LECTURES ON LOOP TORSORS IN ARITHMETIC GEOMETRY

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The four lectures are based on recent and running joint work with Chernousov and Pianzola. Given an algebraic group G and an algebraic variety X, loop G-torsors are those arising from change of groups from the algebraic fundamental groups of X. After a few generalities, we should discuss the case of split tori and the classification of loop torsors in this case. Next we shall see that the definitions extend well in the so-called Abhyankar setting, that is, over a localization of a henselian regular local ring with respect to a normal crossing divisor. In this case, there is a weak classi cation of loop torsors (involving Galois cohomology and Bruhat-Tits theory). It applies to the study of local-global principles for non-abelian Galois of semiglobal fields (joint work with Parimala).

- Lecture 1: Introduction with special regards to Topology;
- Lecture 2: Loop torsors in Algebraic Geometry;
- Lecture 3: Loop torsors in Arithmetic Geometry, for example for the ring $\mathbb{Z}_p[[t]]\left[\frac{1}{p}, \frac{1}{t}\right]$;
- Lecture 4: Application to semiglobal fields and open questions.

We warn the reader that the setting in the lectures is quite often not the most general one.

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Lecture 1: Loop torsors in Topology

The notion of *Torsors* (equivalenty principal homogeneous space) in Algebraic Geometry were introduced by Grothendieck and Serre in 1958 [70] by analogies with Topology, where the similar notion it that of *Principal G-bundle*. An intermediate step between Topology and Algebraic Geometry is the framework of *Analytic Geometry*, see for example the book of Kobayashi [51]. It is also of high interest but will not be discussed today. We start with the topological theory for which we recommend Kawakubo's book [48].

1. G-bundles

Let X be a topological space and let G be a topological group.

Definition 1.1. *A principal G-bundle (or principal homogenous space under G) is a continous map $f : E \to X$ between two topological spaces where E is equipped with a continuous right action of G satisfying:*

(i) $\pi(e\, g) = \pi(e)$ for each $e \in E$, each $g \in G$;

(ii) there exists an open cover $(U_i)_{i\in I}$ and a continous G-homeomorphism

$$U_i \times G \xrightarrow{\sim} f^{-1}(U_i)$$

where $U_i \times G$ is equipped with the right translation of G.

Remarks 1.2. (a) It follows that the natural map $E/G \to X$ is a homeomorphism.

(b) The space $X \times G$ equipped with the right translation is a principal G-bundle. It is called the trivial G-bundle.

(c) The second condition can be seen as a local triviality condition.

(d) There is a natural notion of morphism of principal G-bundles, those are homeomorphisms.

Example 1.3. (a) $G = \mathrm{GL}_n(\mathbb{R})$ for $n \geq 1$ [48, §2.2]. There is an equivalence of categories

$$\text{Principal } \mathrm{GL}_n(\mathbb{R})\text{-bundles over } X \quad <-- > \quad \mathrm{Vect}^n(X),$$

where $\mathrm{Vect}^n(X)$ stands for the groupoid of real vector bundles over X. It means that we consider only isomorphisms of vector bundles. The correspondance goes as follows. Given a principal G-bundle E, we associate the contracted product[1]

$$V(E) = E \times^G \mathbb{R}^n = (E \times \mathbb{R}^n)/G$$

where the action of G is defined by $(e, v).g = (e.g, g^{-1}.v)$ (we use the standard action of G on $\mathbb{R}^n$). In the way around, given a real vector bundle $V \to X$ of relative dimension n, we attach to it the *variety of frames* $E(V)$. It is defined by using a trivializing cover $(U_i)_{i\in I}$ of V and by glueing the $E_i = \mathrm{Isom}(U_i \times \mathbb{R}^n, V_{U_i})$'s; here

[1] There are other ways to denote it in the literature for example $E \times_G \mathbb{R}^n$.

$U_i \times \mathbb{R}^n$ stands for the trivial vector bundle over U_i and V_{U_i} for the pull-back of V on U_i.

(b) The case $G = O_n(\mathbb{R})$ of the orthogonal group for $n \geq 1$. In this case we have an equivalence of categories between the category of principal $O_n(\mathbb{R})$-bundles over X and the groupoid $\mathrm{Quad}^n(X)$ of quadratic vector bundles, whose objects are pairs (V, q) where V is a vector bundle of rank n over X and a family of positive definite quadratic form $q_x : V_x \to \mathbb{R}$ which varies continuously in x.

Example 1.4. If X is compact (and more generally paracompact), there is an equivalence of categories between the groupoid $\mathrm{Vect}^n(X)$ and the groupoid of projective $C^0(X, \mathbb{R})$-modules of rank n. This is the Swan-Serre's correspondence [75, Theorems 1.2].

The functoriality is essential in this theory.

1.1. **Pull-back.** We are given a principal G-bundle $f : E \to X$ and a continous map $h : Y \to X$ where Y is a topological space. We define the pull-back

$$u_*(E) = E \times_X Y = \big\{(e, x) \ \mid \ f(e) = h(x)\big\}$$

The right action of G on E induces a right action of G on $u_*(E)$ and the second projection induces a continuous map $p_2 : u_*(E) \to Y$ which is G-equivariant (for Y equipped withe the trivial action). We need to check that $p_2 : u_*(E) \to Y$ is a principal G-bundle. The first rule is satisfies and to establish the second one, we consider an open cover $(U_i)_{i \in I}$ of X which trivializes f, that is, we have G-equivariant homeomorphisms $\phi_i : U_i \times G \xrightarrow{\sim} E \times_X U_i$. Then $(V_i = h^{-1}(U_i))_{i \in I}$ is an open cover of Y and each ϕ_i induces a G-equivariant homeomorphism $\psi_i : V_i \times G \xrightarrow{\sim} u^*(E) \times_Y V_i = (E \times_X U_i) \times_X Y$.

1.2. **Push-forward.** We are given a continous homomorphism $v : G \to G'$ of topological groups. The push-forward $v_*(E)$ is a principal G'-bundle over X. We put

$$v_*(E) = E \times^G G' = (E \times G')/G,$$

that is the contracted product of E and G'. Precisely it is the quotient space of $E \times G'$ by the right action of G defined by $(e, g)\, g' = (e\, g, g^{-1}\, g')$. The map $f : E \to X$ induces a map $p : v_*(E) \to X$ and the right action of G' on $E \times G'$ gives rise to a right action on $v_*(E)$ which is G'-equivariant to the map p (with X equipped with the trivial G-action.

It remains to check that p is indeed a principal G'-space and the proof goes one again by localizing. It works since the preceding constructions commute to localization so that we are reduced to the case of the trivial principal G-bundle $E_0 = G \times X$. Indeed we have $E_0 \times^G G' = (G \times X) \times^G G' \xrightarrow{\sim} X \times G'$.

For more detailed proofs, see [48, §2.3, 2.4]. An important thing is that these two operations commute and extend to isomorphism classes.

We denote by $H^1(X,G)$ the set of isomorphism classes of principal G-bundles over X. It is pointed by the class of $[X \times G]$. The above discussion leads to the following commutative diagram of pointed sets

$$\begin{array}{ccc} H^1(X,G) & \xrightarrow{u_*} & H^1(X,G') \\ \downarrow{\scriptstyle v_*} & & \downarrow{\scriptstyle v_*} \\ H^1(Y,G) & \xrightarrow{u_*} & H^1(Y,G'). \end{array}$$

1.3. **Universal cover.** For dealing with the theory of universal cover, we assume from now on that X is connected and choose a base point x_0. We assume further that X is locally path connected so that the theory of the universal cover is available. There exists a universal cover (X^{sc}, x_0^{sc}) of (X, x_0) where $p : X^{sc} \to X$ is a cover and X^{sc} is connected and simply connected. The point $x_0^{sc} \in X^{sc}$ lies in $p^{-1}(x_0)$ and the fundamental group of $(X,_0)$ is

$$\pi_1(X, x_0) = \mathrm{Aut}_X(X^{sc})^{op}.$$

It acts on the right on X^{sc} and the point is that p is a principal $\pi_1(X, x_0)$-bundle where $\pi_1(X, x_0)$ is equipped with the discrete topology. Now we can apply the push-forward construction to a homomorphism $f : \pi_1(X, x_0) \to G$ (which is automatically continuous) where G is a topological group. This give rise to the principal G-bundle $E_f = f_*(X^{sc})$ of X.

Definition 1.5. *A* <u>*loop*</u> *G-bundle is a principal G-bundle which is isomorphic to some E_f for a homomorphism $f : \pi_1(X, x_0) \to G$.*

Our terminoloy is not standard, it is related with the fact that $\pi_1(X, x_0)$ parameterizes loops on X. The classical terminology is "induced from the universal cover of X". In term of classes, it gives rise to the map

$$\mathrm{Hom}(\pi_1(X, x_0), G)/G \;\to\; H^1(X,G). \tag{1.1}$$

Example 1.6. (Moebius strip) We consider the double cover $q : Y = S^1 \to S^1 = X$, $z \mapsto z^2$. Then

$$Y \times^{\mathbb{Z}/2\mathbb{Z}} \mathbb{R}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}$ by the sign, is a line real bundle over $X = S^1$. It is nothing but the Moebius strip [48, Example 2 page 65]. The "cutting process" can be understood as pulling back by q. The associated principal $\mathbb{R}^\times$-bundle is loop.

