Affine group schemes I

We shall work over a base ring R (commutative and unital).

2. Sorites

2.1. **R-Functors.** We denote by $\mathcal{A}ff_R$ the category of affine R-schemes. We are interested in R–functors, i.e. covariant functors from $\mathcal{A}ff_R$ to the category of sets. If $\mathfrak{X}$ an R-scheme, it defines a covariant R–functor

$$h_{\mathfrak{X}} : \ \mathcal{A}ff_R \ \to \ Sets, \ \ S \mapsto \mathfrak{X}(S).$$

Given a map $f : \mathfrak{Y} \to \mathfrak{X}$ of R-schemes, there is a natural morphism of functors $f_* : h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ of R-functors.

We recall now Yoneda's lemma in our setting. Let F be an R-functor. If $\mathfrak{X} = \operatorname{Spec}(R[\mathfrak{X}])$ is an affine R–scheme and $\zeta \in F(R[\mathfrak{X}])$, we define a morphism of R-functors

$$\phi(\zeta) : h_{\mathfrak{X}} \to F$$

by $\phi(\zeta)(S) : h_{\mathfrak{X}}(S) = \operatorname{Hom}_R(R[\mathfrak{X}], S) \to F(S)$, $x \mapsto F(f_x)(\zeta)$ for each R-ring S where $f_x \in \operatorname{Hom}_R(R[\mathfrak{X}], S)$ is the evaluation function at x.

2.1.1. **Lemma.** *(Yoneda lemma)*

(1) The assignment $\zeta \to \phi(\zeta)$ induces a bijection

$$F(R[\mathfrak{X}]) \xrightarrow{\sim} \operatorname{Hom}_{R-func}(h_{\mathfrak{X}}, F).$$

(2) Let $\mathfrak{Y}$ be an R–scheme. Then we have

$$\operatorname{Hom}_{R-sch}(\mathfrak{X}, \mathfrak{Y}) = h_{\mathfrak{Y}}(R[\mathfrak{X}]) \xrightarrow{\sim} \operatorname{Hom}_{R-func}(h_{\mathfrak{X}}, h_{\mathfrak{Y}}).$$

Proof. (1) The strategy is to construct the inverse map. We are given $\alpha \in \operatorname{Hom}_{R-func}(h_{\mathfrak{X}}, F)$, it gives rise to a map $\alpha_{R[\mathfrak{X}]} : h_{\mathfrak{X}}(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$ so that the universal point $x^{univ} \in h_{\mathfrak{X}}(R[\mathfrak{X}]) = \operatorname{Hom}_R(R[\mathfrak{X}], R[\mathfrak{X}])$ defines an element $\psi(\alpha) = \alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) \in F(R[\mathfrak{X}])$ or for short $\alpha(id_{R[\mathfrak{X}]}$.

Step 1: $\psi \circ \phi = id_{F(R[\mathfrak{X}])}$. Let $\zeta \in F(R[\mathfrak{X}])$. We apply $\phi(\zeta)_{R[\mathfrak{X}]} : h_{\mathfrak{X}}(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$ to $R[\mathfrak{X}]$ and obtain $\psi(\phi(\zeta)) = F(id_{R[\mathfrak{X}]})(\zeta) = \zeta$.

Step 2: $\phi \circ \psi = id_{\operatorname{Hom}_{R-func}(h_{\mathfrak{X}}, F)}$. Let $\alpha \in \operatorname{Hom}_{R-func}(h_{\mathfrak{X}}, F)$. Then $\psi(\alpha) = \alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]}) \in F(R[\mathfrak{X}])$ and we consider the element $\eta = \phi(\psi(\alpha)) \in \operatorname{Hom}_R(h_{\mathfrak{X}}, F)$ defined as follows. For each $f_x \in \operatorname{Hom}_R(R[\mathfrak{X}], S)$, $\eta(S) : h_X(S) \to F(S)$ applies f_x to

$$F(f_x)\big(\psi(\alpha)\big) = F(f_x)\big(\alpha_{R[\mathfrak{X}]}(id_{R[\mathfrak{X}]})\big) = \alpha(f_x \circ id_{R[\mathfrak{X}]}) = \alpha(f_x)$$

where we used the functorial property in the second equality. Thus $\phi \circ \psi = id_{\operatorname{Hom}_{R-func}(h_{\mathfrak{X}}, F)}$.

(2) We apply (1) to $F = h_{\mathfrak{Y}}$. $\square$

2.1.2. **Remarks.** (a) The formula $F(f_x)\big(\psi(\alpha)\big) = \alpha(f_x)$ arising in the proof expresses the fact that an R–functor $h_X \to F$ is determined by its value on the universal point of X.

(b) For more on the Yoneda lemma, see [Wa, §1.2], [GW, §4.2] or [Vi, §2.1]. Part (2) holds then for general R-schemes.

An R-functor F is representable by an R scheme (resp. an affine R–scheme) if there exists an R-scheme $\mathfrak{X}$ (resp. an affine R–scheme $\mathfrak{X}$) together with an isomorphism of functors $h_X \to F$. We say that $\mathfrak{X}$ represents F.

If $\mathfrak{X}$ is affine, the isomorphism $h_X \to F$ comes from an element $\zeta \in F(R[\mathfrak{X}])$ which is called the universal element of $F(R[\mathfrak{X}])$. The pair $(\mathfrak{X}, \zeta)$ satisfies the following universal property:

For each affine R-scheme $\mathfrak{T}$ and for each $\eta \in F(R[\mathfrak{T}])$, there exists a unique morphism $u : \mathfrak{T} \to \mathfrak{X}$ such that $F(u^*)(\zeta) = \eta$.

Given a morphism of rings $j : R \to R'$, an R–functor F defines by restriction an R'–functor denoted by j_*F or $F_{R'}$. If $F = h_{\mathfrak{X}}$ for an affine R-scheme $\mathfrak{X}$, we have $F_{R'} = h_{\mathfrak{X} \times_R R'}$.

2.1.3. **Examples.** We will see later more non representable R-functors.

(a) The empty R–functor is not representable by an affine R-scheme (and not actually by any R-scheme). Denote by F the empty functor and assume that $h_{\mathfrak{X}} \cong F$ for an R-scheme $\mathfrak{X}$. Then $id_{\mathfrak{X}} \in h_{\mathfrak{X}}(R[\mathfrak{X}])$ contradicting the fact that F is the empty R-functor.

(b) We consider the R-functor $F(S) = S^{(\mathbb{N})}$ and claim that it not representable by an affine R–scheme. Assume that $h_{\mathfrak{X}} \cong F$ so that $\mathrm{Hom}_R(R[\mathfrak{X}], R[\mathfrak{X}]) \cong R[\mathfrak{X}]^{(\mathbb{N})}$. Then the image of $id_{R[\mathfrak{X}]}$ has bounded support d so that $F(S) \subset S^d \subset S^{(\mathbb{N})}$ for each R–ring S. This is a contradiction.

2.1.4. **Remark.** We denote by $F_0(S) = \{\bullet\}$ for each R–ring S. Let F be an R-functor. Then there is a canonical map $F \to F_0$; in other words F_0 is a terminal object of the category of R–functors.

2.2. **Monomorphisms.** The fibered product of R-functors is defined as follows. For $\alpha_1 : F_1 \to E$ and $\alpha_2 : F_1 \to E$ two morphisms of R-functors, we set $(F_1 \times_E F_2)(S) = F_1(S) \times_{E(S)} F_2(S)$ for each R–ring S.

2.2.1. **Lemma.** *Let $\alpha : F \to E$ be a morphism of R-functors. The following conditions are equivalent:*

(i) α is a monomorphism;

(ii) the diagonal $\Delta : F \to F \times_E F$ is an isomorphism;

(iii) $F(S) \to E(S)$ is injective for each R–ring S.

Proof. $(i) \Longrightarrow (ii)$. We consider the projections $p_i : F \times_E F \to F$ for $i = 1, 2$. Since $\alpha \circ p_1 = \alpha \circ p_2$, we obtain that $p_1 = p_2$. Thus p_1 is an isomorphism and so is Δ.

$(ii) \Longrightarrow (i)$. We are given $\beta_1, \beta_2 : G \to F$ be morphisms of R–functors such that $\alpha \circ \beta_1 = \alpha \circ \beta_2$. This defines a map $\beta : G \to F \times_E F \overset{\sim}{\longleftarrow} F$, so that $\beta_1 = \beta_2$.

$(iii) \Longrightarrow (ii)$. For each R-ring S, we have $F(S) \overset{\sim}{\longrightarrow} F(S) \times_{E(S)} F(S)$ so that Δ is an isomorphism of R-functors.

$(ii) \Longrightarrow (iii)$. Obvious.

□

We consider now the case of schemes.

2.2.2. **Lemma.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of R-schemes. The following conditions are equivalent:*

(i) f is a monomorphism;

(i') The R–functor $h_f : h_{\mathfrak{X}} \to h_{\mathfrak{Y}}$ is a monomorphism;

(ii) the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is an isomorphism;

(iii) $F(S) \to E(S)$ is injective for each R–ring S.

Proof. The proof of the implications $(i) \Longleftrightarrow (ii) \Longrightarrow (iii)$ is similar with the previous lemma. The implication $(iii) \Longrightarrow (ii)$ Lemma 2.2.1, $(iii) \Longrightarrow (i)$ yields the implication $(iii) \Longrightarrow (i')$.

It remains to establish the implication $(i') \Longrightarrow (ii)$. Lemma 2.2.1, $(i) \Longrightarrow (ii)$ shows that the diagonal $h_{\mathfrak{X}} \to h_{\mathfrak{X}} \times_{h_{\mathfrak{Y}}} h_{\mathfrak{X}}$ is a an isomorphism of R–functors. Let $\mathfrak{Z}$ be an R-scheme, we need to establish that the diagonal map $\mathfrak{X}(\mathfrak{Z}) \to \mathfrak{X}(\mathfrak{Z}) \times_{\mathfrak{Y}(\mathfrak{Z})} \mathfrak{X}(\mathfrak{Z})$ is an isomorphism. If $\mathfrak{Z}$ is affine over R it is true. Let $g, h \in \mathfrak{X}(\mathfrak{Z})$ mapping to the same element of $\mathfrak{Y}(\mathfrak{Z})$.

We consider then an affine cover $(\mathfrak{U}_i)_{i \in I}$ of $\mathfrak{Z}$ so that the restrictions $g_i : \mathfrak{U}_i \subset \mathfrak{Z} \to \mathfrak{X}$ $h_i : \mathfrak{U}_i \subset \mathfrak{Z} \to \mathfrak{X}$ define an unique element $f_i \in \mathfrak{X}(\mathfrak{U}_i)$. Since the diagonal is split by the first projection, f_i and f_j agree on $\mathfrak{U}_i \cap \mathfrak{U}_j$ so that define $f : \mathfrak{Z} \to \mathfrak{X}$. Then $f = g = h$ and we are done.

□

2.2.3. **Remark.** The equivalence $(i) \Longleftrightarrow (ii)$ in (1) holds in any category with fiber products, see [Sta, Tag 01L3].

