map $\big(I/I^2\big)^\vee \overset{id \times 0}{\to} \big(I/I^2\big)^\vee \oplus \big(I/I^2\big)^\vee \to \big(I/I^2\big)^\vee$ is the identity. It is the same for the second summand, so we conclude that that the left vertical composite map is the addition. $\square$

8.4.2. **Remark.** If $\mathfrak{G}$ is an R–subgroup of some GL_n, the proof of Lemma 8.4.1 boils down to the case of GL_n. In this case

8.4.3. **Example.** Let M be an R-module and consider the R-vector group scheme $\mathfrak{V}(M)$. For each S/R, we have

$$\mathfrak{V}(M)(S[\epsilon]) = \mathrm{Hom}_{S[\epsilon]}(M \otimes_R S[\epsilon], S[\epsilon]) = \mathrm{Hom}_R(M, S[\epsilon]) = \mathfrak{V}(M)^2(S),$$

hence an R-isomorphism $\mathfrak{V}(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{V}(M))$.

8.4.4. **Remarks.** (a) The natural map $\mathrm{Lie}(\mathfrak{G})(R) \otimes_R S \to \mathrm{Lie}(\mathfrak{G})(S)$ is not bijective in general (for example consider the case of a DVR and $\mathfrak{G} = \mathfrak{V}(R/\pi R)$). We have $\mathrm{Lie}(\mathfrak{G})(R) = \mathrm{Hom}_{R[\mathfrak{G}]}(\Omega^1_{R[\mathfrak{G}]/R}, R^{(1)})$.

(b) If $\Omega^1_{R[\mathfrak{G}]/R}$ is a finite locally free $R[\mathfrak{G}]$–module, we claim that the formation of the Lie algebra commutes with arbitrary base change. Writing $\Omega^1_{R[\mathfrak{G}]/R} \oplus N' = S^n$ we have that $\mathrm{Lie}(\mathfrak{G})(R) = \mathrm{Hom}_{R[\mathfrak{G}]}(\Omega^1_{R[\mathfrak{G}]/R}, R^{(1)})$ is a direct summand of $\mathrm{Hom}_{R[\mathfrak{G}]}(R[\mathfrak{G}]^n, R^{(1)}) = R^n$. This behaves well under base change.

(c) The preceding fact applies obviously when R is a field and also when when $\mathfrak{G}$ is smooth over R (due to Lemma 8.3.5.(2))

(d) The condition that the $R[\mathfrak{G}]$–module $\Omega^1_{R[\mathfrak{G}]/R}$ is *f.g.* projective is actually necessary for having this base change property in general, see [DG, §II.4.8].

More generally, we can define the Lie algebra R–functor of a group R–functor F by putting

$$\mathrm{Lie}(F)(S) = \ker\Big(F(S[\epsilon]) \to F(S)\Big).$$

It is a subgroup equipped with a map $\mathrm{Lie}(F)(R) \otimes_R S \to \mathrm{Lie}(F)(S)$ coming from the base change $\epsilon \mapsto \lambda\epsilon$. In that generality, we are actually mainly interested in the following examples.

8.4.5. **Lemma.** *Let M be an R–module. Then $W(M) \xrightarrow{\sim} \mathrm{Lie}(W(M))$ and $\mathrm{End}_S(M \otimes_R S) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{GL}(M))(S)$ for each S/R.*

Proof. The first thing is similar as example 8.4.3. For each S/R, we have indeed a split exact sequence of abstract groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{End}_S(M \otimes_R S) \arrow[r] & \mathrm{GL}(M)(S[\epsilon]) \arrow[r] & \mathrm{GL}(M)(S) \arrow[r] & 1. \\ & f \arrow[r, mapsto] & Id + \epsilon f & & \end{tikzcd}$$

□

If $f : \mathfrak{G} \to \mathfrak{H}$ is a morphism of affine R-group schemes, we have a map $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{H})$ of R-vector groups and the commutativity property $f\big(\exp(\epsilon X)\big) = \exp\Big(\epsilon \, . \, \mathrm{Lie}(f)(X)\Big)$.

The exact sequence defines an action of $\mathfrak{G}(R)$ on $\mathrm{Lie}(\mathfrak{G})(R)$ and actually a homomorphism $\mathfrak{G}(R) \to \mathrm{Aut}_{R-lin}(\mathrm{Lie}(\mathfrak{G})(R))$ called the adjoint representation.