The fact that the pull-back of a loop G-bundle to X^{sc} is trivial is in general fact. To explain it, we consider the following commutative diagram

$$\begin{array}{ccc} & & X^{sc} \\ & \overset{id}{\nearrow} & \downarrow{\scriptstyle p} \\ h : Y = X^{sc} & \xrightarrow{p} & X. \end{array}$$

It turns out that the $\pi_1(X,x_0)$-bundle $h_*(X^{sc}) \to Y$ is trivial. In other words the $\pi_1(X,x_0)$-bundle $X^{sc} \to X$ is trivialized after pull-back by $X^{sc} \to X$. Since pull-backs and push-forward commute, we deduce the following important fact.

Corollary 1.7. *Each loop G–bundle over X is trivialized after pull-back to its simply connected cover X^{sc}.*

Definition 1.8. *We say that a principal G-bundle over X is virtually trivial if its pull-back over X^{sc} is trivial.*

We have then inclusions

$$\{\text{loop } G\text{-bundles }\} \subseteq \{\text{virtually trivial } G\text{-bundles }\} \subseteq \{\text{principal } G\text{-bundles }\}.$$

The case of the principal $\mathrm{GL}_2(\mathbb{R})$-bundle attached to the tangent bundle of the sphere S^2 shows that the right inclusion is strict in general. Indeed S^2 is simply connected to that virtually trivial G-bundles are trivial and the tangent bundle of S^2 is non trivial (hairy ball theorem). A natural question is whether the two first sets coincide or in other words whether a virtually trivial G-bundle is loop.

This question has been investigated by Milnor [55]. It appeals to homotopy theory and an excellent reference consists of Neeb's lecture notes [59, 60]. Milnor provided an interesting example, it is of dimension 2 (in dimension 1, the two families coincide). We take X to be the compact oriented surface of genus $g \geq 2$. We know that X^{sc} is the Poincaré half plane. It is a contractible (paracompact) space so that $H^1(X^{sc}, G) = 1$ for each topological group G [46, Corollary 10.3]; all principal G-bundles are then virtually trivial. We recall that

$$\pi_1(X, x_0) = \big\langle a_1, \dots, a_g, b_1, \dots, b_g, \; [a_1, b_1] \dots [a_g, b_g] = 1 \big\rangle.$$

In other words, $\pi_1(X,x_0)$ is the quotient of the free group in $2g$ generators given by a single relation. Let G be a connected Lie group, it is pointed by 1. Its simply connected cover G^{sc} is also a Lie group and we have a central exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$$

with the abuse of notation $\pi_1(G) = \pi_1(G,1)$. We can use the boundary map defined by Grothendieck [40, §5.4]

$$\delta : H^1(X,G) \to H^2(X, \pi_1(G))$$

where the right handside is the singular cohomology [4, §V.6]. Since X is of dimension 2, the map δ is bijective [59, Remark IV.11 and Proposition IV.14]. By the universal coefficient theorem (*ibid*, Corollary 7.2), we have a (non canonical) decomposition

$$H^2(X, \pi_1(G)) = \mathrm{Ext}_{ab}(H_1(X,\mathbb{Z}), \pi_1(G)) \oplus \mathrm{Hom}(H_2(X,\mathbb{Z}), \pi_1(G)).$$

In our case, we have $H_1(X,\mathbb{Z}) = \mathbb{Z}^{2g}$ and $H_2(X,\mathbb{Z}) = \mathbb{Z}$ [3, §II.4, Example 2] so that

$$H^2(X, \pi_1(G)) = \mathrm{Ext}_{ab}(\mathbb{Z}, \pi_1(G))^{2g} \oplus \pi_1(G).$$

We take $G = \mathrm{GL}_2(\mathbb{R})^+$ (i.e. the group of matrices of positive determinant) which is homeomorphic to

$$S^1 \times \{A \in M_2(\mathbb{R}) \mid A = {}^tA,\, A \text{ positive definite}\}$$

by means of the polar decomposition. It follows that $\pi_1(G) = \pi_1(S^1) = \mathbb{Z}$ so that $H^2(X, \pi_1(G)) = \mathbb{Z}$. Summarizing we have a bijection

$$H^1(X, G) \xrightarrow{\sim} \mathbb{Z}.$$

Milnor shows that the classes of loop G-torsors corresponds to $\{n \in \mathbb{Z},\ |n| \geq g\}$ [55, Theorems 1,2] so that there are non loop torsors which are virtually trivial.

Remarks 1.9. (a) We can deal also with $\mathrm{SL}_2(\mathbb{R})$, see [59, Example I.13].

(b) Since X is C^∞, loop G-bundles are characterized by existence of a flat connection, see [55, Lemma 1].

2. Loop torsors in Galois Cohomology

There are strong analogies between the theory of covers in Topology and Galois theory in algebras, see [76].

Let k be a field and consider $k \subset k_s \subset \overline{k}$, respectively a separable closure of k and an algebraic closure of k. The absolute Galois group of k is $\Gamma_k = \mathrm{Aut}_k(k_s)$, it is a profinite group and plays the role of the fundamental group in Topology.

Definition 2.1. *Let G be an (affine) algebraic group. A G-torsor (or a principal homogeneous G-space) is a k-scheme X equipped with a right k-action of G such that there exists a trivialization*

$$G \times_k \overline{k} \xrightarrow{\sim} X \times_k \overline{k}$$

which is G-equivariant for the action by right translation of the left handside.

As in the topological case, G equipped with the right action of G is called the trivial G-torsor. Also if X is a G-torsor, we have that $X(k) \neq \emptyset$ iff X is isomorphic to the trivial G-torsor. We denote by $H^1(k, G)$ the pointed set of isomorphism classes of G-torsors over k.

Examples 2.2. (a) $G = \mathrm{GL}_n$. Then $H^1(k, \mathrm{GL}_n) = 1$, this is Hilbert 90 theorem.

(b) $G = O_n$, that is, the orthogonal group of the quadratic form $\sum_{i=1}^{n} x_i^2$ ($\mathrm{char}(k) \neq 2$). In this case $H^1(k, O_n)$ classifies the isometry classes of regular quadratic forms of dimension n.

(c) $G = \mu_n$ the k–subgroup of $\mathbb{G}_m$ of n-roots of unity. We have $H^1(k, \mu_n) = k^\times/(k^\times)^n$. To each $c \in k^\times$, we associate the μ_n-torsor $\{x^n = c\}$.

The notion of *virtually trivial torsor* translates by saying that a G-torsor has a k_s-point. If G is smooth, then X is smooth and satisfies then $X(k_s) \neq \emptyset$.

The G-torsors having a k_s-point are classified by the pointed set

$$\ker\big(H^1(k,G) \to H^1(k_s,G)\big)$$

which injects in the Galois cohomology set $H^1(\Gamma_k, G(k_s))$ made of Galois cocycles. The fact that we can attach to a Galois cocycle a G-torsor is an avatar of Galois descent and it follows that $\ker\big(H^1(k,G) \to H^1(k_s,G)\big) \xrightarrow{\sim} H^1(\Gamma_k, G(k_s))$.

We have seen already that $H^1(k_s,G) = 1$ if G is smooth, another case for this vanishing fact is when k is perfect.

The first approach for defining loop torsors in this framework is to consider a continuous group homomorphism $\phi : \Gamma_k \to G(k)$. It permits to define a Galois 1-cocycle $z_\sigma = \phi(\sigma)$ ($\sigma \in \Gamma_k$) with values in $G(k_s)$. This gives rise to a map

$$\mathrm{Hom}_{ct}(\Gamma_k, G(k))/G(k) \;\to\; H^1(k,G) \tag{2.1}$$

which is similar to (1.1). The continuity is important here, it means that $\Gamma_k \to G(k)$ factorizes through $\mathrm{Gal}(K/k)$ for a finite Galois extension K of k. Classes arising from the maps (2.1) are called *strongly loop* in this lecture.

This approach fits well for the orthogonal group O_n which comes equipped with the diagonal k-subgroup μ_2^n. The induced map

$$\big(k^\times/(k^\times)^2\big)^n = H^1(k,\mu_2)^n \to H^1(k,O_n)$$

applies an n-uple $(a_1,\dots,a_n)$ to the isometry class of diagonal quadratic form $\sum\limits_{i=1}^{n} a_i\, x_i^2$. Gauss diagonalization implies then that O_n-torsors are strongly loop.