We consider now the epimorphisms of R–functors. If $\alpha : F \to E$ satisfies that $F(S) \to E(S)$ is surjective for each R–ring S, we claim that α is an epimorphism.

Let $\gamma_1, \gamma_2 : E \to D$ be morphisms of R–functors such that $\gamma_1 \circ \alpha = \gamma_2 \circ \alpha$. Then $\gamma_1 : E(S) \to D(S)$ agrees with $\gamma_2 : E(S) \to D(S)$ for each R–ring S so that $\beta_1 = \beta_2$. Thus α is an epimorphism.

It can be shown by using coproducts that the epimorphisms are all of that shape, see [KS, §2, Ex. 2.4, 2.23] or [SGA3, §I.1.4]; those references put also the monomorphism case in a much wider setting.

In the category of R–schemes, we have to pay attention that there are epimorphisms whose associated functor is not surjective, see [GW, Ex. 8.2.(d)] for the construction of a bunch of epimorphisms. A concrete example is with $k = \mathbb{R}$ and the morphism $u : \mathfrak{X} = \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R}) = \mathfrak{Y}$.

Let $\mathfrak{Z}$ be an $\mathbb{R}$-scheme and let $f_1, f_2 : \mathfrak{Y} \to \mathfrak{Z}$ such that $f_1 \circ u = f_2 \circ u$. In other words we have two points $z_1, z_2 \in \mathfrak{Z}(\mathbb{R})$ which coincide as complex points. Since $\mathfrak{Z}(\mathbb{R})$ injects in $\mathfrak{Z}(\mathfrak{C})$, it follows that $z_1 = z_2$ so that u is an epimorphism. The fact that $\mathfrak{Z}(\mathbb{R})$ injects in $\mathfrak{Z}(\mathbb{C})$ reduces to an affine scheme $\mathrm{Spec}(A)$ for which we have $\mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R}) \subset \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(A, \mathbb{C})$.

2.3. **Zariski sheaves.** We say that an R–functor F is a Zariski sheaf if it satisfies the following requirements:

(A) for each R–ring S and each decomposition $1 = f_1 + \cdots + f_n$ in S, then

$$F(S) \xrightarrow{\sim} \Big\{ (\alpha_i) \in \prod_{i=1,..,n} F(S_{f_i}) \mid (\alpha_i)_{S_{f_i f_j}} = (\alpha_j)_{S_{f_i f_j}} \text{ for } \ i,j = 1, ..., n \Big\}.$$

(B) $F(0) = \{\bullet\}$.

2.3.1. **Lemma.** *Let F be an R–functor F which a Zariski sheaf. Then F is additive, i.e. the map $F(S_1 \times S_2) \to F(S_1) \times F(S_2)$ is bijective for each pair (S_1, S_2) of R–algebras.*

Proof. We are given an R–ring $S = S_1 \times S_2$; we write it $S = S_1 \times S_2 = Se_1 + Se_2$ where e_1, e_2 are idempotents satisfying $e_1 + e_2 = 1$, we have $S_1 = S_{e_1}$, $S_2 = S_{e_2}$ and $S_{e_1 e_2} = 0$ [Sta, Tag 00ED]. Then

$$F(S) \xrightarrow{\sim} \big\{ (\alpha_1, \alpha_2) \in F(S_1) \times F(S_2) \mid \alpha_{1,0} = \alpha_{2,0} \in F(0) \big\}.$$

Since $F(0) = \{\bullet\}$, we conclude that $F(S) = F(S_1) \times F(S_2)$. □

Representable R-functors are clearly Zariski sheaves. In particular, to be a Zariski sheaf is a necessary condition for an R–functor to be representable.

2.3.2. **Lemma.** *Let $1 = f_1 + \cdots + f_n$. Let F be an R–functor which is a Zariski sheaf and such that $F_{R_{f_i}}$ is representable by an affine R_{f_i}-scheme for $i = 1, ..., n$. Then F is representable by an affine R–scheme.*

Proof. Let $\mathfrak{X}_i$ be an R_{f_i}-scheme together with an isomorphism $\zeta_i : h_{\mathfrak{X}_i} \xrightarrow{\sim} F_{R_{f_i}}$ of R_{f_i}–functors for $i = 1, .., n$. Then for $i \neq j$, $F_{R_{f_i f_j}}$ is represented by $\mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j}$ and $\mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}$. More precisely, the isomorphism $\zeta^{-1}_{j, R_{f_i f_j}} \circ \zeta_{i, R_{f_i f_j}} : h_{\mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j}} \xrightarrow{\sim} h_{\mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}}$ defines an isomorphism $u_{i,j} : \mathfrak{X}_i \times_{R_{f_i}} R_{f_i f_j} \xrightarrow{\sim} \mathfrak{X}_j \times_{R_{f_j}} R_{f_i f_j}$ and we have compatiblities $u_{i,j} \circ u_{j,k} = u_{i,k}$ once restricted to $R_{f_i f_j f_k}$. It follows that the $\mathfrak{X}_i$'s glue in an affine R–scheme $\mathfrak{X}$. Also the map ζ_i^{-1} glue in an R–map $F \to h_{\mathfrak{X}}$. Since F is a Zariski sheaf, we conclude that $F \xrightarrow{\sim} h_{\mathfrak{X}}$. □

2.4. **Functors in groups.**

2.5. **Definition.** An R–group scheme $\mathfrak{G}$ is a group object in the category of R-schemes. It means that $\mathfrak{G}/R$ is an affine scheme equipped with a section $\epsilon : \mathrm{Spec}(R) \to \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \to \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{tikzcd}
(\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} \arrow[r, "m \times id"] \arrow[dd, "\wr"', "can"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[dr, "m"] & \\
& & \mathfrak{G} \\
\mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) \arrow[r, "id \times m"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[ur, "m"] &
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "id \times \epsilon"] \arrow[dr, "can", "\cong"'] & \mathfrak{G} \times_R \mathfrak{G} & \mathrm{Spec}(R) \times_R \mathfrak{G} \arrow[l, "\epsilon \times id"'] \arrow[dl, "can"', "\cong"] \\
& \mathfrak{G} &
\end{tikzcd}$$

Symmetry:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathfrak{G} \arrow[r, "id \times \sigma"] \arrow[d] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"] \\
\mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathfrak{G}.
\end{tikzcd}$$

We say that $\mathfrak{G}$ is commutative if furthermore the following diagram commutes

$$\begin{tikzcd}
& \mathfrak{G} \times_R \mathfrak{G} \arrow[dl, "m"'] \arrow[dd, "switch"] \\
\mathfrak{G} & \\
& \mathfrak{G} \times_R \mathfrak{G} \arrow[ul, "m"]
\end{tikzcd}$$

We will mostly work with affine R–group schemes, that is, when $\mathfrak{G}$ is an affine R–group scheme.

Let $R[\mathfrak{G}]$ be the coordinate ring of $\mathfrak{G}$. We call $\epsilon^* : R[\mathfrak{G}] \to \mathfrak{G}$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \to R[G]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. By means of the dictionary affine schemes/rings, they satisfy the following commutativity rules:

Co-associativity:

$$\begin{tikzcd} & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[r, "id\otimes\Delta"] & R[\mathfrak{G}]\otimes_R (R[\mathfrak{G}]\otimes_R R[\mathfrak{G}]) \arrow[dd, "\wr"] \\ R[\mathfrak{G}] \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\ & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[r, "\Delta\otimes id"] & (R[\mathfrak{G}]\otimes_R R[\mathfrak{G}])\otimes_R R[\mathfrak{G}]. \end{tikzcd}$$

Counit: The following composite maps are $id_{R[\mathfrak{G}]}$

$$R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \xrightarrow{id\otimes\epsilon} R[\mathfrak{G}]\otimes_R R \xrightarrow{\sim} R[\mathfrak{G}]$$

$$R[\mathfrak{G}] \xrightarrow{\Delta} R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \xrightarrow{\epsilon\otimes id} R[\mathfrak{G}]\otimes_R R \xrightarrow{\sim} R[\mathfrak{G}].$$

Cosymmetry:

$$\begin{tikzcd} & R[\mathfrak{G}] \arrow[r, "\Delta"] \arrow[dl, "\epsilon^*"] & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[dd, "id\times\sigma^*"] \\ R \arrow[dr] & & \\ & R[\mathfrak{G}] & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[l, "product"'] \end{tikzcd}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R–algebra[1]. Given an affine R–scheme $\mathfrak{X}$, there is then a one to one correspondence between group structures on $\mathfrak{X}$ and commutative R–algebra structures on $R[\mathfrak{X}]$.

Also $\mathfrak{G}$ is commuative if and only if the following diagram commutes

$$\begin{tikzcd} & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[dd, "switch"] \\ R[\mathfrak{G}] \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \end{tikzcd}$$

[1]This is Waterhouse definition [Wa, §I.4], other people talk about cocommutative coassociative Hopf algebra.

If $\mathfrak{G}/R$ is an (affine) R–group scheme, then for each R–algebra S the abstract group $\mathfrak{G}(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \to \mathfrak{G}(S)$. It means that the functor $h_{\mathfrak{G}}$ is actually a group functor.

2.5.1. **Lemma.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the Yoneda lemma induces a one to one correspondence between group structures on $\mathfrak{X}$ and group structures on $h_{\mathfrak{X}}$.*

In other words, defining a group law on $\mathfrak{X}$ is the same that to define compatible group laws on each $\mathfrak{G}(S)$ for S running over the R-algebras.

Proof. This is an immediate consequence of Yoneda's lemma. We assume that the R-functor h_X is equipped with a group structure. The Yoneda lemma shows that this group structure arises in an unique way of an affine R-group scheme structure. □

2.5.2. **Remark.** We shall encounter certain non-affine group R-schemes. A group scheme $\mathfrak{G}/R$ is a group object in the category of R-schemes. More generally the previous lemma holds for a non affine R–group scheme.

3. Examples

3.1. **Constant group schemes.** Let I be a set and consider theconsider the R–scheme $I_R = \bigsqcup_{\gamma \in I} \operatorname{Spec}(R) = \bigsqcup_{\gamma \in I} U_i$. We claim that its functor of points h_{I_R} identifies with

$$\Big\{ \text{locally constant functions} \operatorname{Spec}(S)_{top} \to I \Big\}.$$

To see this let S be an R–ring and let $f \in h_{I_R}(S) = \operatorname{Hom}_{\operatorname{Spec}(R)}(\operatorname{Spec}(S), I_R)$. By pulling back the open cover (U_i) of I_R, we obtain a decomposition $S = \bigsqcup_{\gamma \in I} S_i$ in open subschemes of R. This defines a locally constant function $\operatorname{Spec}(S)_{top} \to I$ having the value i on each S_i (for more details see [GW, Ex. 4.43] or [Sta, Tag 03YW]).

Next let Γ be an abstract group. We consider the R–scheme $\Gamma_R = \bigsqcup_{\gamma \in \Gamma} \operatorname{Spec}(R)$. Its functor of points h_{Γ_R} identifies with

$$\Big\{ \text{locally constant functions} \operatorname{Spec}(S)_{top} \to \Gamma \Big\}.$$

The group structure on Γ induces an R-group scheme structure on Γ_R. If R is non zero, this group scheme is affine and only if Γ is finite.