8.4.6. **Lemma.** *Let M be a f.g. projective R-module and put $\mathfrak{G} = \mathrm{GL}(M)$. Then $\mathrm{End}_R(M) = \mathrm{Lie}(\mathfrak{G})(R)$ and the adjoint action is*

$$\mathrm{Ad}(g) \, . \, X = g \, X \, g^{-1}.$$

Proof. The R–group scheme $\mathfrak{G}$ is open in $W(\mathrm{End}_R(M))$ so that the tangent space at 1 in $\mathfrak{G}$ is the same than in $W(\mathrm{End}_R(M))$. By example 8.4.3, we get then an R–isomorphism $\mathrm{End}_R(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{G})(R)$. We perform now the computation of $\mathrm{Ad}(g)\ \exp(\epsilon X)$ in $\mathfrak{G}(R[\epsilon]) \subset \mathrm{End}_R(M) \otimes_R R[\epsilon]$. We have $\mathrm{Ad}(g)\ \exp(\epsilon X) = g\,(Id + \epsilon X)\,g^{-1} = Id + \epsilon g X g^{-1} = \exp(\epsilon\, g\, X\, g^{-1})$. $\square$

We assume for simplicity first that $\mathrm{Lie}(\mathfrak{G}) = W(\mathrm{Lie}(\mathfrak{G})(R))$ with $\mathrm{Lie}(G)(R)$ finite locally free (e.g. $\mathfrak{G}$ is smooth over R). We will refer to this property as (LF).

It follows that the adjoint representation functor

$$\mathrm{Ad} : \mathfrak{G} \to \mathrm{GL}(W(\mathrm{Lie}(\mathfrak{G})(R)).$$

is actually a representation of $\mathfrak{G}$ called the adjoint representation. By applying the Lie functor, it induces then a morphism of vector R-group schemes

$$\mathrm{ad} : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}\Big(\mathrm{GL}\big(\mathrm{Lie}(\mathfrak{G})(R)\big)\Big).$$

For each S/R, we have then an S–map

$$\mathrm{ad}(S) : \mathrm{Lie}(\mathfrak{G})(S) \to \mathrm{Lie}\Big(\mathrm{GL}\big(\mathrm{Lie}(\mathfrak{G})\big)\Big)(S) = \mathrm{End}_S\big(\mathrm{Lie}(\mathfrak{G})(R) \otimes_R S\big)$$

in view of the preceding lemma. For each $X, Y \in \mathrm{Lie}(\mathfrak{G})(S)$, we denote by

$$[X,Y] = \mathrm{ad}(S)(X).\ Y \in \mathrm{Lie}(\mathfrak{G})(S) \tag{8.4.7}$$

the Lie bracket of X and Y.

8.4.8. **Lemma.** *(1) Let $f : \mathfrak{G} \to \mathfrak{H}$ be a morphism of affine R–group schemes satisfying both property (LF). For each $X, Y \in \mathrm{Lie}(\mathfrak{G})(R)$, we have*

$$\mathrm{Lie}(f)\,.\,[X,Y] = [\mathrm{Lie}(f)\,.\,X,\ \mathrm{Lie}(f)\,.\,Y\,]\ \in \mathrm{Lie}(\mathfrak{G})(R).$$

(2) In the case $\mathfrak{G} = \mathrm{GL}(M)$ with M f.g. projective, the Lie bracket $\mathrm{End}_R(M) \times \mathrm{End}_R(M) \to \mathrm{End}_R(M)$ reads $[X,Y] = XY - YX$.

Proof. (1) Up to replace f by $id \times f : \mathfrak{G} \to \mathfrak{G} \times \mathfrak{H}$, we may assume that f is a monomorphism. It follows that the R-functor $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{H})$ is a monomorphism. We consider the following diagram of R–functors in groups

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "Ad_{\mathfrak{G}}"] \arrow[dr, "f\circ \mathrm{Ad}_{\mathfrak{H}}"] \arrow[dd, "f"'] & \mathrm{GL}(\mathrm{Lie}(\mathfrak{G})) \\
 & \mathrm{GL}(\mathrm{Lie}(\mathfrak{H}), \mathrm{Lie}(\mathfrak{G})) \arrow[u] \arrow[d] \\
\mathfrak{H} \arrow[r, "Ad_{\mathfrak{H}}"] & \mathrm{GL}(\mathrm{Lie}(\mathfrak{H}))
\end{tikzcd}$$

where $\mathrm{GL}(\mathrm{Lie}(\mathfrak{H}), \mathrm{Lie}(\mathfrak{G}))$ stands for the normalizer functor of $\mathrm{Lie}(\mathfrak{G})$ in $\mathrm{GL}(\mathrm{Lie}(\mathfrak{H}))$ (as defined in some exercise or in [DG, §II.1.3]). We derive it and get

$$\begin{tikzcd}
\mathrm{Lie}(\mathfrak{G}) \arrow[r, "ad_{\mathfrak{G}}"] \arrow[dr, "\mathrm{Lie}(f\circ \mathrm{Ad}_{\mathfrak{H}})"] \arrow[dd, "\mathrm{Lie}(f)"] & \mathrm{End}(\mathrm{Lie}(\mathfrak{G})) \\
 & \mathrm{End}(\mathrm{Lie}(\mathfrak{H}), \mathrm{Lie}(\mathfrak{G})) \arrow[u] \arrow[d] \\
\mathrm{Lie}(\mathfrak{H}) \arrow[r, "ad_{\mathfrak{H}}"] & \mathrm{End}(\mathrm{Lie}(\mathfrak{H}))
\end{tikzcd}$$

whence the desired compatibility.