It is quite restrictive, we can see it on the following example. In PGL_n ($n \in k^\times$), we have the k-subgroup $\mu_n \times \mathbb{Z}/n\mathbb{Z}$ where μ_n sits as

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \omega^{n-1} \end{bmatrix}$$

and $\mathbb{Z}/n\mathbb{Z}$ is the subgroup generated by the element

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & & 0 \\ 0 & 1 & 0 & \dots & & 0 \\ & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

Note that the two matrices above commute in $\mathrm{PGL}_n(k)$. The induced map $H^1(k,\mu_n) \times H^1(k,\mathbb{Z}/n\mathbb{Z}) \to H^1(k,\mathrm{PGL}_n)$ applies a pair (a,χ) where $(a) \in H^1(k,\mu_n)$ and a character $\chi : \Gamma_k \to \mathbb{Z}/n\mathbb{Z}$ to the class of the cyclic central simple k-algebra (a, χ) [37, §2.5]. This construction will be strongly loop if k contains the n-roots of unity. We set the definition as follows.

Definition 2.3. *A G-torsor E over k is finite (resp. finite étale) if it admits a reduction to a finite (resp. finite étale) k-subgroup F of G.*

Based on ideas by Bogomolov in Geometric Invariant Theory, the following statement shows that in the reductive case, all G-torsors are finite.

Theorem 2.4. [14, Theorem 1.1] *(and* [15, Theorem 1.2]*) Let G be a reductive k-subgroup of G. Then there exists a finite k-subgroup S of G such that*

$$H^1(F, S) \to H^1(F, G)$$

is onto for each field extension F/k. In particular, if k is of characteristic zero, all G-torsors are finite étale.

Remarks 2.5. (a) The k-subgroup S is a finite subgroup of the normalizer $N_G(T)$ of a maximal k-torus of G. It is an extension of the Weyl group $N_G(T)/T$ by ${}_NS$ for some integer N.

(b) If N is prime to p, then S is étale so that in this case all G-torsors are finite étale.

(c) This S is not optimal. For example, for the split group of type G_2, we can take the subgroup $\mu_2 \times \mu_2 \times \mathbb{Z}/2\mathbb{Z}$.

For PGL_n, we can take $(\mu_n)^n/\mu_n \rtimes S_n$.

We did not define the notion of loop torsors here since we want to be coherent with the next lecture. We warn the reader that the forthcoming notion of loop torsors on varieties is trivial when restricted over $\mathrm{Spec}(k)$.

Lecture 2: Loop torsors in Algebraic Geometry

3. The étale viewpoint

Let k be a field. The generalization of the theory of principal homogeneous spaces in Algebraic Geometry is due to Grothendieck and Serre in 1958 [70]. They used the terminology *espaces fibrés algébriques* but we use now *torsor*. It went together with the notion of *étale* morphisms, that is, of smooth morphisms of algebraic varieties of relative dimension zero. An étale cover of an algebraic variety[2] X over k is a finite collection of étale maps $f_1 : U_1 \to X, \dots, f_n : U_n \to X$ such that $f_1(U_1) \cup \cdots \cup f_n(U_n) = X$.

Zariski covers are obviously étale covers and the simplest example of an étale cover which is not a Zariski cover is the Kummer morphism $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$, $t \to t^n$ (for n invertible in k).

3.1. **Torsors.** Let X be a k-variety and let G be an affine algebraic k–group.

Definition 3.1. *A G-torsor Y over X is an algebraic k-variety Y equipped with a right action of G together with a map $f : Y \to X$ which is G-invariant and such that X admits an étale cover $(U_i)_{i\in I}$ and trivializations (which are G-equivariant)*

$$\phi_i : U_i \times_X G \xrightarrow{\sim} Y \times_X U_i$$

where the action of G on $U_i \times_X G$ is by right translation.

Examples 3.2. (a) In the case of GL_n, a GL_n–torsor is the same thing than an algebraic vector bundle of rank n, this is the algebraic version of Swan-Serre's correspondence. In that case, Zariski topology is enough because projective modules of finite rank are locally free.

(b) In the case of the orthogonal group O_d of the quadratic form $x_1^2 + \cdots + x_d^2$, a O_d-torsor is the same thing than an quadratric vector bundle of rank d.

(c) We are back with the rank one torus $\mathbb{G}_{m,k} = \mathrm{GL}_1 = \mathrm{Spec}(k[t^{\pm 1}])$. Assuming here that $2 \in k^\times$, the Kummer cover $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$, $x \mapsto x^2$ is a $\mathbb{Z}/2\mathbb{Z}$–torsor. It is not locally split for the Zariski topology.

We denote by $H^1(X, G)$ the set of isomorphism classes of G–torsors over X. It is pointed by the class of $[G]$.

Remark 3.3. There is also a notion of sheaf G-torsors over X for the étale topology (and for other Grothendieck topologies), see [54, §III.4]. A G-torsor gives E rise to a sheaf G-torsor $\underline{E}$ and $\underline{E}$ determines E. The problem to decide whether a sheaf G-torsor $\mathcal{F}$ arises from a G-torsor; in that is called $\mathcal{F}$ is called representable. Since G is affine, a consequence of the faithfully flat descent theorem is that G-torsors are

[2] In the lectures, a variety is a separated k-scheme of finite type.

representable [54, III, Theorem 4.3]. We can then use indifferently use both settings; in particular we can refer to the literature on sheaf torsors for torsors.

(c) If Γ is a finite group, it is also an algebraic k–group. In that case Γ-torsors $Y \to X$ with X, Y connected are nothing but Galois covers in the Grothendieck theory of fundamental group in algebraic geometry.

We deal now with the functoriality. As in the topological case we have pull-back functors with base change on X and a push-forward functors attached to algebraic group homomorphisms $G \to G'$. It involves quotient by free actions of G and this technical point is actually discussed in detail in [56, Proposition 7.1] and is based on descent theory. Summarizing the formation of $H^1(X, G)$ is functorial in X and in G and these two operations commute.

Let Γ be a finite group and consider a Γ-torsor $p : Y \to X$ and a homomorphism $u : \Gamma \to G$, we can associate the G-torsor $u_*(Y) = Y \wedge^\Gamma G$ by taking the quotient in the algebraic sense (that is using Galois descent theory in this case).

This is an important example of loop torsor (as we will define later). As in the topological case, the base change $u_*(Y) \times_X Y$ is a trivial G–torsor.

We say that a G-torsor E over X is *isotrivial* if there exists a finite étale cover $X' \to X$ such that $E \times_X X'$ is a trivial G-torsor. We will see that loop torsors are then isotrivial but what about the converse?

As for Galois cohomology, the set of isotrivial torsors is described by the exact sequence [28, §2.9]

$$1 \to H^1(\Gamma, G(Y)) \to H^1(X, G) \to H^1(Y, G) \tag{3.1}$$

where $H^1(\Gamma, G(Y))$ is the non-abelian group cohomology set with respect to the natural action of Γ on $G(Y)$. It fits well with the torsor u_*Y above. Its class comes from the natural map

$$\mathrm{Hom}_{gp}(\Gamma, G(k))/G(k) \to H^1(\Gamma, G(Y)) \to H^1(X, G). \tag{3.2}$$

The point is that the homomorphism $\sigma \to u(\sigma) \in G(k) \subseteq G(Y)$ is a 1-cocycle since $G(k) \subseteq G(Y)^\Gamma$.

The G-torsors arising by this construction are called strongly loop.

3.2. **Loop torsors.** Let us be now more precise by giving the definition of loop torsors. Let X be a k-variety (separated k-scheme of finite type) assumed geometrically integral. Let K be a k_s-field which is separably closed and let $x \in X(K)$. Such a point is called a *quasi geometrical* point. Grothendieck's theory of fundamental groups [68] associates to (X, x) a projective limit of connected pointed schemes

$$(X^{sc}, x^{sc}) = \varprojlim_{i \in I} (X_i, x_i)$$

where X_i runs over finite étale covers, $x_i \in X_i(K)$, and $x^{sc} \in X^{sc}(K)$.

We can refine this limit by limiting ourselves to Galois objects, that is, when $X_i \to X$ is a Γ_i-torsor for a finite group Γ_i. The algebraic fundamental group of (X,x) is defined by using fibre functors; concretely [76, Corollary 5.4.8], it is the profinite group

$$\pi_1(X,x) = \varprojlim_{i \in I} \mathrm{Aut}_X(X_i)^{op}.$$

Examples 3.4. (a) The universal cover of $\mathbb{G}_{m,\mathbb{C}}$ is the projective limit of Kummer covers $f_n : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$, $t \to t^n$ [35, Lemma 2.8]. In terms of rings it is the inductive limit of the $\mathbb{C}[t^{\frac{1}{n}}]$. The algebraic fundamental group of $(\mathbb{G}_{m,\mathbb{C}}, 1)$ is isomorphic to $\widehat{\mathbb{Z}}$.
(b) Let A be a complex abelian variety (e.g. an elliptic curve). Then its universal cover is the projective limit of the multiplication by n isogenies $A \xrightarrow{\times n} A$. The algebraic fundamental group of $(A,0)$ is isomorphic to $(\widehat{\mathbb{Z}})^{2\dim(A)}$.