3.2. **Vector groups.** Let N be an R–module. We consider the commutative group functors

$$V_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto \operatorname{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee,$$

$$W_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto N \otimes_R S.$$

3.2.1. **Lemma.** *The R–group functor V_N is representable by the affine R–scheme $\mathfrak{V}(N) = \mathrm{Spec}(S^*(N))$ which is then a commutative R–group scheme. Furthermore if the R–module N is of finite presentation then the R–scheme $\mathfrak{V}(N)$ is of finite presentation.*

Proof. It follows readily of the universal property of the symmetric algebra

$$\mathrm{Hom}_{R'-mod}(N \otimes_R R', R') \xleftarrow{\sim} \mathrm{Hom}_{R-mod}(N, R') \xrightarrow{\sim} \mathrm{Hom}_{R-alg}(S^*(N), R')$$

for each R-algebra R'.

We assume that the R–module N is finitely presented, that is, there exists an exact sequence $0 \to M \to R^n \to N \to 0$ where M is a finitely generated R–module. According to [Sta, Tag 00DO] the kernel I of the surjective map $S^*(R^n) \to S^*(N)$ is generated by M (seen in degree one) so is a finitely generated $S^*(R^n)$-module. Since $S^*(R^n) = R[t_1, \dots, t_n]$, we conclude that the R–algebra $S^*(N)$ is of finite presentation.

□

3.2.2. **Remark.** The converse of the last assertion holds as well by using the limit characterizations of the finite presentation property, see [Sta, Tags 0G8P, 00QO].

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N. We note that $N = \mathfrak{V}(N)(R)$. In the special case $N = R^d$, this is nothing but the affine space $\mathbf{A}^d_R$ of relative dimension d.

Its group law on the R–group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \to S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The cosymmetry is $\sigma^* : S^*(N) \to S^*(N)$, $X \mapsto -X$ and the counit is the augmentation map $S^*(N) \to R$.

If $N = R$, we get the affine line over R. Given a map $f : N \to N'$ of R–modules, there is a natural map $f^* : \mathfrak{V}(N') \to \mathfrak{V}(N)$ of R–group schemes.

3.2.3. **Lemma.** *The assignement $N \to \mathfrak{V}(N)$ is a faithful contravariant (essentially surjective) functor from the category of R-modules and that of vector group R-schemes.*

Proof. Since this functor is essentially surjective, it is enough to show that it is faithful. Given two R–modules N, N' we want to show that the morphism

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_{R-gp}\big(\mathfrak{V}(N'), \mathfrak{V}(N)\big), \quad f \mapsto f^*$$

is injective. This is clear since $f_* : S^*(N) \to S^*(N')$ is a graded morphism and applies N to N' by f. □

3.2.4. **Remark.** Let k be a field of characteristic $p > 0$ and consider the Frobenius morphism $\mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$, $x \mapsto x^p$. It is a k–group homomorphism but is linear. This shows that the functor above is not fully faithful and then not an anti-equivalence of categories. For obtaining an anti-equivalence of categories, we need to restrict the morphisms to linear morphisms, see [SGA3, I.4.6.2].

We consider also the R-functor $W(N)$ defined by $W(N)(S) = N \otimes_R S$. If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that the R–functor $W(N)$ is representable by an affine group scheme.

3.2.5. **Theorem.** *The R–functor $W(N)$ is representable if and only if N is projective and finitely generated.*

If R is noetherian, this is due to [Ni04]. The general case has been handled by Romagny [Ro, Thm. 5.4.5]. Note that it is coherent with the example 2.1.3.(b).

3.3. **Group of invertible elements, linear groups.** Let A/R be an algebra (unital, associative). We consider the R-functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times.$$

3.3.1. **Lemma.** *If A/R is finitely generated projective, then* $\mathrm{GL}_1(A)$ *is representable by an affine group scheme. Furthermore,* $\mathrm{GL}_1(A)$ *is of finite presentation.*

Proof. Up to localize for the Zariski topology (Lemma 2.3.2), we can assume that A is a free R–module of rank d.

We shall use the norm map $N : A \to R$ defined by $a \mapsto \det(L_a)$ where $L_a : A \to A$ is the R-endomorphism of A defined by the left translation by A. We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a. More precisely, let $P_a(X) = X^d - \mathrm{Tr}(L_a)X^{d-1} + \cdots + (-1)^{d-1}c_{d-1}(L_a)X + (-1)^d \det(L_a) \in R[X]$ be the characteristic polynomial of L_a; according to the Cayley-Hamilton theorem we have $P_a(L_a) = 0$ [Bbk1, III, §11] so that $L_{P_a(a)} = 0$ and $P_a(a) = 0$. If $\det(L_a) \in R^\times$, it follows that

$$a\left(a^{d-1} - \mathrm{Tr}(L_a)a^{d-2} + \cdots + (-1)^{d-1}c_{d-1}(L_a)a\right) = (-1)^{d+1}\det(A)$$

so that $ab = ba = 1$ with $b = (-1)^{d+1}\det(A)^{-1}\Big(a^{d-1} - \mathrm{Tr}(L_a)a^{d-2} + (-1)^{d-1}c_{d-1}(L_a)a\Big)$.

The same is true after tensoring by S, so that

$$\mathrm{GL}_1(A)(S) = \Big\{a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \quad | \quad N(a) \in S^\times\Big\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{tikzcd} \mathfrak{G} \arrow[r] \arrow[d] & \mathfrak{W}(A) \arrow[d, "N"'] \\ \mathbb{G}_{m,R} \arrow[r] & \mathfrak{W}(R). \end{tikzcd}$$

$\square$

Given an R–module N, we consider the R–group functor

$$S \mapsto \mathrm{GL}(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S) = \mathrm{End}_S(N \otimes_R S)^{\times}.$$

So if N is finitely generated projective. then $\mathrm{GL}(N)$ is representable by an affine R–group scheme. Furthermore $\mathrm{GL}(N)$ is of finite presentation.

3.3.2. **Remark.** If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

3.4. **Diagonalizable group schemes.** Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R–algebra of A. As R-module, we have

$$R[A] = \bigoplus_{a \in A} R\, e_a$$

and the multiplication is given by $e_a\, e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R. We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R-algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \to R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$,

Antipode: $\sigma^* : R[A] \to R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \to R$, $\epsilon\big(\sum_{a \in A} r_a\, e_a\big) = r_0$.

We can check easily that it satisfies the axioms of affine commutative group schemes. One important example is that of $A = \mathbb{Z}$. In this case, we find $\mathbb{G}_{m,R} = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. Another one is $A = \mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$ for which we have $\mu_{n,R} = \mathrm{Spec}(R[T]/(T^n - 1)$ called the R–scheme of n–roots of unity.

3.4.1. **Definition.** *We denote by* $\mathfrak{D}(A)/R$ *(or* $\widehat{A}$*) the affine commutative group scheme* $\mathrm{Spec}(R[A])$*. It is called the diagonalizable R–group scheme of base A. An affine R–group scheme is diagonalizable if it is isomorphic to some* $\mathfrak{D}(B)$*.*

We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$.

If $f : B \to A$ is a morphism of abelian groups, it induces a group homomorphism $f^* : \mathfrak{D}(A) \to \mathfrak{D}(B)$. In particular, when taking $B = \mathbb{Z}$, we have a natural mapping

$$\eta_A : A \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathbb{G}_m).$$

3.4.2. **Remark.** For $a \in A$, put $\chi_a = \eta_A(a) : \mathfrak{D}(A) \to \mathbb{G}_m$. The map $\chi_a^* : R[t, t^{-1}] \to R[A]$ applies t to e_a. Using the commutative diagram

$$\begin{tikzcd}
\mathfrak{D}(A)(R[A]) \arrow[r, "\chi_a"] \arrow[d, "\wr"] & \mathbb{G}_m(R[A]) \arrow[d, "\wr"] \\
\mathrm{Hom}_R(R[A], R[A]) \arrow[r] & \mathrm{Hom}_R(R[t, t^{-1}], R[A]) = R[A]^{\times},
\end{tikzcd}$$

we see that the universal element of $\mathfrak{D}(A)$ maps to χ_a^* which corresponds to e_a.

3.4.3. **Lemma.** *If R is connected, η_A is bijective.*

Proof. We establish first the injectivity. If $\eta_A(a) = 0$, it means that the map $R[T, T^{-1}] \to R[A]$, $T \mapsto e_a$ factorises by the augmentation $R[T, T^{-1}] \to R$ hence $a = 0$.

For the surjectivity, let $f : \mathfrak{D}(A) \to \mathbb{G}_m$ be a morphism of R–group schemes. Equivalently it is given by the map $f^* : R[T, T^{-1}] \to R[A]$ of Hopf algebra which satisfies in particular the following compatibility

$$\begin{tikzcd} R[T,T^{-1}] \arrow[r, "f^*"] \arrow[d, "\Delta"] & R[A] \arrow[d, "\Delta_A"] \\ R[T,T^{-1}] \otimes_R R[T,T^{-1}] \arrow[r, "f^* \otimes f^*"] & R[A] \otimes_R R[A]. \end{tikzcd}$$

In other words, it is determined by the function $X = f^*(T) \in R[A]^\times$ satisfying $\Delta(X) = X \otimes X$. Writing $X = \sum_{a \in A} r_a e_a$, we have

$$\sum_{a \in A} r_a \, e_a \otimes e_a = \sum_{a, a' \in A} r_a \, r_{a'} \, e_a \otimes e_{a'}.$$

It follows that $r_a \, r_b = 0$ if $a \neq b$ and $r_a \, r_a = r_a$. Since the ring is connected, 0 and 1 are the only idempotents so that $r_a = 0$ or $r_a = 1$. Then there exists a unique a such that $r_a = 1$ and $r_b = 0$ for $b \neq a$. This shows that the map η_A is surjective. We conclude that η_A is bijective. $\square$

3.4.4. **Proposition.** *(Cartier duality) Assume that R is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable R–group schemes.*

Proof. It is enough to contruct the inverse map $\mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{D}(B)) \to \mathrm{Hom}(A, B)$ for abelian groups A, B. We are given a group homomorphism $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R-gp}(\mathfrak{D}(B), \mathbb{G}_m) \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathbb{G}_m),$$

hence a map $B \to A$. It is routine to check that the two functors are inverse of each other. $\square$

3.4.5. **Lemma.** *Assume that R is connected. The following are equivalent:*

(i) A is finitely generated;

(ii) $\mathfrak{D}(A)/R$ is of finite presentation;

(iii) $\mathfrak{D}(A)/R$ is of finite type.

Proof. $(i) \implies (ii)$. We use the structure theorem of abelian groups $A \cong$: $\mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \cdots \times \mathbb{Z}/n_c\mathbb{Z}$. Using the compatibility with products we are reduced to the case of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ which correspond to $\mathbb{G}_{m,R}$ and $\mu_{n,R}$. Both are finitely presented over R.

$(ii) \Longrightarrow (iii)$. Obvious.