(2) We consider the adjoint representation $\mathrm{Ad}(R) : \mathrm{GL}(M)(R) \to \mathrm{GL}(\mathrm{End}_R(M))(R)$ known to be $\mathrm{Ad}(g).X = gXg^{-1}$. We consider $\mathrm{Ad}(R[\epsilon]) : \mathrm{GL}(M)(R[\epsilon]) \to \mathrm{GL}(\mathrm{End}_R(M))(R[\epsilon])$; for $X, Y \in \mathrm{End}_R(M)$ we compute inside $(\mathrm{End}_R(M))(R[\epsilon])$ using Lemma 8.4.6

$$\begin{aligned}
\mathrm{Ad}(R[\epsilon])(\exp(\epsilon X))\,.\,Y &= (1+\epsilon X)Y(1+\epsilon X)^{-1} \\
&= (1+\epsilon X)Y(1-\epsilon X) \\
&= Y + \epsilon(XY - YX).
\end{aligned}$$

We conclude that $[X, Y] = XY - YX$. □

8.4.9. **Proposition.** *Assume that $\mathfrak{G}$ satisfies the property (LF) and that $\mathfrak{G}$ admits a faithful linear representation in some GL_n. The Lie bracket defines a Lie R-algebra structure on the R–module $\mathrm{Lie}(\mathfrak{G})(R)$, that is*

(i) the bracket is R–bilinear and alternating;

(ii) (Jacobi identity) For each $X, Y, Z \in \mathrm{Lie}(\mathfrak{G})(R)$, we have

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0.$$

We give here a short non orthodox proof specific to affine group schemes; for a more general setting, see [DG, II.4.4.3] and [SGA3, Exp. II].

Proof. Let us start with the case where $\mathfrak{G}$ admits a faithful representation in $\mathrm{GL}(R^n)$. Then the R–map $\mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathrm{GL}(M))$ is a monomorphism. From Lemma 8.4.8, it is then enough to check it for the linear group GL_n. That case is straightforward, we have $\mathrm{Lie}(\mathrm{GL}_n)(R) = M_n(R)$ and the bracket is $[X, Y] = XY - XY$(lemma 8.4.8). □

The theory is actually much wider since there is no need of the (LF) condition and also there is need to assume that $\mathfrak{G}$ admits a faithful embedding in some GL_n. Using §3.2, we have an anti-isomorphism of R–functors $\mathrm{Aut}_{lin}(\mathfrak{V}(I/I^2)) \xrightarrow{\sim} \mathrm{GL}(I/I^2)$. This induces an isomorphism of R–functors in abelian groups

$$\mathrm{Lie}\Big(\mathrm{Aut}_{lin}(\mathfrak{V}(I/I^2))\Big) \xrightarrow{\sim} \mathrm{Lie}\Big(\mathrm{GL}(I/I^2)\Big)$$

which is nothing but $W(\mathrm{End}_R(I/I^2))$ in view of Lemma 8.4.5 and also $\mathrm{Hom}_{lin}(\mathrm{Lie}(\mathfrak{G}), \mathrm{Lie}(\mathfrak{G}))$ [DG, II.4.4.1]. This permits to define the Lie bracket as a morphism of R–functors

$$[\, ,\,] : \mathrm{Lie}(\mathfrak{G}) \times \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{G})$$

with the above formula (8.4.7). It turns out that $\mathrm{Lie}(\mathfrak{G})(R)$ is indeed a Lie algebra [DG, II.4.4.5]. The main idea is to embed $\mathrm{Lie}(\mathfrak{G})(R) = \mathrm{Der}_R(R[\mathfrak{G}], R^{(1)})$ in the algebra $\mathrm{Der}_R(R[\mathfrak{G}], R[\mathfrak{G}])$ which is a Lie R-subalgebra of $\mathrm{End}_R(R[\mathfrak{G}], R[\mathfrak{G}])$ [Bbk1, §III.10.4]. In the field case, there is a short proof of this approach in [KMRT, §21.A].

8.4.10. **Remark.** If $j : R \to S$ is a finite locally free morphism and $\mathfrak{H}/S$ a group scheme over S, it is a natural question to determine the Lie algebra of $\mathfrak{G}$. It is done in [CGP, A.7.6]. and we have $\mathrm{Lie}(\mathfrak{G}) = j_*\mathrm{Lie}(\mathfrak{H})$, that is $\mathrm{Lie}(\mathfrak{G})(R') = \mathrm{Lie}(\mathfrak{H})(S \otimes_R R')$ for each R'/R.

8.4.11. **Examples.** If k is a field of characteristic $p > 0$, $\mathrm{Lie}(\mu_p)(k) = k$ with trivial Lie structure.