Since $G(X^{sc}) = \varinjlim_{i\in I} G(X_i)$, we can pass to the limit the previous exact sequence and obtain

$$1 \to H^1(\pi_1(X,x), G(X^{sc})) \to H^1(X,G) \to H^1(X^{sc},G). \tag{3.3}$$

Similarly we have a map

$$\mathrm{Hom}_{ct}(\pi_1(X,x), G(k))/G(k) \to H^1(\pi_1(X,x), G(X^{sc})) \to H^1(X,G). \tag{3.4}$$

We recall the fundamental homotopy exact sequence of profinite groups [76, Proposition 5.6.1]

$$1 \to \pi(X_{k_s}, x) \to \pi_1(X,x) \to \mathrm{Gal}(k_s/k) \to 1. \tag{3.5}$$

This projection map permits to see the subgroup $G(k_s) \subseteq G(X^{sc})$ as a $\pi_1(X,x)$-subgroup. We have then a map

$$H^1(\mathrm{Gal}(k_s/k), G(k_s)) \to H^1(\pi_1(X,x), G(X^{sc})) \to H^1(X,G). \tag{3.6}$$

Definition 3.5. *(1) The image of the map $H^1(\mathrm{Gal}(k_s/k), G(k_s)) \to H^1(X,G)$ above is denoted by $H^1(X,G)_{loop}$ and are called loop classes.*
(2) We say that a G-torsor E is loop if $[E] \in H^1(X,G)_{loop}$.

That definition says that all constant classes (i.e. arising from $H^1(\mathrm{Gal}(k_s/k), G(k_s)))$ are loop.

Examples 3.6. (a) If G is finite étale, all G–torsors are loop. Firstly $G_{X^{sc}}$ is constant finite so that $H^1(X^{sc}, G) = 1$ since it classifies Galois covers of X^{sc} which is then simply connected (see [77, Definition 6 and Proposition 3.4] for explanations). Since X^{sc} is connected, we have $G(k_s) = G(X^{sc})$ so that $H^1(\Pi_1(X,x), G(k_s)) = H^1(\Pi_1(X,x), G(X^{sc}) = H^1(X,G)$ by using the sequence (3.3).
(b) Assume that $G = \mathbb{G}_{m,k}$ with k of characteristic zero. In this case $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$ and we claim that $H^1(X, \mathbb{G}_m)_{loop} = \mathrm{Pic}(X)_{torsion}$. It is left as an exercise.

3.3. **Case of a proper variety.** We assume here furthermore that X is proper. In the case of GL_n, i.e. for vector bundles, it has been investigated by Nori [63].

Each X_i is proper and we put $k_i = H^0(X_i, \mathcal{O}_{X_i})$, it is a finite Galois extension of k. Since G is affine, we have $G(k_i) = G(X_i)$ for each $i \in I$ so that $G(k_s) = G(X^{sc})$. In this case

$$H^1(\pi_1(X,x), G(k_s)) = H^1(X,G). \tag{3.7}$$

The loop torsors are then exactly the isotrivial torsors making then a difference with the topological case. If k is separably closed, it simplifies as follows

$$\mathrm{Hom}_{ct}(\pi_1(X,x), G(k))/G(k) \; = \; H^1(\pi_1(X,x), G(k)) = H^1(X,G). \tag{3.8}$$

In this case the notion of loop torsors coincide then with the first approach of that notion.

In the case of an abelian variety A of dimension g, it is then related with finite abelian subgroups of $G(k)$ of rank $\leq g$. The case of GL_n (resp. PGL_n) is of special interest since it relates to Mukai theory [57] (resp. Mumford's theory of Θ-groups [58, §IV.3]). More generally those torsors occur in Brion's theory of homogeneous G-torsors [5, 6].

3.4. **Case $X(k) \neq \emptyset$.** We pick $x_0 \in X(k)$ and assume that $x = x_0 \in X(k_s)$. This point defines a continuous section of the homotopy sequence (3.5) so that we have a semi-direct product of profinite groups

$$\Pi_1(X,x) = \Pi_1(X_{k_s}, x) \rtimes \mathrm{Gal}(k_s/k).$$

A class of $H^1(\Pi_1(X,x), G(k_s))$ is represented by a 1-cocycle $\phi : \Pi_1(X,x) \to G(k_s)$ called a *loop cocycle*. Its restriction to $\mathrm{Gal}(k_s/k)$ is called the arithmetic part and is denoted by ϕ^{ar} (or simply z).

Its restriction $\phi^{geo} : \Pi_1(X,x) \to G(k_s)$ is a group homomorphism but does not land in $G(k)$. However we claim that is takes values in the subgroup ${}_zG(k)$ where ${}_zG$ stands for the twisted k-group by z. Recall that ${}_zG(k_s)$ is the group $G(k_s)$ equipped with the following Galois action

$$\sigma \,.\, g = z_\sigma \, \sigma(g) \, z_\sigma^{-1} \qquad (\sigma \in \mathrm{Gal}(k_s/k))$$

and that

$${}_zG(k) = \big\{ g \in G(k_s) \mid g = z_\sigma \, \sigma(g) z_\sigma^{-1} \quad \forall \sigma \in \mathrm{Gal}(k_s/k) \big\}.$$

Indeed for $\tau \in \Pi_1(X,x)$ and $\sigma \in \mathrm{Gal}(k_s/k)$, we have

$$\begin{aligned}
\phi^{geo}_{\sigma\tau\sigma^{-1}} &= \phi_{\sigma\tau\sigma^{-1}} = \phi_\sigma \, {}^\sigma(\phi_{\tau\sigma^{-1}}) && [\sigma \text{ is a cocycle}] \\
&= z_\sigma \, {}^\sigma(\eta_{\tau\sigma^{-1}}) \\
&= z_\sigma \, {}^\sigma(\phi^{geo}(\tau) \, z_{\sigma^{-1}}) && [\eta \text{ is a cocycle and } \tau \text{ acts trivially on } \mathbb{G}(k_s)] \\
&= z_\sigma \, {}^\sigma(\eta^{\mathrm{geo}}(\tau)) \, z_\sigma^{-1} && [1 = z_\sigma \, \sigma(z_\sigma^{-1})].
\end{aligned}$$

What we obtained is a dictionnary based on the fact that a loop cocycle is the same thing that a section of $G(k_s) \rtimes \Pi_1(X,x) \to \Pi_1(X,x)$.

Lemma 3.7. [30, Lemme 3.2]

(1) The assignation $\phi \to (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z^1\big(\pi_1(X,x), G(k_s)\big)$ and the couples (z,ϕ) where $z \in Z^1\big(\mathrm{Gal}(k_s/k), G(k_s)\big)$ and $\phi : \pi_1(X,x) \to {}_zG$ is a k–group homomorphism.

(2) We have an exact sequence of pointed sets

(3.9)

$$1 \to \mathrm{Hom}_{k-gr}(\pi_1(X_{k_s},x),G)/G(k) \to H^1\big(\pi_1(X,x),G(k_s)\big) \xrightarrow{Res_x} H^1\big(\mathrm{Gal}(k_s/k),G(k_s)\big) \to 1.$$

Furthermore the first map is injective.

(3) We have a decomposition

$$H^1\big(\pi_1(X,x),G(k_s)\big) = \coprod_{[z]\in H^1(k,G)} \mathrm{Hom}_{k-gr}(\pi_1(X_{k_s},x),{}_zG)/{}_zG(k).$$

4. The case of Laurent polynomials

We assume there that k is of characteristic zero and will comment later about the positive characteristic case investigated in [13]. We present first a bunch of results and then enter into the technicalities.

4.1. Covers and isotriviality.

We can refine further the Galois tower of the Laurent polynomial ring $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ as follows.

A basic tame cover of R_n is $R_{n,m} \otimes_k l = l[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}]$ where $m \geq 1$ and l is a finite Galois field extension of k containing a primitive m-root of unity. It is a Galois cover of group $\mu_m(l)^n \rtimes \mathrm{Gal}(l/k)$ and we know that R_n^{sc} is the inductive limit of the $R_{n,m} \otimes_k l$'s [35, Lemma 2.8]. The motivation for investigating this case is related with the theory of Kac-Moody algebras and of Extended Affine Lie Algebras.

A first result is that R_n-torsors under a reductive group G are isotrivial [35, Corollary 2.16]. The previous question is then to see whether loop torsors exhaust the torsors.

The loop objects are quite usual, for example consider the quadratic form case since the set $H^1(R_n, \mathbf{O}_d)$ classifies regular quadratic forms of rank d over R_n.

Proposition 4.1.1. [12, Proposition 5.7] *For each subset $I \subseteq \{1,...,d\}$, we put $t_I = \prod_{i\in I} t_i \in R_n^\times$ with the convention $t_\emptyset = 1$. Each class in $H^1(R_n, \mathbf{O}_n)_{loop}$ is represented by an R_n-quadratic form of the shape*

$$\bigoplus_{I\subseteq\{1,...,d\}} \langle t_I \rangle \otimes q_{I,R_n}$$

where all q_I's with $I \neq \emptyset$ are nonsingular quadratic k-forms such that

$$d = \bigoplus_{I \subseteq \{1,\dots,d\}} \dim(q_I).$$

Since $R_n^\times$ is generated by the t_I and $k^\times$, it rephrases by saying that the loop quadratic forms are the diagonalizable ones.