$(iii) \Longrightarrow (i)$. We assume that $R[A]$ is a finitely generated R–ring. We write $A = \varinjlim_i A_i$ as the inductive limit of finitely generated subgroups. We have $R[A] = \varinjlim_i R[A_i]$. Since the ring $R[A]$ is finitely generated over R, the identity $\mathbb{Z}[A] \to \mathbb{Z}[A]$ factorizes through $\mathbb{Z}[A_i]$ for some i. It implies that $\mathbb{Z}[A_i] \xrightarrow{\sim} \mathbb{Z}[A]$. Cartier duality shows that $A_i \xrightarrow{\sim} A$. Thus A is finitely generated. □

There are other notable properties of Cartier duality, see [SGA3, VIII.2.1]. In practice we will work with finiteness assumptions, however it is remarkable that the theory holds for arbitrary abelian groups.

3.5. **Monomorphisms of group schemes.** We recall that a morphism of R–functors $f : F \to F'$ is a monomorphism if $f(S) : F(S) \to F'(S)$ is injective for each R–algebra S/R (§2.2). If F and F' are functors in groups and f respects the group structure, the kernel of f is the R–group functor defined by $\ker(f)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S.

We recall that a morphism $f : \mathfrak{G} \to \mathfrak{H}$ of affine R-group schemes is a monomorphism if h_f is a monomorphism (Lemma 2.2.2).

3.5.1. **Lemma.** *Let $f : \mathfrak{G} \to \mathfrak{H}$ be a morphism of R–group schemes. Then the R–functor $\ker(f)$ is representable by a closed subgroup scheme of $\mathfrak{G}$.*

Proof. Indeed the carthesian product

$$\begin{array}{ccc} \mathfrak{N} & \longrightarrow & \mathfrak{G} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(R) & \xrightarrow{\epsilon'} & \mathfrak{H} \end{array}$$

does the job. □

Summarizing $f : \mathfrak{G} \to \mathfrak{H}$ is a monomorphism if and only if the kernel R-group scheme $\ker(f)$ is the trivial group scheme.

Over a field F, we know that a monomorphism of algebraic groups is a closed immersion [SGA3, VI_B.1.4.2].

Over a DVR, it is not true in general that an open immersion (and a fortiori a monomorphism as seen in the exercise session) of group schemes of finite type is a closed immersion. We consider the following example [SGA3, VIII.7]. Assume that R is a DVR and consider the constant group scheme $\mathfrak{H} = (\mathbb{Z}/2\mathbb{Z})_R$. Now let $\mathfrak{G}$ be the open subgroup scheme of $\mathfrak{H}$ which is the complement of the closed point 1 in the closed fiber. By construction $\mathfrak{G}$ is dense in $\mathfrak{H}$, so that the immersion $\mathfrak{G} \to \mathfrak{H}$ is not closed. Raynaud constructed a more elaborated example where $\mathfrak{H}$ and $\mathfrak{G}$ are both affine over $\mathbf{F}_2[[t]]$ and a monomorphism which is not an immersion [SGA3, XVI.1.1.c].

However diagonalizable groups have a wonderful behaviour with that respect by using Cartier duality (Proposition 3.4.4).

3.5.2. **Proposition.** *Assume that R is connected. Let $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ be a group homomorphism of diagonalizable R–group schemes. Then the following are equivalent:*

(i) $f^* : A \to B$ is onto;

(ii) f is a closed immersion;

(iii) f is a monomorphism.

Proof. $(i) \Longrightarrow (ii)$: Then $R[B]$ is a quotient of $R[A]$ so that $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is a closed immersion.

$(ii) \Longrightarrow (iii)$: obvious.

$(iii) \Longrightarrow (i)$: We denote by $B_0 \subset B$ the image of $f^* : A \to B$. We consider the compositum

$$\begin{tikzcd}
\mathfrak{D}(B/B_0) \arrow[r] \arrow[rr, bend right, "v"'] & \mathfrak{D}(B) \arrow[r, "v"] \arrow[rr, bend left, "f"] & \mathfrak{D}(B_0) \arrow[r, "w"] & \mathfrak{D}(A).
\end{tikzcd}$$

We observe that it is the trivial morphism (v is trivial) and is a monomorphism as compositum of the monomorphisms u and f. It follows that $\mathfrak{D}(B/B_0) = \mathrm{Spec}(R)$ and we conclude that $B_0 = B$ by Cartier duality. $\square$

Of the same flavour, the kernel of a map $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is isomorphic to $\mathfrak{D}(f(A))$. The case of vector groups is more subtle.

3.5.3. **Proposition.** *Let $f : N_1 \to N_2$ be a morphism of finitely generated projective R-modules. Then the morphism of functors $f_* : W(N_1) \to W(N_2)$ is a monomorphism if and only if f identifies N_1 as a direct summand of N_2. If it the case, $f_* : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$ is a closed immersion.*

Proof. If N_1 is a direct summand of N_2, the morphism $f_* : W(N_1) = V(N_1^\vee) \to W(N_2^\vee)$ is a closed immersion and a fortiori a monomorphism. Conversely we assume that $f_* : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$ is a monomorphism.

Conversely suppose that f_* is a monomorphism. Since $W(N_1)(R)$ injects in $W(N_2)(R)$, we have that $f : N_1 \to N_2$ is injective. We put $N_3 = N_2/f(N_1)$. To show that N_1 is a direct summand of N_2 it is enough to show that N_3 is (finitely generated projective). This is our plan. Since N_2 and N_3 are f.g. projective R–modules, the R–module N_3 is of finite presentation. In view of the characterization of f.g. projective modules [Bbk2, II.5.2], it is enough to show that $N_3 \otimes R_\mathfrak{m}$ is free for each maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{m}$ be a maximal ideal of R.

Applying the criterion of Lemma 2.2.1 to the residue field $S = R/\mathfrak{m}$ we have that the map

$$f_*(R/\mathfrak{m}) : N_1 \otimes_R R/\mathfrak{m} \to N_2 \otimes_R R/\mathfrak{m}$$

is injective. It follows that there exists an $R/\mathfrak{m}$-base $(\overline{w}_1, \dots, \overline{w}_r, \overline{w}_{r+1}, \dots, \overline{w}_n)$ of $N_2 \otimes_R R/\mathfrak{m}$ such that $(\overline{w}_1, ..., \overline{w}_r)$ is a base of $f(N_1 \otimes_R R/\mathfrak{m})$. We have $\overline{w}_i = f(\overline{v}_i)$ for $i = 1, .., r$. We lift the $\overline{v}_i$'s in an arbitrary way in $N_1 \otimes_R R_\mathfrak{m}$ and the $\overline{w}_{r+1}, \dots, \overline{w}_n$ in $N_2 \otimes_R R_\mathfrak{m}$. Then $(v_1, \dots, v_r)$ is an $R_\mathfrak{m}$-base of $N_1 \otimes_R R_\mathfrak{m}$ and $(f(v_1), \dots, f(v_r), w_{r+1}, \dots, w_n)$ is an $R_\mathfrak{m}$–base of $N_2 \otimes_R R_\mathfrak{m}$. Thus $N_3 \otimes_R R_\mathfrak{m}$ is free.

We conclude that f identifies N_1 as a direct summand of N_2. $\square$

4. Sequences of group functors

4.1. **Exactness.** We say that a sequence of R–group functors

$$1 \to F_1 \overset{u}{\to} F_2 \overset{v}{\to} F_3 \to 1$$

is exact if for each R–algebra S, the sequence of abstract groups

$$1 \to F_1(S) \overset{u(S)}{\to} F_2(S) \overset{v(S)}{\to} F_3(S) \to 1$$

is exact. Similarly we can define the exactness of a sequence $1 \to F_1 \to \cdots \to F_n \to 1$. If $w : F \to F'$ is a map of R–group functors, recall the definition of the R–group functor $\ker(w)$ by $\ker(w)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S. Also the cokernel $\mathrm{coker}(w)(S) = \mathrm{coker}(F(S) \to F'(S))$ is an R-functor (but not necessarily an R-functor in groups).

4.1.1. **Example.** We consider an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of finitely generated modules with N_1, N_2 projective. We claim that it induces an exact sequence of R–functors in groups

$$0 \to W(N_1) \to W(N_2) \to W(N_3) \to 0$$

if and only if the starting sequence is split (equivalently N_3 is projective). The converse implication is obvious. If the sequence above of R–functors in groups is exact, then $W(N_1) \to W(N_2)$ is a monomorphism so that Proposition 3.5.3 shows that N_1 is a direct summand of N_2.

We can define also the cokernel of a morphism R–group schemes. But it is very rarely representable. The simplest example is the Kummer morphism $f_n : \mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$, $x \mapsto x^n$ for $n \geq 2$ for $R = \mathbb{C}$, the field of complex numbers. Assume that there exists an affine $\mathbb{C}$–group scheme $\mathfrak{G}$ such that there is a four terms exact sequence of $\mathbb{C}$–functors

$$1 \to h_{\mu_n} \to h_{\mathbb{G}_m} \overset{h_{f_n}}{\to} h_{\mathbb{G}_m} \to h_{\mathfrak{G}} \to 1$$

We denote by T' the parameter for the first $\mathbb{G}_m$ and by $T = (T')^n$ the parameter of the second one. Then $T \in \mathbb{G}_m(R[T, T^{-1}])$ defines a non trivial element of $\mathfrak{G}(R[T, T^{-1}])$ which is trivial in $\mathfrak{G}(R[T', T'^{-1}])$ It is a contradiction.

We provide a criterion.

4.1.2. **Lemma.** *Let*

$$1 \to \mathfrak{G}_1 \xrightarrow{u} \mathfrak{G}_2 \xrightarrow{v} \mathfrak{G}_3 \to 1$$

be a sequence of affine R–group schemes. Then the sequence of R–functors

$$1 \to h_{\mathfrak{G}_1} \to h_{\mathfrak{G}_2} \to h_{\mathfrak{G}_3} \to 1$$

is exact if and only if the following conditions are satisfied:

(i) $u : \mathfrak{G}_1 \to \ker(v)$ is an isomorphism;

(ii) $v : \mathfrak{G}_2 \to \mathfrak{G}_3$ admits a splitting $f : \mathfrak{G}_3 \to \mathfrak{G}_2$ as R–schemes.

4.1.3. **Remark.** Note that if (ii) holds, we have $\mathfrak{G}_2(S) = u(\mathfrak{G}_1(S))f(\mathfrak{G}_3(S))$ for each R-algebra S. Let S be an R–algebra and let $g_2 \in \mathfrak{G}_2(S)$. Since $\mathfrak{G}_1(S) \to \mathfrak{G}_2(S) \to \mathfrak{G}_3(S)$ is exact, $g_2\, f(v(g_2))^{-1} \in \mathfrak{G}_1(S)$. We conclude that $\mathfrak{G}_2(S) = u(\mathfrak{G}_1(S))f(\mathfrak{G}_3(S))$.

We proceed to the proof of Lemma 4.1.2.

Proof. We assume that the sequence of R-functors $1 \to h_{\mathfrak{G}_1} \to h_{\mathfrak{G}_2} \to h_{\mathfrak{G}_3} \to 1$ is exact. We have seen that $\mathfrak{G}_1$ is the kernel of v. This shows (i). The assertion (ii) is an avatar of Yoneda's lemma. We consider the surjective map $\mathfrak{G}_2(R[\mathfrak{G}_3]) \to \mathfrak{G}_3(R[\mathfrak{G}_3])$ and lift the identity of $\mathfrak{G}_3$ to a map $t : \mathfrak{G}_2(R[\mathfrak{G}_3]) = \mathrm{Hom}_{R-sch}(\mathfrak{G}_3, \mathfrak{G}_2)$. Then t is an R–scheme splitting of $v : \mathfrak{G}_2 \to \mathfrak{G}_3$.

Conversely we assume (i) and (ii). Clearly $h_{\mathfrak{G}_1} \to h_{\mathfrak{G}_2}$ is a monomorphism and $h_{\mathfrak{G}_2} \to h_{\mathfrak{G}_3}$ is a epimorphism (see §2.2). We only have to check the exactness of $\mathfrak{G}_1(S) \to \mathfrak{G}_2(S) \to \mathfrak{G}_3(S)$ for each S/R but it follows from (ii). □

4.1.4. **Examples.** (a) It is not obvious to construct examples of exact sequences of group functors which are not split as R–group functors. An example is the exact sequence of Witt vectors groups over $\mathbb{F}_p$ $0 \to W_1 \to W_2 \to W_1 \to 0$. It provides a non split exact sequence of commutative affine $\mathbb{F}_p$–group schemes $0 \to \mathbb{G}_a \to W_2 \to \mathbb{G}_a \to 0$. For other examples see [DG, III.6]. (b) Also it is natural question to ask whether the existence of sections of the map $\mathfrak{G}_2 \to \mathfrak{G}_3$ locally over $\mathfrak{G}_3$ is enough. It is not the case and an example of this phenomenon is by using the $\mathbb{R}$–group scheme G_2 defined as the unit group scheme of the $\mathbb{R}$-algebra $\mathbb{C}$; recall that its functor of points is $G_2(S) = (S \otimes_{\mathbb{R}} \mathbb{C})^\times$ (§3.3). It comes with a norm morphism $N : G_2(S) \to \mathbb{G}_{m,\mathbb{R}}$ and we consider the kernel $G_3 = \ker(N)$. Note that G_2 comes with an involution σ given by the complex conjugation. We consider the sequence of $\mathbb{R}$–group schemes

$$1 \to \mathbb{G}_m \to G_2 \xrightarrow{\sigma - id} G_3 \to 1.$$

The associated sequence for real points is $1 \to \mathbb{R}^\times \to \mathbb{C}^\times \to S^1 \to 1$, where the last map is $z \mapsto \overline{z}/z$. For topological reasons[2], there is no continuous section of the map $\mathbb{C}^\times \to S^1$. A fortiori, there is no algebraic section of

[2]The induced map $\mathbb{Z} = \pi(\mathbb{C}^\times, 1) \to \mathbb{Z} = \pi_1(S^1, 1)$ is the multipliczation by 2.

the map $G_2 \xrightarrow{\sigma - id} G_3$. On the other hand this map admits local splittings, let us explain how it works for example on $\mathfrak{G}_3 \setminus \{(-1,0)\}$. We map $t \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) = (\sigma - 1).(1+ti)$ induces an isomorphism $\mathbb{R}[\mathfrak{G}_3]_{(-1,0)} \xrightarrow{\sim} \mathbb{R}\big[t, \frac{1}{t^2+1}\big]$ and defines a section of $\sigma - id$ on $\mathfrak{G}_3 \setminus \{(-1,0)\}$. The sequence above is not exact in the category of $\mathbb{R}$–functors.

4.2. **Semi-direct product.** Let $\mathfrak{G}/R$ be an affine group scheme acting on another affine group scheme $\mathfrak{H}/R$, that is we are given a morphism of R–functors

$$\theta : h_{\mathfrak{G}} \to \mathrm{Aut}(h_{\mathfrak{H}}).$$

The semi-direct product $h_{\mathfrak{H}} \rtimes^{\theta} h_{\mathfrak{G}}$ is well defined as R–functor.

4.2.1. **Lemma.** *$h_{\mathfrak{H}} \rtimes^{\theta} h_{\mathfrak{G}}$ is representable by an affine R-scheme denote by $\mathfrak{H} \rtimes^{\theta} \mathfrak{G}$. Furthermore we have an exact sequence of affine R-group schemes*

$$1 \to \mathfrak{H} \to \mathfrak{H} \rtimes^{\theta} \mathfrak{G} \to \mathfrak{G} \to 1.$$

Proof. We consider the affine R-scheme $\mathfrak{X} = \mathfrak{H} \times_R \mathfrak{G}$. Then $h_{\mathfrak{X}} = h_{\mathfrak{H}} \rtimes^{\theta} h_{\mathfrak{G}}$ has a group structure so defines a group scheme structure on $\mathfrak{X}$. The sequence holds in view ot the criterion provided by Lemma 4.1.2. $\square$

A nice example of this construction is the "affine group" of a finitely generated. projective R–module N. The R-group scheme $\mathrm{GL}(N)$ acts on the vector R–group $\mathfrak{W}_N$ so that we can form the R–group scheme $\mathfrak{W}_N \rtimes \mathrm{GL}(N)$ of affine transformations of N.

Affine group schemes II

5. Flatness

We will explain in this section why flatness is a somehow a minimal reasonable assumption when studying affine group schemes. This includes a nice behaviour of the dimension of geometric fibers, see Thm. 5.3.1 below.

5.1. **Examples of flat affine group schemes.**

5.1.1. **Lemma.** *Let $\mathfrak{G}$ be an affine R-group scheme. Then $\mathfrak{G}$ is flat if and only if $\mathfrak{G}$ is faithfully flat.*

Proof. Faithfully flat means that the structural morphism $\mathfrak{G} \to \operatorname{Spec}(R)$ is flat and surjective. Since $\mathfrak{G} \to \operatorname{Spec}(R)$ admits the unit section, the structural morphism is surjective. This explains the equivalence between flatness and faithfully flatness in our setting. $\square$

All examples we have seen so far were flat. Constant group schemes are obviously flat. If A is an abelian group, the diagonalizable R–group scheme $\mathfrak{D}(A)$ is R–flat since $R[A]$ is a free R-module.

If N is a finitely generated projective R-module, the affine group schemes $\mathfrak{V}(N)$ and $\mathfrak{W}(N)$ are flat. Indeed, flatness is a local property for the Zariski topology on $\operatorname{Spec}(R)$ [Sta, Tag 00HJ] so that it reduces to the case of the affine space $\mathbb{A}^d_R$ which is clear since the R-module $R[t_1, \dots, t_d]$ is free. A more complicated fact is the following.

5.1.2. **Lemma.** *Let M be an R-module. Then M is flat if and only if $\mathfrak{V}(M)$ is a flat R-scheme.*

Proof. By definition the R-scheme $\mathfrak{V}(M)$ is flat if and only is the symmetric algebra $S^*(M)$ is a flat R–module. Since M is a direct summand of $S^*(M)$ as R-module, the flatness of $S^*(M)$ implies that M is flat.

For the converse we use Lazard's theorem stating that M is isomorphic to a direct limit $\varinjlim_{i\in I} M_i$ of f.g. free R-modules [Sta, Tag 058G]. Since $S^*(M) = \varinjlim_{i\in I} S^*(M_i)$ and each $S^*(M_i)$ is a free R-module (so a fortiori flat), it follows that $S^*(M)$ is a flat R–algebra in view of [Sta, Tag 05UU] (use the case $R_i = R$ for all i).

$\square$

Finally the group scheme of invertible elements $U(A)$ of an algebra A/R f.g. projective is flat. We have seen that $U(A)$ is principal open in $\mathfrak{W}(A)$ so that $R[U(A)]$ is flat over $R[\mathfrak{W}(A)]$ [Sta, Tag 00HT]. Since flatness behave well for composition [GW, prop. 14.3], we conclude that the affine R-scheme $U(A)$ is flat.

5.2. **The DVR case.** Assume that R is a DVR with uniformizing parameter π and denote by K its field of fractions. We recall the following well-known fact.

5.2.1. **Lemma.** *Let M be an R–module. Then the following are equivalent:*

(i) M is flat;

(ii) M is torsion free, that is $\times\pi : M \to M$ is injective;

(iii) $M \to M \otimes_R K$ is injective.

Furthermore, if M is finitely generated, this is equivalent to $M \cong R^n$.

Proof. $(i) \Longrightarrow (ii)$. It means that the functor $\otimes_R M$ is exact. Since $\pi : R \to R$ is injective, it follows that $\times\pi : M \to M$.

$(ii) \Longrightarrow (i)$. The R module M is the filtered inductive limit of its finitely generated submodules. Also, submodules of torsionfree modules are torsionfree, and inductive limits of flat modules are flat [Sta, Tag 05UU]. This is why it suffices to prove that finitely generated torsionfree R-modules are flat, or even free. We assume then that M is a finitely generated R-module. Choose $m_1, \dots, m_n \in M$ such that $\overline{m_1}, \dots, \overline{m_n}$ is a k-basis of the k–vector space $M \otimes_R k$. By Nakayama's Lemma, $m_1, \dots, m_n$ is a generating set of M; in other words we have a surjective R–map $f : R^n \to M$. Consider a non zero relation $f(r_1, \dots, r_n) = \sum_{i=1}^n r_i\, m_i = 0$. Since M is torsionfree, dividing the r_i' by the largest possible power π^c occuring so that we get a non-trivial relation $\sum_{i=1}^n \overline{r_i}\, \overline{m}_i = 0$. This is a contradiction.

$(ii) \Longrightarrow (iii)$. Once again this reduces to the finitely generated case which is free. Since $R^n \to K^n$ is injective, we are done.

$(iii) \Longrightarrow (ii)$. Obvious.

□

Note that there are generalization to Dedekind domains and valuation rings [Sta, Tags 0AUW, 0539]. From the lemma, we know that an affine scheme $\mathfrak{X}/R$ is flat, that is, $R[\mathfrak{X}]$ is torsionfree or equivalently that $R[\mathfrak{X}]$ embeds in $K[\mathfrak{X}]$.

5.2.2. **Proposition.** [EGA4, 2.8.1] *(see also* [GW, §14.3]*)*

Let $\mathfrak{X}/R$ be a flat affine R-group scheme. There is a one to one correspondence between the flat closed R-subschemes of $\mathfrak{X}$ and the closed K–subschemes of the generic fiber $\mathfrak{X}_K$.

Furthermore this correspondence commutes with fibered products over R and is functorial with respect to R-morphisms $\mathfrak{X} \to \mathfrak{X}'$ of flat R–schemes.