4.2. **The case $n = 1$.**

Theorem 4.1. [10] *Let G be a reductive k-group. Then all G-torsors are loop and the map*

$$H^1(R_1, G) \to H^1(k((t)), G)$$

is bijective.

In certain cases $H^1(k((t)), G)$ can be computed by Bruhat-Tits decomposition which is somehow a generalization of Springer's decomposition for $k((t))$-quadratic forms [9]. In the case of PGL_d-torsors, there is a more complicated statement for Azumaya algebras is [36][Theorem 13.4].

The proof is based on Bruhat-Tits twin buildings and can be seen as generalization of the case of the affine line. It is the opportunity to quote Raghunathan-Ramanathan's theorem stating that $H^1(k, G) = H^1(k[t], G)$ [65]; by the way, this result is presented in the notes [31].

4.3. **The case $n = 2$.** The simplest example of (isotrivial) non loop torsor is for $G = \mathrm{PGL}_2$ and $n = 2$; the example is explicit. In this case $H^1(R_2, \mathrm{PGL}_2)$ classifies quaternion Azumaya R_2-algebras. We first list $H^1(R_2, \mathrm{PGL}_2)_{loop}$ which arises from

$$\mathrm{Hom}_{\mathbb{C}-gp}(\widehat{\mathbb{Z}}^2, \mathrm{PGL}_2(\mathbb{C})) / \mathrm{PGL}_2(\mathbb{C}).$$

In other words a loop cocycle is given by two commuting elements g_1, g_2 of finite order of $\mathrm{PGL}_2(\mathbb{C})$. If g_1, g_2 belong to a common subtorus $T \cong \mathbb{G}_{m,\mathbb{C}}$ of PGL_2, then the associated loop torsor is trivial since its class belongs in the image of the map $1 = H^1(R_2, T) \to H^1(R_2, \mathrm{PGL}_2)$. It is well-known that $\mathrm{PGL}_2(\mathbb{C})$ carries a unique (up to conjugacy) finite abelian subgroup which is non toral. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{PGL}_2(k)$ applying $(1,0)$ on $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $(0,1)$ on $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It follows that $H^1(R_2, \mathrm{PGL}_2)$ has at least a non-trivial element. The loop object $\mathcal{Q}$ associated to the above homomorphism is defined by the presentation

$$T_1^2 = t_1,\ T_2^2 = t_2,\ T_1T_2 + T_2T_1 = 0.$$

It is an R_2-quaternion algebra and it not split since it not split over $\mathbb{C}((t_1))((t_2))$. Thus $H^1(R_2, \mathrm{PGL}_2)$ has two elements, the trivial class and $[\mathcal{Q}]$.

The idea for constructing an exotic object is due to Ojanguren-Sridharan. We take $\mathcal{Q}' = \mathrm{End}_{\mathcal{Q}}(\mathcal{L})$ for a non-free invertible right $\mathcal{Q}$-module $\mathcal{L}$. Since $\mathrm{Pic}(R_2) = 1$, we have that $\mathcal{Q} \cong \mathcal{Q}'$ if and only if $\mathcal{L}$ is trivial.

We define $\mathcal{L}$ as the kernel of the split map of right $\mathcal{Q}$-modules

$$\Phi : \mathcal{Q}^2 \to \mathcal{Q},\ (p,q) \mapsto (1+T_1)p - (1+T_2)q.$$

It works because

$$\begin{aligned}\Phi(1+T_2, 1+T_1) &= (1+T_1)(1+T_2) - (1+T_2)(1+T_1)\\ &= T_1T_2 - T_2T_1 = 2T_1T_2 \in \mathcal{Q}^\times.\end{aligned}$$

We have $\mathcal{L} \oplus \mathcal{Q} \xrightarrow{\sim} \mathcal{Q}^2$ and by computations, we can show that $\mathcal{L}$ is not free [34, §3.6]. Thus $[\mathcal{Q}']$ defines a non-loop class of $H^1(R_2, \mathrm{PGL}_2)$. Actually $H^1(R_2, \mathrm{PGL}_2)$ is infinite (and even has the cardinality of $\mathbb{C}$).

4.4. **Loop brothers.** Since invertible modules are locally free, the counterexample coincides with a loop torsor locally for the Zariski topology. This is a general phenomenon.

Theorem 4.2. [34, Theorem 1.1] *Let E be a G-torsor over R_n. Then E admits a unique (up to isomorphism) loop brother E_0, that is, a loop G-torsor E_0 such that E and E_0 coincide locally for the Zariski topology.*

For example, if q is a quadratic form of rank d over R_n, there exists a unique loop quadratic form q_{loop} as in Proposition 4.1.1 such that q_{loop} and q are isometric locally for the Zariski topology.

Given a G-torsor E, we see it as a Zariski form of its loop brother E_0. The R_n-group scheme $\mathrm{Aut}_G(E_0)$ is the twist $\mathfrak{G} = {}^{E_0}G$ of G by the R_n-torsor E_0 so that E defines a class in $H^1_{Zar}(R_n, \mathfrak{G})$.

4.5. **The conjecture.** The above existence of exotic torsors did happen when the generic fiber of the relevant twisted group scheme is anisotropic (this was $\mathrm{PGL}_1(\mathcal{Q}')$ in the example). We conjectured that loop torsors exhaust the torsors away of isotropic issues. Stavrova proved the conjecture.

Theorem 4.3. [74, Corollary 1.3] *Assume that G is semisimple and absolutely simple. Let E be a G-torsor over R_n and denote by E_0 its loop brother. Assume that the twisted R_n-group scheme ${}^{E_0}G$ is isotropic (i.e. contains a $\mathbb{G}_m$). Then $H^1_{Zar}(R_n, {}^{E_0}G) = 1$ and E is a loop torsor over R_n.*

4.6. **The acyclicity statement.** The main intermediate statement is a generalization of Theorem 4.1 for $n \geq 2$. We consider the iterated Laurent serie field $F_n = k((t_1))\dots((t_n))$.

Theorem 4.4. *Let G be a reductive k-group. Then the map $H^1(R_n, G) \to H^1(F_n, G)$ induces a bijection*

$$H^1(R_n, G)_{loop} \xrightarrow{\sim} H^1(F_n, G).$$

We call it the *acyclicity* statement. It is the way to prove that loop quadratic forms are the diagonalizable ones (see Example 5.3.(b)).

The injectivity is the hard thing and the surjectivity is an application of the statement on the reduction of G-torsors to finite subgroups discussed in the first lecture. We have seen that there exists a finite k-subgroup S such that $H^1(F_n, S) \to H^1(F_n, G)$ is onto. We are reduced to check that $H^1(R_n, S) \to H^1(F_n, S)$ is onto. The point is that R_n and F_n have the same Galois theory. In particular $R_n^{sc} \otimes_{R_n} F_n$ is a Galois closure of F_n and $\mathrm{Gal}(F_{n,s}/F_n) \xrightarrow{\sim} \Pi_1(R_n, 1)$. The map $H^1(R_n, S) \to H^1(F_n, S)$ is

$$H^1(\Pi_1(R_n, 1), S(R_n^{sc})) \to H^1\big(\mathrm{Gal}(F_{n,s}/F_n), S(F_{n,s})\big).$$

But S is finite so that $S(k_s) = S(R_n^{sc}) = S(F_{n,s})$ so that the above map is even bijective.

The injectivity 4.4 involves reduction to the anisotropic case à la Borel-Tits. Here comes the connection with Bruhat-Tits' theory presented by Gopal Prasad in his lectures. A key step is the following.

Proposition 4.5. *Let P be a parabolic k-subgroup of G. Let E be a loop G-torsor. If E_{F_n} admits a reduction to P, then E admits a reduction to P.*

Proof. The proof goes by induction on $n \geq 1$, the case $n = 0$ being obvious. The class $[E] \in H^1(R_n, G)_{loop}$ is given by a loop cocycle $\phi : \pi_1(R_n, 1) \to G(k_s)$. We reason at finite level by taking a basic cover $R_{n,m} \otimes_k l$ so that we have $\phi : \mu_m(l)^r \rtimes \mathrm{Gal}(l/k) \to G(k_s)$. We use the dictionnary and get a 1-cocycle $z : \mathrm{Gal}(l/k) \to G(l)$ and a homomorphism $\phi^{geo} : \mu_m^r \to {}_zG$.

We have $[E]_{R_{n,m}} = [z]$ so that $[z]_{F_{n,m}}$ admits a reduction to P. We denote by E_0 the G-torsor associated to z and consider the variety $X = E_0/P$ of reductions of E_0 to P. We have $X(F_{n,m}) \neq \emptyset$ so that $X(F_n) \neq \emptyset$. Since X is projective, we have

$$X(F_n) = X(F_{n-1}((t_n))) = X(F_{n-1}[[t_n]]);$$

specializing at $t_n = 0$ yields that $X(F_{n-1}) \neq \emptyset$. By induction we have $X(k) \neq \emptyset$ hence $[z]$ admits a reduction to P. Without loss of generality we can assume that z takes values in $P(l)$ and the classical twisting argument boils down to the case of $z = 1$.