The correspondence goes as follows. In one way we take the generic fiber and in the way around we take the schematic closure (in the sense of the scheme theoretic image of the immersion map $Y \subset \mathfrak{X}_K \hookrightarrow \mathfrak{X}$ [Sta, Tag 01R7]). The schematic closure $\mathfrak{Y}$ of Y in $\mathfrak{X}$ is the smallest closed subscheme $\mathfrak{X}$ such that $Y \subset \mathfrak{X}_K \hookrightarrow \mathfrak{X}$ factorizes through $\mathfrak{Y}$. Let us explain its construction in terms of rings. If Y/K is a closed K–subscheme of X/K, it is defined by the ideal $I(Y) = \mathrm{Ker}(K[\mathfrak{X}] \to K[Y])$ of $K[\mathfrak{X}]$. Similarly we deal with the ideal $I(\mathfrak{Y}) = \mathrm{Ker}(R[\mathfrak{X}] \to R[\mathfrak{Y}])$ of $R[\mathfrak{X}]$. This fits in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I(\mathfrak{Y}) \arrow[r] \arrow[d] & R[\mathfrak{X}] \arrow[r] \arrow[d, hook] & R[\mathfrak{Y}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I(Y) \arrow[r] & K[\mathfrak{X}] \arrow[r] & K[Y] \arrow[r] & 0
\end{tikzcd}$$

The ideal $I(\mathfrak{Y})$ of $R[\mathfrak{X}]$ is the smallest ideal which maps in $I(Y)$, it follows that $I(\mathfrak{Y}) = I \cap R[\mathfrak{X}]$. Since $I(\mathfrak{Y}) \otimes_R K = I(Y)$, we have $R[\mathfrak{Y}] \otimes_R K = K[Y]$, that is, $\mathfrak{Y} \times_R K = Y_K$. Also the map $R[\mathfrak{Y}] \to K[Y]$ is injective, i.e. $\mathfrak{Y}$ is a flat affine R–scheme. It remains to show that the other composite is the identity and also the functorial properties. We proceed then to the end of the proof of Proposition 5.2.2.

Proof. Given $\mathfrak{Y} \subset \mathfrak{X}$ a flat closed R–subcheme, we consider the ideal $I(\mathfrak{Y}) = \mathrm{Ker}(R[\mathfrak{X}] \to R[\mathfrak{Y}])$. We denote by $\mathfrak{Y}' \subset \mathfrak{X}$ the schematic closure of $\mathfrak{Y}_K \subset \mathfrak{X}$. We have $I(\mathfrak{Y}') = I(\mathfrak{Y}_K) \cap R[\mathfrak{X}]$. We consider the commutative diagram of exact sequences of R-modules

$$\begin{tikzcd}
0 \arrow[r] & I(\mathfrak{Y}) \arrow[r] \arrow[d, hook] & R[\mathfrak{X}] \arrow[r] \arrow[d, hook] & R[\mathfrak{Y}] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I(\mathfrak{Y}_K) \arrow[r] & K[\mathfrak{X}] \arrow[r] & K[\mathfrak{Y}] \arrow[r] & 0
\end{tikzcd}$$

where the two vertical maps on the right express flatness of $\mathfrak{X}$ and $\mathfrak{Y}$. By diagram chase we have $I(\mathfrak{Y}) = I(\mathfrak{Y}')$.

We examine now the behaviour for fibered products, We are given two affine flat R–schemes $\mathfrak{X}_1$, $\mathfrak{X}_2$ with closed flat R-subschemes $\mathfrak{Y}_1 \subset \mathfrak{X}_1$ and $\mathfrak{Y}_2 \subset \mathfrak{X}_2$. Then $\mathfrak{Y}_1 \times_R \mathfrak{Y}_2$ is a flat closed R–subscheme (using that flatness behave well with tensor products, see [Bbk2, §I.7]) of $\mathfrak{X}_1 \times_R \mathfrak{X}_2$ and of generic fiber $\mathfrak{Y}_{1,K} \times_K \mathfrak{Y}_{2,K}$ so is the schematic closure of $\mathfrak{Y}_{1,K} \times_K \mathfrak{Y}_{2,K}$ in $\mathfrak{X}_1 \times_R \mathfrak{X}_2$.

Next we deal with a morphism $f : \mathfrak{X} \to \mathfrak{X}'$ of affine flat R–schemes. For an affine flat closed R–subcheme $\mathfrak{Y} \subset \mathfrak{X}$ (resp. $\mathfrak{Y}' \subset \mathfrak{X}'$), if f induces a morphism $\mathfrak{Y} \to \mathfrak{Y}'$ then f_K induces a map $\mathfrak{Y}_K \to \mathfrak{Y}'_K$. Conversely assume that f_K induces a map $f_K : Y' \to Y$ where $Y \subset \mathfrak{X}_K$ (resp. $Y' \subset \mathfrak{X}'_K$) and denote by $\mathfrak{Y} \subset \mathfrak{X}$ (resp. $\mathfrak{Y}' \subset \mathfrak{X}'$) the schematic adherence of Y. We need to check that f induces a map $\mathfrak{Y} \to \mathfrak{Y}'$. We consider the diagram

$$\begin{tikzcd} R[\mathfrak{X}'] \arrow[r, "f^*"] \arrow[d, hook] & R[\mathfrak{X}] \arrow[d, hook] \\ K[\mathfrak{X}'] \arrow[r, "f_K^*"] \arrow[d, phantom, "\bigcup"] & K[\mathfrak{X}] \arrow[d, phantom, "\bigcup"] \\ I(Y') \arrow[r] & I(Y) \end{tikzcd}$$

It shows that $f^*(R[\mathfrak{X}'] \cap I(Y')) \subseteq R[\mathfrak{X}] \cap I(Y)$ whence $f^*(R[\mathfrak{Y}']) \subseteq R[\mathfrak{Y}]$ as desired. $\square$

In particular, if $\mathfrak{G}/R$ is a flat group scheme, it induces a one to one correspondence between flat closed R-subgroup schemes of $\mathfrak{G}$ and closed K–subgroup schemes of $\mathfrak{G}_K$[3].

5.2.3. **Example.** We consider the centralizer closed subgroup scheme of $\mathrm{GL}_{2,R}$

$$\mathfrak{Z} = \Big\{ g \in \mathrm{GL}_{2,R} \quad | \quad g\,A = A\,g \Big\}$$

of the element $A = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$. Then $\mathfrak{Z} \times_R R/\pi R \xrightarrow{\sim} \mathrm{GL}_{2,R}$ and

$$\mathfrak{Z} \times_R K = \mathbb{G}_{m,K} \times_K \mathbb{G}_{a,K} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\}$$

Then the closure of $\mathfrak{Z}_K$ in $\mathrm{GL}_{2,R}$ is $\mathbb{G}_{m,R} \times_R \mathbb{G}_{a,R}$, so does not contain the special fiber of $\mathfrak{Z}$. We conclude that $\mathfrak{Z}$ is not flat.

5.3. **A necessary condition.** In the above example, the geometrical fibers were of dimension 4 and 2 respectively. It illustrates then the following general result.

5.3.1. **Theorem.** [SGA3, VI_B.4.3] *Let R be a ring and let $\mathfrak{G}/R$ be a flat group scheme of finite presentation. Then the dimension of the geometrical fibers is locally constant.*

It means that the dimension of the fibers cannot jump by specialization.

6. Representations

Let $\mathfrak{G}/R$ be an a affine group scheme.

6.0.1. **Definition.** *A (left) $R - \mathfrak{G}$-module (or $\mathfrak{G}$-module for short) is an R–module M equipped with a morphism of group functors*

$$\rho : h_{\mathfrak{G}} \to \mathrm{Aut}_{lin}(W(M)).$$

We say that the $\mathfrak{G}$–module M is faithful is ρ is a monomorphism.

[3]Warning: the fact that the schematic closure of a group scheme is a group scheme is specific to Dedekind rings.

Here $\mathrm{Aut}_{lin}(W(M))$ stands for for linear automorphisms of the functor $W(M)$, that is, $\mathrm{Aut}_{lin}(W(M))(S) = \mathrm{End}_S(M \otimes_R S)^\times$ for each R–algebra S. We denote by $\mathrm{GL}(M)$ and we bear in mind that is not necessarily representable.

If $M = R^n$, then $\mathrm{GL}(M)$ is representable by $\mathrm{GL}_{n,R}$ so that it corresponds to an R–group homomorphism $\mathfrak{G} \to \mathrm{GL}_{n,R}$ and faithfulness corresponds to the triviality of the kernel.

Coming back to the general setting, it means that for each algebra S/R, we are given an action of $\mathfrak{G}(S)$ on $W(M)(S) = M \otimes_R S$. We use again Yoneda lemma. The mapping ρ is defined by the image of the universal point $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ provides an element called the coaction

$$c_\rho \in \mathrm{Hom}_R\Big(M, M \otimes_R R[\mathfrak{G}]\Big) \xrightarrow{\sim} \mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big).$$

Yoneda lemma implies that c_ρ determines ρ. We denote $\widetilde{c}_\rho$ its image in $\mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big)$. For $g \in \mathfrak{G}(R)$, we use the evaluation $\epsilon_g : R[\mathfrak{G}] \to \mathfrak{R}$ and have by functoriality the bottom of the following commutative diagram

(6.0.2)
$$\begin{tikzcd}
M \arrow[d, "id"] \arrow[dr, "c_\rho"] \arrow[dd, bend right=60, "id"'] & \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "\widetilde{c}_\rho"] \arrow[d, "id\otimes\epsilon_g"] & M \otimes_R R[G] \arrow[d, "id\otimes\epsilon_g"] \\
M \arrow[r, "\rho(g)"] & M.
\end{tikzcd}$$

In other words we have

(6.0.3)
$$\rho(g).m = \epsilon_g(c_\rho.m) \qquad (g \in \mathfrak{G}(R), m \in M).$$

6.0.4. **Remark.** For the trivial representation, we have that $\tilde{c}_{triv} = id_{M\otimes_R R[\mathfrak{G}]}$ so that $c_{triv}(m) = m \otimes 1$.

6.0.5. **Proposition.** *(1) Both diagrams*

(6.0.6)
$$\begin{tikzcd}
M \arrow[r, "c_\rho"] \arrow[d, "c_\rho"'] & M \otimes_R R[\mathfrak{G}] \arrow[d, "id\otimes\Delta_{\mathfrak{G}}"'] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "c_\rho\otimes id"] & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}],
\end{tikzcd}$$

(6.0.7)
$$\begin{tikzcd}
M \arrow[r, "c_\rho"] \arrow[d, "id"'] & M \otimes_R R[\mathfrak{G}] \arrow[dl, "id\times\epsilon^*"] \\
M &
\end{tikzcd}$$

commute.

(2) Conversely, if an R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above, there is a unique representation $\rho_c : h_{\mathfrak{G}} \to \mathrm{GL}(W(M))$ such that $c_{\rho_c} = c$.

A module M equipped with a R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above is called a $\mathfrak{G}$-module (and also a comodule over the Hopf algebra $R[\mathfrak{G}]$). The proposition shows that it is the same to talk about representations of $\mathfrak{G}$ or about $\mathfrak{G}$-modules (or also $R - \mathfrak{G}$-modules).

6.0.8. **Remark.** There is of course a compatibility with the inverse map but it follows from the other rules.