We are reduced to the case of purely geometric loop cocycle. The proof goes by a fixed point argument and the idea is to prove that

$$(G/P)^{\phi^{geo}}(k) \neq \emptyset, \tag{4.1}$$

that is, the action of $(\mu_m)^n$ on the flag variety G/P has a rational fixed point. Assuming it works, such a point define a parabolic subgroup Q of G which is $G(k)$-conjugated to P and such that ϕ has value in $Q(l)$. Summarizing the Claim 4.1 is enough to complete the proof.

Let us establish the fixed point fact (4.1) We have

$$\emptyset \neq (E/P)(F_n) = \Big\{ x \in (G/P)(F_{n,m} \otimes_k l) \mid x = \phi(\sigma)\, \sigma(x) \quad \forall\, \sigma \in \mu_m^n(l) \rtimes \mathrm{Gal}(l/k) \Big\}.$$

Since the cocycle is purely loop, it simplifies as

$$\emptyset \neq (E/P)(F_n) = \Big\{ x \in (G/P)(F_{n,m}) \mid x = \phi^{geo}(\sigma)\, \sigma(x) \quad \forall\, \sigma \in \mu_m^n(l) \Big\}.$$

We write

$$(G/P)(F_{n,m}) = (G/P)\Big(F_{n-1,m}((t_n^{\frac{1}{m}}))\Big) = (G/P)\Big(F_{n-1,m}[[t_n^{\frac{1}{m}}]]\Big).$$

The point is that the specialization map at $t_n^{\frac{1}{m}} = 0$ is $\mu_m(l)^r \rtimes \mathrm{Gal}(l/k)$ equivariant so that

$$\emptyset \neq \Big\{ x \in (G/P)(F_{n-1,m}) \mid x = \phi^{geo}(\sigma)\, \sigma(x) \quad \forall\, \sigma \in \mu_m^n(l) \Big\}.$$

Repeating this argument yields $(G/P)^{\phi^{geo}}(k) \neq \emptyset$ as desired. □

The reason why parabolic and isotropic reductions are involved is to obtain some control on the loop cocycles. Another step of the proof of Theorem 4.4 is the following.

Proposition 4.6. [36] *Let $\phi : \pi_1(R_n, 1) \to G(k_s)$ be a purely geometric loop cocycle which is anisotropic, i.e. the action of the centralizer $C_G(\phi^{geo})^0$ is an anisotropic reductive k–group.*

Let $\psi : \pi_1(R_n, 1) \to G(k_s)$ be another loop cocycle. If $[\phi]_{F_n} = [\psi]_{F_n} \in H^1(F_n, G)$, then $[\phi] = [\psi] \in H^1(\pi_1(R_n, 1), G(k_s))$.

The proof goes in a similar manner but is more involved. The induction step requires the Bruhat-Tits of $G_{F_{n-1}((t_n^{\frac{1}{m}}))}$ and the Rousseau's tamely ramified descent theorem.

4.7. **Positive characteristic case.** In the recent paper [13], we investigated the case of a base field of characteristic $p > 0$. The theory works essentially the same if we restrict to the Galois covers which are tame. It leads to the notion of *tame loop* torsor we will encounter also in the next lecture.

Lecture 3: lectures in Arithmetic Geometry

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The main goal is to provide an analogous theory for quite specific rings which are related to Abhyankar's theory of tame covers [1]. This includes $\mathbb{Z}_p[[t]]\big[\frac{1}{p}, \frac{1}{t}\big]$ but also the case of $k[[x_1, ..., x_n]]\big[\frac{1}{x_1}, \dots, \frac{1}{x_n}\big]$ which is close of the Laurent polynomial ring R_n over k.

5. Abhyankar's setting

5.1. **Tame ramification.** The Abhyankar lemma and its applications are ubiquitous, see for example the Stacks project [73, Tag 0EXT] and toric geometry. We will follow the presentation of [68] in the specific case of interest for us.

Let A be an henselian (e.g. complete) regular local ring of dimension $r \geq 1$, of residue field k and fraction field K. We denote by $p \geq 0$ be its characteristic. We put $\widehat{\mathbb{Z}}' = \prod_{l \neq p} \mathbb{Z}_l$.

Let K_s be a separable closure of K. It determines a base point $\xi : \mathrm{Spec}(K) \to X = \mathrm{Spec}(A)$ so that we can deal with the Grothendieck fundamental group $\Pi_1(X, \xi)$. Since A is henselian, there is an equivalence of categories between finite étale covers of A and étale k–algebras [26, 18.5.15]; it follows that the profinite group $\Pi_1(X, x)$ is isomorphic to $\mathrm{Gal}(k_s/k)$.

Let $(f_1, \dots, f_r)$ be a regular sequence of A and consider the divisor $D = \sum_{i=1}^{r} D_i = \sum_{i=1}^{r} \mathrm{div}(f_i)$, it has strict normal crossings. We put $U = X \setminus D = \mathrm{Spec}(A_D)$.

Examples 5.1. (a) $A = k[[x_1, ..., x_r]]$, and $A_D = \big[\frac{1}{x_1}, \dots, \frac{1}{x_r}\big]$.
(b) $A = \mathbb{Z}_p[[t]]$ and $A_D = \mathbb{Z}_p[\frac{1}{p}, \frac{1}{t}]$.

We recall that a finite Galois cover $V \to U$ is *tamely ramified* with respect to D if the associated Galois extension L of K is is tamely ramified at the $D_i's$. It means that the inertia group associated to the valuation v_{D_i} has order prime[3] to p [68, XIII.2.0].

One can define the tame (*modéré* in French) fundamental group $\Pi_1^D(U, \xi)$ with respect to $U \subset X$ see [68, XIII.2.1.3] and [41, §2]. Concretly, it is the projective limit of Galois groups of pointed Galois covers (X_i, x_i) which are tamely ramified.

The group $\Pi_1^D(U, \xi)$ occurs then as a profinite quotient of $\Pi_1(U, \xi)$; the quotients by open subgroups provides the finite Galois tame covers of U.

Let $V \to U$ be Galois tame cover. In this case Abhyankar's lemma states that there exists a Kummer cover $X' = \mathrm{Spec}(A') \to X$ where

$$A' = A[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r)$$

[3] Of course if p is of characteristic zero, this condition is empty

and the n_i's are coprime to p such that $V' = V \times_X X' \to X'$ extends uniquely to a finite étale cover $Y' \to X'$ [68, XIII.5.2].

Without loss of generality we can assume that the n_i's are equal to some n prime to p. According to [26, 18.5.10], the finite A–ring A' is a finite product of henselian local rings. We observe that $A' \otimes_A k = k[T_1, \dots, T_r]/(T_1^{n_1}, \dots, T_r^{n_r})$ is a local Artinian algebra so that A' is connected. It follows that A' is a henselian local ring. Its maximal ideal is $\mathfrak{m}' = \mathfrak{m} \otimes_A A' + \langle T_1, \dots, T_r \rangle$ so that $A'/\mathfrak{m}' = k$. Since there is an equivalence of categories between finite étale covers of A (resp. A') and étale k–algebras [26, 18.5.15], the base change from A to A' provides an equivalence of categories between the category of finite étale covers of A and that of A'.

It follows that $Y' \to X'$ descends uniquely to a finite étale cover $f_\sharp : \mathrm{Spec}(B_\sharp) = Y_\sharp \to X$. It implies that

$$H^0(V', \mathcal{O}_{V'}) = B_\sharp[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r).$$

roots of unity for $i = 1, ..., r$. It follows that $V \to U$ is a quotient of a Galois cover of the shape

$$B_n = B[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is Galois cover of A containing a primitive n–th root of unity. We notice that B_n is the localization at $T_1 \dots T_r$ of $B_n' = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$. We have

$$\mathrm{Gal}(B_n/A_D) = \big(\prod_{i=1}^r \mu_n(B)\big) \rtimes \mathrm{Gal}(B/A).$$

Passing to the limit we obtain an isomorphism

$$\pi_1^t(U, \xi) \cong \big(\prod_{i=1}^r \widehat{\mathbb{Z}}'(1)\big) \rtimes \pi_1(X, \xi).$$

We denote by $f : U^{sc,t} \to U$ the profinite étale cover associated to the quotient $\pi_1^t(U, \xi)$ of $\pi_1(U, \xi)$. According to [41, Theorem 2.4.2], it is the universal tamely ramified cover of U.

5.2. **Tame loop torsors.** Let G be an affine A-group scheme of finite presentation. G–torsors are defined the same way over an A-scheme than in the field case (see Definition 3.1. For each A-scheme Z, we denote by $H^1(Z, G)$ the set of equivalence classes of G-torsors over Z. The formation of $H^1(Z, G)$ is functorial in Z and in G.

For each B_n as above, we have an exact sequence of pointed sets

$$1 \to H^1(\mathrm{Gal}(B_n/B), G(B_n)) \to H^1(A_D, G) \to H^1(B_n, G) \tag{5.1}$$

Definition 5.2. *Let E be a G-torsor over A_D.*

(1) We say that E is tame isotrivial if there exists some basic tame cover B_n as above such that $E \times_{A_D} B_n$ is trivial.

(2) We say that E is a tame loop G–torsor if there exists some cover B_n as above such that the class $[E]$ belongs to the image of

$$H^1(\mathrm{Gal}(B_n/B), G(B)) \to H^1(\mathrm{Gal}(B_n/B), G(B_n)) \hookrightarrow H^1(A_D, G).$$

We denote by $H^1_{\substack{tame\\loop}}(A_D, G)$ the subset of consisting in classes of tame loop G-torsors. The natural question is to investigate whether tame isotrivial torsors are tame loop. A nice case when it holds is when G is finite étale of prime to p rank.

When $r = 1$, then $A_D = K$ is a henselian discrete valued field and the fact that tame loop torsors exhaust torsors annihilated by base change to the tamely ramified closure K^{tr} follows from reduction to finite subgroup in Galois cohomology [14] in good residual characteristic and requires some Bruhat-Tits theory for small p. The case $r \geq 2$ is work in progress with Chernousov and Pianzola; we expect that it holds $r \leq 2$ and that fails for $r \geq 3$.

Examples 5.3. We discuss our favorite special cases.

(a) $G = \mathrm{GL}_d$. It is then about projective A_D-modules of rank d. Using algebraic K-theory, we know that those modules are stably free. It $r = 1$, this is the DVR case and the f.g. projective A_D-modules are free. If $r = 2$, a projective A_D-module of rank d extends to a projective A-module of rank d so is free; this uses that reflexive modules are locally free (see [22, Lemma 2.2]).

If $r \leq 3$, a special case of Gabber's purity theorem [27, Theorem 2.3] tells us that a projective A_D-module of rank d extends to a projective A-module of rank d so is free.

(b) $G = O_d$ for $q = \sum\limits_{i=1}^{d} x_i^2$ with $p \neq 2$. As for Laurent polynomials, tame loop quadratic forms are the diagonalizable ones [30, Proposition 7.4.(4)].

To state the main result requires the next fundamental construction.

5.3. **Blow-up.** We follow a blowing-up construction arising from [26, lemma 15.1.1.6]. We denote by $\widehat{X}$ the blow-up of $X = \mathrm{Spec}(A)$ at his closed point, this is a regular scheme [52, §8.1, Theorem 1.19] and the exceptional divisor $E \subset \widehat{X}$ is a Cartier divisor isomorphic to $\mathbb{P}^{r-1}_k$. We denote by $R = \mathcal{O}_{\widehat{X},\eta}$ the local ring at the generic point η of E. The ring R is a DVR of fraction field K and of residue field $F = k(E) = k(t_1, \dots, t_{r-1})$ where t_i is the image of $\frac{f_i}{f_r} \in R$ by the specialization map. At first glance it seems remarkable that F is a purely transcendental extension over k but it is clear from the geometrical viewpoint. Indeed F is the function field of the exceptional divisor $E \cong \mathbb{P}^{n-1}_k$.

We denote by $v : K^\times \to \mathbb{Z}$ the discrete valuation associated to R. Denoting by $\mathfrak{m} = \langle f_1, \dots, f_r \rangle$ the maximal ideal of A, a simpler way to define this valuation over $A \setminus \{0\}$ is

$$v(x) = \mathrm{Max}\big\{ n \geq 0 \mid x \in \mathfrak{m}^n \big\}.$$

We denote by R_v the completion of R and by K_v its fraction field.

Examples 5.4. (a) $A = k[[x_1, ..., x_r]]$, and $A_D = \bigl[\frac{1}{x_1}, \dots, \frac{1}{x_r}\bigr]$. Then the valuation ring of the Grothendieck valuation is $R = k[[x_1, ..., x_r]]\Bigl[(\frac{x_2}{x_1})^{\pm 1}, \dots, (\frac{x_r}{x_1})^{\pm 1}\Bigr]$ and $K_v \cong k\bigl(\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}\bigr)((x_1))$.

(b) $A = \mathbb{Z}_p[[t]]$ and $A_D = \mathbb{Z}_p\bigl[\frac{1}{p}, \frac{1}{t}\bigr]$. We have $R = \mathbb{Z}_p[[t]]\Bigl[(\frac{t}{p})^{\pm 1}\Bigr]$ and K_v is a complete discrete valued field such that p is an uniformizer parameter and such that its residue field is $k(u)$.

5.4. **The main result.**

Theorem 5.5. [30, Theorem 6.9] *Let G be a reductive A-group scheme. Then the natural map $H^1(A_D, G) \to H^1(K_v, G)$ induces an injective map*

$$H^1_{\substack{tame\\loop}}(A_D, G) \hookrightarrow H^1(K_v, G).$$

Except for $r = 1$, this map has no reason to be injective. Again $H^1(K_v, G)$ can be understood in many cases by the third part of Bruhat-Tits' theory [9]. It is based on several steps. The first one is parabolic reduction and goes by a fixed point argument.

5.5. **Fixed points method.** Let $\phi : (\prod_{i=1}^{r} \mu_n)(B) \rtimes \mathrm{Gal}(B/A) \to G(B)$ be a loop cocycle. As in the geometrical case, it is determined by its restrictions to $\mathrm{Gal}(B/A)$ and $(\prod_{i=1}^{r} \mu_n)(B)$ called respectively the arithmetic part $\phi^{ar} : \mathrm{Gal}(B/A) \to G(B)$ and the geometric part $\phi^{geo} : \prod_{i=1}^{r} \to G(B)$. It tunrs out that the cocycle conditions rephrases by saying that ϕ^{geo} arises from a A-group homomorphism $(\mu_{n,A})^r \to {}_{\phi^{ar}}G$ denoted also ϕ^{geo} where ${}_{\phi^{ar}}G$ stands for the twisted A–group scheme attached to ϕ^{ar}.

Next assume that the A–group scheme G acts on an A–scheme Z. We have then an action of ${}_{\phi^{ar}}G$ on the twisted A–scheme ${}_{\phi^{ar}}Z$ (which is an A-scheme [73, Tag 0CCJ, case (1)]).

It gives rise by precomposition by ϕ^{geo} to an A–action of $(\mu_{n,A})^r$ on ${}_{\phi^{ar}}Z$ We denote by $({}_{\phi^{ar}}Z)^{\phi^{geo}}$ the fixed point locus for this action, it is representable by a closed A–subscheme of ${}_{\phi^{ar}}Z$ [23, A.8.10.(1)]. We have a closed embedding $({}_{\phi^{ar}}Z)^{\phi^{geo}} \times_X U \subset {}_{\phi}Z$ of U-schemes.

Theorem 5.6. *The following are equivalent:*

(i) $({}_{\phi}Z)(K_v) \neq \emptyset$;

(ii) $Y(k) \neq \emptyset$;

(iii) $Y(U) \neq \emptyset$;

(iv) $({}_{\phi}Z)(U) \neq \emptyset$.

This is quite similar with the fixed point theorem [36, Thm. 7.1] which is mild generalization of Proposition 4.5. The following example makes the connection.

Example 5.7. We assume that $A = k[[t_1,\dots,t_r]]$ for a field k and $k[U] = k[[t_1,\dots,t_n]]\big[\frac{1}{t_1},\dots,\frac{1}{t_r}\big]$. We consider an affine algebraic k–group G acting on a smooth proper k–scheme Z. In this case $K = k((t_1,\dots,t_r))$ and A embeds in $k\big(\frac{t_1}{t_r},\dots,\frac{t_{r-1}}{t_r}\big)[[t_r]]$ so that K embeds in $k\big(\frac{t_1}{t_n},\dots,\frac{t_{r-1}}{t_r}\big)((t_r))$ which is nothing but the complete field K_v. If Q is a loop G-torsor over U, the statement is then that ${}^QZ(U) \neq \emptyset$ if and only if ${}^QZ(K_v) \neq \emptyset$. Taking a cocycle $\phi \in Z^1(\pi_1(U)^t, G(k_s))$ for E, this rephrases by the equivalence between $({}_\phi Z)(U) \neq \emptyset$ and $({}_\phi Z)(K_v) \neq \emptyset$.

What we have from [36, Thm. 7.1] (in characteristic zero but this extends to this tame setting) is the equivalence between $({}_\phi Z)(k[t_1^{\pm 1},\dots,t_r^{\pm 1}]) \neq \emptyset$ and $({}_\phi Z)\big(k((t_1))\dots((t_r))\big) \neq \emptyset$. Since $({}_\phi Z)(k[t_1^{\pm 1},\dots,t_r^{\pm 1}]) \subset ({}_\phi Z)(U)$ and $({}_\phi Z)\big(K_v\big) \subset ({}_\phi Z)\big(k((t_1))\dots((t_r))\big)$, it follows that this special case of Theorem 5.6 is a consequence of the fixed point result [36, Thm. 7.1].