In particular, the comultiplication $R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ defines a $\mathfrak{G}$-structure on the R–module $R[\mathfrak{G}]$. It is called the regular representation and is studied more closely in Example 6.0.9. We proceed to the proof of Proposition 6.0.5.

Proof. (1) We double the notation by putting $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$. We consider the following commutative diagram

$$\begin{tikzcd}
\mathfrak{G}(R[\mathfrak{G}_1]) \times \mathfrak{G}(R[\mathfrak{G}_2]) \arrow[r, "\rho\times\rho"] \arrow[d] & \mathrm{GL}(M)(R[\mathfrak{G}_1]) \times \mathrm{GL}(M)(R[\mathfrak{G}_2]) \arrow[d] \\
\mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \arrow[r, "\rho\times\rho"] \arrow[d, "m"'] & \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \times \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \arrow[d, "m"'] \\
\mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2]) \arrow[r, "\rho"] & \mathrm{GL}(M)(R[\mathfrak{G}_1 \times \mathfrak{G}_2])
\end{tikzcd}$$

and consider the image $\eta \in \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2])$ of the couple (ζ_1, ζ_2) of universal elements by the left vertical map. Then η is defined by the ring homomorphism $\eta^* : R[\mathfrak{G}] \overset{\Delta_{\mathfrak{G}}}{\to} R[\mathfrak{G} \times \mathfrak{G}] \xrightarrow{\sim} R[\mathfrak{G}_1 \times \mathfrak{G}_2]$ so that $\rho(\eta)$ is defined by the following commutative diagram (in view of the compatibility (6.0.2))

$$\begin{tikzcd}
M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \arrow[r, "\rho(\eta)"] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
M \otimes_R R[\mathfrak{G}] \arrow[r, "\tilde{c}_\rho"] \arrow[u, "id_M\otimes\Delta"] & M \otimes_R R[\mathfrak{G}] \arrow[u, "id_M\otimes\Delta"]
\end{tikzcd}$$

On the other hand we have that $\rho(\eta) = \widetilde{c}_{\rho,2} \circ \widetilde{c}_{\rho,1}$ where we did not write the extensions to $R[\mathfrak{G}_1 \times \mathfrak{G}_2]$. Reporting that fact in the diagram above provides the commutative diagram

$$\begin{tikzcd}
M \arrow[r] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \arrow[r, "\tilde{c}_{\rho_2}\circ\tilde{c}_{\rho_1}"] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\
M \arrow[r] \arrow[u, "id"] & M \otimes_R R[\mathfrak{G}] \arrow[r, "\tilde{c}_\rho"] \arrow[u, "id_M\otimes\Delta"] & M \otimes_R R[\mathfrak{G}]. \arrow[u, "id_M\otimes\Delta"]
\end{tikzcd}$$

By restricting to M, we get the commutative square

$$\begin{tikzcd} M \otimes_R R[\mathfrak{G}_1] \arrow[r, "c_{\rho_2} \otimes id_{R[\mathfrak{G}_1]}"] & M \otimes_R R[\mathfrak{G}_1 \times \mathfrak{G}_2] \\ M \arrow[u, "c_{\rho,1}"] \arrow[r, "c_\rho"] & M \otimes_R R[\mathfrak{G}] \arrow[u, "\Delta"'] \end{tikzcd}$$

as desired. The other rule comes from the fact that $1 \in G(R)$ acts trivially on M and is a special case of the diagram (6.0.2).

(2) We are given $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules. We define first a morphism of R–functors $h_{\mathfrak{G}} \to W(End_R(M))$. According to Yoneda lemma 2.1.1, we have

$$\mathrm{Hom}_{R-func}\big(h_{\mathfrak{G}}, W(End_R(M))\big) = W(End_R(M))(R[\mathfrak{G}])$$

$$= \mathrm{Hom}_{R[\mathfrak{G}]}(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]) \xleftarrow{\sim} \mathrm{Hom}_R(M, M \otimes_R R[\mathfrak{G}]).$$

It follows that c defines a (unique) morphism of R–functors $\rho_c : h_{\mathfrak{G}} \to W(End_R(M))$ such that the universal element of $\mathfrak{G}$ is applied to $\tilde{c}$. The first rule insures the multiplicativity (check it) and the second rule says that the unit element $1 \in \mathfrak{G}(R)$ is applied to id_M. It follows that ρ_c factorizes through the subfunctor $\mathrm{GL}(M)$ of $W(End_R(M))$ and induces a homomorphism of R–group functors $h_{\mathfrak{G}} \to \mathrm{GL}(M)$. $\square$

6.0.9. **Example.** We claim that the regular representation is nothing but the right translation on $R[\mathfrak{G}]$ and that it is faithful. We consider the $\mathfrak{G}$–module $A = R[\mathfrak{G}]$ defined by the comultiplication $\Delta : A \to A \otimes_R R[\mathfrak{G}]$. It defines the regular representation $\rho : \mathfrak{G} \to \mathrm{GL}(A)$. Given $g \in \mathfrak{G}(R)$, we consider the following diagram (special case of the diagram (6.0.2))

(6.0.10)
$$\begin{tikzcd} A \arrow[d, "id"] \arrow[dr, "\Delta"] & \\ A \otimes_R R[\mathfrak{G}] \arrow[r, "\widetilde{\Delta}"] \arrow[d, "id \otimes \epsilon_g"] & A \otimes_R R[\mathfrak{G}] \arrow[d, "id \otimes \epsilon_g"] \\ A \arrow[r, "\rho(g)"] & A. \end{tikzcd}$$

where ϵ_g is the evaluation at g and where the bottom is the compatibility (6.0.2). In terms of schemes, the map below is $\mathfrak{G} = \mathfrak{G} \times_R \mathrm{Spec}(R) \xrightarrow{id_{\mathfrak{G}} \times g} \mathfrak{G} \times_R \mathfrak{G} \xrightarrow{product} \mathfrak{G}$. It follows that $\rho(g).a = a \circ R_g = R_g^*(a)$ for each $a \in A = R[\mathfrak{G}]$ where $R_g : \mathfrak{G} \to \mathfrak{G}$ is the right translation by g, $x \mapsto xg$.

Let us show that the regular representation $R[\mathfrak{G}]$ is faithful. Let S be an R–ring and let $g \in \mathfrak{G}(S)$ acting trivially on $S[\mathfrak{G}]$. It means that $f \circ R_g = f$ for each $f \in S[\mathfrak{G}]$ hence $f(g) = f(1)$ for each $f \in S[\mathfrak{G}]$. But

$\mathfrak{G}(S) = \mathrm{Hom}_S(S[\mathfrak{G}], S)$, so that $g = 1$. This shows that the regular representation is faithful.

A morphism of $\mathfrak{G}$-modules is an R–morphism $f : M \to M'$ such that $f(S) \circ \rho(g) = \rho'(g) \circ f(S) \in \mathrm{Hom}_S(M \otimes_R S, M' \otimes_R S)$ for each S/R and for $g \in \mathfrak{G}(S)$. Equivalenty, this is to require the commutativty of the following diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "c_\rho"] & M' \arrow[d, "c_{\rho'}"] \\ M \otimes_R R[\mathfrak{G}] \arrow[r, "f\otimes id"] & M' \otimes_R R[\mathfrak{G}]. \end{tikzcd} \tag{6.0.11}$$

It is clear that the R–module $\mathrm{coker}(f)$ is equipped then with a natural structure of $\mathfrak{G}$-module. For the kernel $\ker(f)$, we cannot proceed similarly because the mapping $\ker(f) \otimes_R S \to \ker(M \otimes_R S \stackrel{f(S)}{\to} M' \otimes_R S)$ is not necessarily injective. One tries to use the module viewpoint by considering the following commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r] & M \arrow[r, "f"] \arrow[d, "c_\rho"'] & M' \arrow[d, "c_{\rho'}"'] \\ & \ker(f) \otimes_R R[\mathfrak{G}] \arrow[r] & M \otimes_R R[\mathfrak{G}] \arrow[r, "f\otimes id"] & M' \otimes_R R[\mathfrak{G}]. \end{tikzcd}$$

If $\mathfrak{G}$ is flat, then the left bottom map is injective, and the diagram defines a map $c : \ker(f) \to \ker(f) \otimes_R R[\mathfrak{G}]$. This map c satisfies the two compatibilities and define then a $\mathfrak{G}$-module structure on $\ker(f)$. We have proven the important fact.

6.0.12. **Proposition.** *Assume that $\mathfrak{G}/R$ is flat. Then the category of $\mathfrak{G}$-modules is an abelian category.*

6.0.13. **Remark.** It is actually more than an abelian category since it carries tensor products.

6.1. **Representations of diagonalizable group schemes.** Let $\mathfrak{G} = \mathfrak{D}(A)/R$ be a diagonalizable group scheme. For each $a \in A$, we can attach a character $\chi_a = \eta_A(a) : \mathfrak{D}(A) \to \mathbb{G}_m = \mathrm{GL}_1(R)$. It defines then a $\mathfrak{G}$–structure on the R–module R.

To identify the relevant coaction, we use again Yoneda's technique by considering the homomorphism $\chi_{a,*} = \mathfrak{D}(A)(R[A]) \to \mathbb{G}_m(R[A]) = R[A]^\times$ and the image of the universal element which is e_a in view of Remark 3.4.2. It follows that the coaction is defined by $\widetilde{c}_a : R[A] \xrightarrow{\sim} R[A]$, $u \mapsto e_a\, u$ so that we have $c_a(r) = r \otimes e_a \in R \otimes_R R[A] = R[A]$.

If $M = \oplus_{a\in A} M_a$ is an A-graded R–module, the group scheme $\mathfrak{D}(A)$ acts diagonally on it by χ_a on each piece M_a.

We have constructed a covariant functor from the category of graded A–modules to the category of representations of $\mathfrak{D}(A)$.

6.1.1. **Proposition.** *The functor above is an equivalence of abelian categories from the category of A–graded R-modules to the category of $R-\mathfrak{D}(A)$-modules.*

Proof. Step 1: full faithfulness. Let $M_\bullet$ and $N_\bullet$ be A–graded modules. We have maps

$$\mathrm{Hom}_{A-gr}(M_\bullet, N_\bullet) \prod_{a\in A} \mathrm{Hom}_R(M_a, N_a) \to \mathrm{Hom}_{\mathfrak{D}(A)-mod}(M_\bullet, N_\bullet) \hookrightarrow \prod_{a,b\in A} \mathrm{Hom}_R(M_a, N_b).$$

It is then enough to show that $\mathrm{Hom}_R(M_a, N_b) = 0$ if $a \neq b$. For $a \neq b$, let $f : M_a \to N_b$ be a morphism of $\mathfrak{D}(A)$-modules. Then for $l \in M_a$, we have $c_{N_b}(f(m)) = f(c_{M_a}(m))$ so that $f(m) \otimes e_b = f(m \otimes e_a) = f(m) \otimes e_a \in N_b \otimes R[A]$. Since $R[A] = \bigoplus_{a\in A} Re_a$, we conclude that $f(m) = 0$. We conclude that $\mathrm{Hom}_R(M_a, N_b) = 0$ if $a \neq b$.