Remark 5.8. An important case is when G is reductive and $Z = G/P$ where P is a parabolic A-subgroup scheme of G. In this case the condition $({}_\phi Z)(U) \neq \emptyset$ means that the twisted A_D–group scheme ${}_\phi Z$ admits an A_D-parabolic subgroup scheme of same type as P.

We proceed to the proof of Theorem 5.6.

Proof. We denote by L the fraction field of B and by L_n that of B_n. We have $[L_n : L] = n^r$. We want to extend the valuation v to L and to L_n. We put $\Gamma = \mathrm{Gal}(B_n/A_D) = \mathrm{Gal}(L_n/K)$.

According to [23, A.8.10.(1)], $Y = \big({}_{\phi^{ar}}(Z^{\phi^{geo}})\big)$ is a closed A–scheme of ${}_{\phi^{ar}}Z$ so it is projective. It is smooth over X according to point (2) of the same reference. Up to replacing G by ${}_{\phi^{ar}G}$ and G by ${}_{\phi^{ar}Z}$, we can assume that $\phi^{ar} = 1$ without lost of generality.

$(ii) \Longrightarrow (iii)$. Since Y_k is the special fiber of the smooth X–scheme Y, Hensel's lemma shows that $Y(A) \to Y(k)$ is onto. Since $Y(k)$ is not empty, it follows that $Y(A)$ is not empty and so is $Y(U)$.

$(iii) \Longrightarrow (iv)$. Since $Y(U) \subset ({}_\phi Z = (U)$, $Y(U) \neq \emptyset$ implies that $({}_\phi Z)(U) \neq \emptyset$.

$(iv) \Longrightarrow (i)$. This is obvious.

$(i) \Longrightarrow (ii)$. We assume that $({}_\phi Z)(K_v) \neq \emptyset$. For understanding the set $({}_\phi Z)(K_v)$, we need to describe the Galois L_n–algebra $K_v \otimes_K L_n$ of group Γ One checks that it is a field extension which comes equipped with the unique valuation w_n extending v, we denote it by L_{w_n}. Actually w_n restricted to $B[f_1^{1/n},\dots,f_r^{1/n}]$ is nothing but the Grothendieck valuation of that regular local ring.

By definition we have

$$({}_\phi Z)(K_v) = \big\{ z \in Z(L_{w_n}) \mid \phi(\sigma).\sigma(z) = z \ \ \forall \sigma \in \Gamma) \big\}$$

and our assumption is that this set is non-empty. Let O_{w_n} be the valuation ring of L_{w_n} and denote by $F_{l,n} = l(t_1^{1/n}, \dots, t_{r-1}^{1/n})$ its residue field. Since Z is projective over A, we have a specialization map $Z(L_{w_n}) = Z(O_{w_n}) \to Z_k(F_{l,n})$. We get that the set

$$\big\{z \in Z_k(F_{l,n}) \mid \phi(\sigma).\sigma(z) = z \ \ \forall \sigma \in \Gamma\big\}$$

is not empty. Since we have an embedding

$$F_{l,n} = l\big(t_1^{1/n}, \dots, t_{r-1}\big) \ \hookrightarrow \ l\big(\big(t_1^{1/n}\big)\big) \dots \big(\big(t_{r-1}^{1/n}\big)\big)$$

in a higher field of Laurent series. Successive specializations along the coordinates $t_1^{1/n}$, ..., $t_{r-1}^{1/n}$ as in the proof of Proposition 4.5 show that the set

$$\Big\{z \in (Z_k)\big(l\big) \ \mid \ \phi(\sigma).\sigma(z) = z \ \ \forall \sigma \in \Gamma\Big\} \tag{5.2}$$

is not empty. Since $\phi^{ar} = 1$, this set is $(Z_k)^{\phi^{geo}}(k)$. Thus $Y(k) = (Z_k)^{\phi^{geo}}(k)$ is non empty. □

Lecture 4: Applications and open questions

6. Infinite dimensional Lie theory, essential dimension

The theory of loop torsors over R_n is an important ingredient in the classification of extended affine Lie algebras (EALA for short), especially for conjugacy issues for maximal diagonalizable algebras. EALA are a generalization of affine Kac-Moody algebras, see the survey by Neher [61] and the papers [16, 17].

Informally speaking, the essential dimension of an algebraic object is the minimal number of independent parameters one needs to define it, see the survey talk [66]. An important case is that of torsors under an algebraic group G over a field k. The essential dimension of loop torsors is sometimes computable. For example the essential dimension of the generic quadratic form $\sum_{i=1}^n t_i\, x_i^2$ over the field $k(t_1, ..., t_n)$ is exactly n, see [67, Theorem 10.3]. More interesting cases have been investigated by Chernousov and Serre [18] and a systematic recent study is due to Ofek [64].

7. Arithmetic of semi-global fields

We will explain now how the tame loop torsors occur in the study of algebraic groups (and related objects) over a semi-global field F.

Let T be complete DVR of fraction field K and of residue field $k = T/tT$. We denote by p the characteristic exponent of k. Let X be a smooth projective geometrical aalgebraic K-curve; its function field $F = K(X)$ is a semi-global field. There are many results due to Harbater-Hartmann-Krashen [42, 43, 44], by Colliot-Thélène, Parimala and Suresh [21], by the reunion of the two teams [19, 20] and by others.

The field F comes with many valuations and here we are interested in the divisorial valuations namely those coming from regular models of X. More precisely, a divisorial valuation of X is the valuation associated to an irreducible divisor D of a regular T-model of X (that is flat, projective over T of relative dimension 1). We deal then with local global principles for torsors and other varieties with respect to the F_v's where v runs over all divisorial valuations. My goal is to explain on an specific point (see Lemma 7.1 below). how loop torsors techniques matter. More precisely it is about a step in the proof of the local-global principle for twisted flag varieties obtained recently with Parimala [32].

For simplicity we deal with a semisimple adjoint split F-group G_0 and with an inner form G of G_0. It means that there is a G_0-torsor E over F such that $G \cong {}^E(G_0)$. Given a parabolic F-subgroup P_0 of G_0, we consider the twisted flag variety ${}^E(G_0/P_0) = E/P_0$. Since $Z(F)$ parameterizes the F-parabolic subgroups of G, it called also the variety of parabolics of G of same type as P_0.

The starting point is to use the finite étale (p must be then large enough) F-subgroup $S_0 \subset G_0$ of the first lecture. Then the map $H^1(F, S_0) \to H^1(F, G_0)$ is onto so that we can work with a S_0-torsor Q of push-forward E. The theory of models

(Abhyankar, Lipman) provides a regular model $\mathfrak{X}$ of X, an open subset $\mathfrak{V}$ and a class $[\mathfrak{Q}] \in H^1(\mathfrak{V}, S_0)$ extending $[S]$ such that $\mathfrak{X} \setminus \mathfrak{V}$ is a strict normal crossing divisor. Let $Y = \mathfrak{X}_k$ be the special fiber of $\mathfrak{X}$. Without loss of generality we can assume that $Y \cap \mathfrak{V} = \emptyset$.

Lemma 7.1. [32, Proposition 6.2]. *Assume that $Z(F_v)$ for all divisorial places v. of F. Then there exists a regular proper model $\mathfrak{X}$ of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $Z(F_y) \neq \emptyset$.*

The statement needs explanations. If y is the generic point of an irreducible component Y_i of Y, then F_y is the completion of F with repsect the divisorial valuation on F associated to Y_i. In this case the statement is obvious. If y is a closed point of Y, then F_y is the fraction field of the completion A of the local ring $\mathcal{O}_{\mathfrak{X},y}$ with respect to its maximal ideal. We proceed to the proof of Lemma 7.1.

Proof. Taking two regular parameters f_1, f_2 of A, the S_0-torsor $\mathfrak{Q}$ over A_D is a tame loop torsor and so is $\mathfrak{E}$. In particular we can deal with the A_D-scheme $\mathfrak{Z} =^{\mathfrak{E}} (G_0/P_0)$.

Let v be the Grothendieck valuation on the field F_y. Its completion $F_{y,v}$ is nothing but the completion on the divisorial valuation on $\mathcal{O}_{\mathfrak{X},y}$ so that $Z(F_{y,v}) \neq \emptyset$ by assumption. The fixed point statement 5.6 yields that $Z(A_D) \neq \emptyset$. A fortiori we have $Z(F_y) \neq \emptyset$ as desired. □

Remark 7.2. The interest of the lemma is is to catch up the patching setting called also the HHK method (for Harbater, Hartmann and Krashen).

8. Open questions

Most of the open questions consist to try to generalize results from the Laurent ring case to the Abhyankar setting.

A first question is whether G–torsors over A_D are isotrivial under mild conditions of ramification. Starting with the case of projective A_D–modules, this question is then widely open.

Another question is whether tame isotrivial G-torsors admit loop brothers (that is, a loop class which agrees Zariski locally). Further we can think about generalization of Stavrova's theorem 4.3 for A_D. It would mean that under an isotropy assumption, isotrivial loop G-torsors over A_D are tame loop.

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