Step 2: Essential surjectivity. Let M be an $R - \mathfrak{D}(A)$-module and consider the underlying map $c : M \to M \otimes_R R[A]$. We write $c(m) = \sum_{a\in A} \varphi_a(m) \otimes e_a$. We apply the first rule (6.0.6), that is, the commutativity of

$$\begin{array}{ccc} M & \xrightarrow{c} & M \otimes_R R[A] \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle id\otimes\Delta} \\ M \otimes_R R[A] & \xrightarrow{c\otimes id} & M \otimes_R R[A] \otimes_R R[A]. \end{array} \tag{6.1.2}$$

We have then

$$(c \otimes id)(c(m)) = (c \otimes id)\Big(\sum_{a\in A} \varphi_a(m) \otimes e_a\Big) = \sum_{b\in A}\sum_{a\in A} \varphi_b(\varphi_a(m))\, e_b \otimes e_a.$$

On the other hand we have

$$(id \otimes \Delta)(c(m)) = (id \otimes \Delta)\Big(\sum_{a\in A} \varphi_a(m) \otimes e_a\Big) = \sum_{a\in A} \varphi_a(m)\, e_a \otimes e_a.$$

It follows that

$$\varphi_b \circ \varphi_a = \delta_{a,b}\, \varphi_a \qquad (a, b \in A)$$

We consider also the other compatibility (6.0.7)

$$\begin{array}{ccc} M & \xrightarrow{c} & M \otimes_R R[A] \\ {\scriptstyle id}\big\downarrow & \swarrow{\scriptstyle id\times\epsilon^*} & \\ M & & \end{array} \tag{6.1.3}$$

It implies that

$$m = id \times \epsilon^*\Big(\sum_{a\in A} \varphi_a(m)\, e_a\Big) = \sum_{a\in A} \varphi_a(m).$$

We obtain that
$$\sum_{a\in A} \varphi_a = id_M.$$
Hence the φ_a's are pairwise orthogonal projectors whose sum is the identity. Thus $M = \bigoplus_{a\in A} \varphi_a(M)$ which decomposes a direct summand of eigenspaces as desired. □

6.1.4. **Corollary.** *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of* $R - \mathfrak{D}(A)$*–modules.*

(1) For each $a \in A$*, it induces an exact sequence* $0 \to (M_1)_a \to (M_2)_a \to (M_3)_a \to 0$.

(2) The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ *splits as sequence of* $R - \mathfrak{D}(A)$*–modules if and only if it splits as sequence of* R*–modules.*

Proof. (1) It readily follows of the equivalence of categories stated in Proposition 6.1.1.

(2) The direct sense is obvious. Conversely, let $s : M_3 \to M_2$ be a splitting. Then for each $a \in A$, the composite $(M_3)_a \to M_3 \overset{s}{\to} M_2 \overset{\varphi_a}{\to} (M_2)_a$ provides the splitting of $(M_2)_a \to (M_3)_a$. □

We record also the following property.

6.1.5. **Corollary.** *Let* M *be an* $R - \mathfrak{G}$*-module. Then for each* S/R *and for each* $a \in A$*, we have* $M_a \otimes_R S = (M_a \otimes_R S)_a$.

6.1.6. **Corollary.** *Assume that* R *is a field. Then the category of representations of* $\mathfrak{D}(A)$ *is semisimple abelian category, that is, all short semisimple exact sequences split* [KS, 8.3.16].

Proof. Since the category of k–vector spaces is semisimple so is the category of A-graded vector spaces. Proposition 6.1.1 shows that the category of representations of $\mathfrak{D}(A)$ is semisimple. □

It is also of interest to know kernels of representations.

6.1.7. **Lemma.** *Let* $A^\sharp$ *be a finite subset of* A *and denote by* A_0 *the subgroup generated by* $A^\sharp$*. We consider the representation* $M = \oplus_{a\in A^\sharp} R^{n_a}$ *of* $\mathfrak{G} = \mathfrak{D}(A)$*, with* $n_a \geq 1$*. Then the representation* $\rho : \mathfrak{G} \to \mathrm{GL}(M)$ *factorizes as*
$$\mathfrak{G} = \mathfrak{D}(A) \to \mathfrak{D}(A_0) \overset{\rho_0}{\to} \mathrm{GL}(M)$$
where ρ_0 *is a closed immersion. Furthermore* $\ker(\rho) = \mathfrak{D}(A/A_0)$ *is a closed subgroup scheme.*

Proof. First case: $A = A_0$. Then the map $\mathfrak{G} \to \mathrm{GL}(M)$ factorizes by the closed subgroup scheme $T = \prod_{a\in A^\sharp} \mathbb{G}_{m,R}^{n_a}$. Since the map $\widehat{T} \to A_0 = A$ is onto, the map $\mathfrak{G} \to \prod_{a\in A^\sharp} \mathbb{G}_{m,R}^{n_a}$ is a closed immersion (Proposition 3.5.2).

A composite of closed immersion being a closed immersion, ρ is a closed immersion.

General case. The representation $\rho : \mathfrak{G} \to \mathrm{GL}(M)$ factorizes as

$$\mathfrak{G} = \mathfrak{D}(A) \overset{q}{\to} \mathfrak{D}(A_0) \overset{\rho_0}{\to} \mathrm{GL}(M)$$

where ρ_0 is a closed immersion. It follows that $\ker(\rho) = \ker(q)$. This kernel $\ker(q)$ is $\mathfrak{D}(A/A_0)$ and is a closed subgroup scheme of $\mathfrak{G}$ (*ibid*). □

6.1.8. **Remark.** If R is a field, all finite dimensional representations of $\mathfrak{D}(A)$ are of this shape, so one knows the kernel of each finite dimensional representation.

6.2. **Existence of faithful finite dimensional representations (field case).** Let k be a field and let $\mathfrak{G}$ be an affine k-group.

6.2.1. **Theorem.** *Let V be a $k - \mathfrak{G}$-representation. Then $V = \varinjlim_{i \in I} V_i$ where V_i runs over the f.d. subrepresentations of V.*

Proof. We write $c : V \to V \otimes_k k[\mathfrak{G}]$ for the coaction. A sum of f.d. subrepresentations of $\mathfrak{G}$ is again one, so it is enough to show that each $v \in V$ belongs in some finite-dimensional subrepresentation. Let $(a_i)_{i \in I}$ be a basis of the k–vector space A. We write $c(v) = \sum_{i \in I} v_i \otimes a_i$, where all but finitely many v_i's are zero. Next we have $\Delta(a_i) = \sum_{j,l \in I} r_{i,j,l}\, a_j \otimes_k a_l$. Using the first rule (6.0.6) of comodules we have

$$\sum_{i \in I} c(v_i) \otimes a_i = (c \otimes id)(c(v)) = (id \otimes \Delta)c(v) = \sum_{i,j,l} r_{i,j,l}\; v_i \otimes a_j \otimes a_l.$$

Comparing the coefficients, we get $c(v_l) = \sum_{i,j \in I} r_{i,j,l}\; v_i \otimes a_j$. Hence the subspace W spanned by v and the v_i's is a subrepresentation. □

6.2.2. **Theorem.** *Assume that $\mathfrak{G}$ is algebraic, that is, the k–algebra $k[\mathfrak{G}]$ admits a finite dimensional faithful k–representation V.*

Proof. We start with the regular representation V of G which is faithful in view of Example 6.0.9. We write $V = \varinjlim_{i \in I} V_i$ as in the previous theorem where the V_i's are finite dimensional. We put $\mathfrak{H}_i = \ker(\mathfrak{G} \to \mathrm{GL}(V_i))$, this is a closed k-subgroup of G. For each k–algebra S, we have

$$\bigcap_i \mathfrak{H}_i(S) = \ker\Big(\mathfrak{G}(S) \to \mathrm{GL}(V)(S)\Big) = 1.$$

We put $\mathfrak{H} = \bigcap_i \mathfrak{H}_i$, this is a closed k–subgroup of $\mathfrak{G}$ with trivial functor of points so that $H = 1$. We write $k[\mathfrak{H}_i] = k[\mathfrak{G}]/J_i$. Then $\ker(R[\mathfrak{G}] \to R) = \sum_{i \in I} J_i$. Since the ring $k[\mathfrak{G}]$ is a noetherian ring, its ideals are finitely generated so that there exists $i \in I$ $\ker(R[\mathfrak{G}] \to R) = J_i$. Thus $H_i = 1$ and V_i is a faithful representation of $\mathfrak{G}$. □

6.2.3. **Remark.** We will see later that a monomorphism of affine algebraic k-group is a closed immersion, see also [DG, §III.7.2] or [Mi2, thm. 3.34]. An easier thing ro do is to upgrade Theorem 6.2.2 by requiring that the homomorphism is a closed immersion, see [Wa, Thm. 3.4].

6.3. **Existence of faithful finite rank representations.** This question is rather delicate for general groups and general rings, see [SGA3, VI_B.13] and the paper [Th] by Thomason. Over a field or a Dedekind ring, faithful representations occur.

6.3.1. **Theorem.** *Assume that R or a Dedekind ring (e.g. DVR). Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a faithful $\mathfrak{G}$-module M which is f.g. free as R-module.*

The key thing is the following fact due to Serre [Se4, §1.5, prop. 2].

6.3.2. **Proposition.** *Assume that R is noetherian and let $\mathfrak{G}/R$ be an affine flat group scheme. Let M be a $\mathfrak{G}$-module. Let N be an R-submodule of M of finite type. Then there exists an $R-\mathfrak{G}$-submodule $\widetilde{N}$ of M which contains N and is f.g. as R-module.*

We can now proceed to the proof of Theorem 6.3.1. We take $M = R[\mathfrak{G}]$ seen as the regular representation $\mathfrak{G}$–module, it is faithful (Example 6.0.9). The proposition shows that M is the direct limit of the family of $\mathfrak{G}$-submodules $(M_i)_{i\in I}$ which are f.g. as R-modules. The M_i are torsion–free so are flats. Hence the M_i are projective.

We look at the kernel $\mathfrak{N}_i/R$ of the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$. The regular representation is faithful and its kernel is the intersection of the $\mathfrak{N}_i$. Since $\mathfrak{G}$ is a noetherian scheme, there is an index i such that $\mathfrak{N}_i = 1$. In other words, the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$ is faithful. Now M_i is a direct summand of a free module R^n, i.e. $R^n = M_i \oplus M_i'$. It provides a representation $\mathfrak{G} \to \mathrm{GL}(M_i) \to \mathrm{GL}(M_i \oplus M_i')$ which is faithful and such that the underlying module is free.

An alternative proof is §1.4.5 of [BT2] which shows that the provided representation $\mathfrak{G} \to \mathrm{GL}(M)$ is actually a closed immersion. This occurs as special case of the following result.

6.3.3. **Theorem.** *(Raynaud-Gabber* [SGA3, VI_B.13.2]*) Assume that R is a regular noetherian ring of dimension ≤ 2. Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a $\mathfrak{G}$-module M isomorphic to R^n as R–module such that $\rho_M : \mathfrak{G} \to \mathrm{GL}_n(R)$ is a closed immersion.*