INTRODUCTION TO AFFINE ALGEBRAIC GROUPS IN POSITIVE CHARACTERISTIC

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Contents

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1. Introduction

The theory of affine algebraic groups over an algebraically closed field of characteristic zero is well spread out and is rather close of the theory of complex Lie groups. When dealing with semisimple real Lie groups, we deal real algebraic groups and that theory extends well to a base field of characteristic zero. When dealing in the positive characteristic case, new objects (as non smooth groups for examples) and new phenomenons (as failure of reducibility for linear representations of GL_2) occur. Technically speaking, it is also harder since the language of varieties is not anymore adapted (and quite dangerous) and the natural framework is that of group schemes.

The theory of affine algebraic groups over a field of positive characteristic was significantly extended recently by the theory of pseudo-reductive groups of Conrad-Gabber-Prasad; it will be presented in the course of B. Conrad.

To start with this topic, I recommend "Basic theory of affine group schemes" by Milne [M1] before reading the Demazure-Gabriel's book.

Let me discuss basic examples for motivating the lectures. We denote by k a base field and by $p \geq 1$ its characteristic exponent.

1.1. Over an algebraically closed field of characteristic $p > 0$.

1.1.1. *There are non trivial commutative extensions of* $\mathbb{G}_a$ *by itself.* Such an example is provided by the k–group W_2 of Witt vectors of length 2. As k-variety, we have $W_2 = \mathbf{A}_k^2$ and the (commutative) rule law is given on k^2 by $(x_0, x_1) + (y_0, y_1) = \Big(x_0 + y_0, x_1 + y_1 \,-\, S_1(x_0, y_0)\Big)$ where

$$S_1(x, y) = \sum_{i=1}^{p-1} \frac{i!(p-1)!}{(p-i)!}\, x^i y^{p-i}.$$

The projection map $W_2 \to \mathbb{G}_a$ is a group homomorphism and we have an exact sequence (a precise sense will be provided, see §4) of k–groups

$$0 \to \mathbb{G}_a \to W_2 \to \mathbb{G}_a \to 0$$

This sequence is not split. For simplicity we consider the case $p = 2$. It the sequence splits, we would have a k–group morphism $\mathbb{G}_a \to W_2$, $t \mapsto (t, f(t))$ where $f \in k[t]$. Such a f satisfies the rule $f(x+y) = f(x) + f(y) + xy$ and xy cannot be written as a difference $f(x+y) - f(x) - f(y)$. Another way for $k = \mathbb{F}_p$ is to see that $W_2(\mathbb{F}_p) \cong \mathbb{Z}/p^2\mathbb{Z}$.

In constrast, recall that in characteristic zero, any commutative k–group which is extension of $\mathbb{G}_a$ by itself is trivial [DG, 2.4.2].

1.1.2. *Intersection of smooth subgroups are not necessarily smooth.* In the additive group $G = \mathbb{G}_a^2$, we consider the closed k–subgroup H_1 given by the equation $x^p + x = y$. Then H_1 is isomorphic to $\mathbb{G}_a$, hence is smooth. In the other hand, we consider the additive subgroup H_2 of G defined by the equation $x = y$. Then the intersection $H = H_1 \cap H_2$ is given by the equations $x = y$ and $x^p = 0$, so is isomorphic to the infinitesimal group α_p whose coordinate ring is $k[t]/t^p$. It is not smooth. In the language of varieties, the intersection is trivial but is not in the scheme-theoretical viewpoint.

1.2. Over a non-perfect field of characteristic $p > 0$.

1.2.1. *Groups with few points.* We start with the following example due to Tits. We put $K = \mathbb{F}_p(t)$ (or $\overline{\mathbb{F}}_p(t)$) and consider the K–subgroup G of $\mathbb{G}_{a,K}^2$ defined by the equation

$$x + tx^p + y^p = 0.$$

Geometrically, G is very nice since $G_{K(\sqrt[p]{t})} \cong \mathbb{G}_{a,K(\sqrt[p]{t})}$; in particular, G is smooth and connected. This illustrates the descent technique.

If $p > 2$, we claim that $G(K) = \{(0,0\}$ (if $p = 2$, $G(K)$ is infinite since G is a smooth curve of genus zero). Let $(x(t), y(t)) \in G(K)$. If $x(t) = 0$, then $y(t) = 0$ so we can deal with the case $x(t) = \frac{P}{Q}$ where P, Q are coprime irreducible polynomials of $\mathbb{F}_p[t]$. By deriving the equation, we get that $P'Q - PQ' + P^pQ^{2-p} = 0$ which is impossible since $p > 2$.

This contrasts with the fact that $H(F)$ is schematically dense in H for every affine algebraic connected group H defined over a field F of characteristic zero [Bo, 18.3].

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Lecture I: Generalities

2. Sorites

We shall work over a base ring R (commutative and unital) and discuss more precisely the case of our base field k.

2.1. **R-Functors.** We denote by $\mathcal{A}ff_R$ the category of affine R-schemes. We are interested in R–functors, i.e. covariant functors from $\mathcal{A}ff_R$ to the category of sets. If $\mathfrak{X}$ an R-scheme, it defines a covariant functor

$$h_{\mathfrak{X}}: \ \mathcal{A}ff_R \ \to \ Sets, \ S \mapsto \mathfrak{X}(S).$$

Given a map $f : \mathfrak{Y} \to \mathfrak{X}$ of R-schemes, there is a natural morphism of functors $f_* : h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ of R-functors.

We recall now Yoneda's lemma. Let F be a R-functor. If $\mathfrak{X} = \mathrm{Spec}(R[\mathfrak{X}])$ is an affine R–scheme and $\zeta \in F(R[\mathfrak{X}])$, we define a morphism of R-functors

$$\widetilde{\zeta} : h_{\mathfrak{X}} \to F$$

by $\widetilde{\zeta}(S) : h_{\mathfrak{X}}(S) = \mathrm{Hom}_R(R[\mathfrak{X}], S) \to F(S), \ f \mapsto F(f)(\zeta)$.

Each morphism $\varphi : h_{\mathfrak{X}} \to F$ is of this shape for a unique $\zeta \in F(R[\mathfrak{X}])$: ζ is the image of $Id_{R[X]}$ by the mapping $\varphi : h_X(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$.

In particular, each morphism of functors $h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ is of the shape h_v for a unique R–morphism $v : \mathfrak{Y} \to \mathfrak{X}$.

A R-functor F is representable by an affine scheme if there exists an affine scheme $\mathfrak{X}$ and an isomorphism of functors $h_X \to F$. We say that $\mathfrak{X}$ represents F. The isomorphism $h_X \xrightarrow{\sim} F$ comes from an element $\zeta \in F(R[\mathfrak{X}])$ which is called the universal element of $F(R[\mathfrak{X}])$. The pair $(\mathfrak{X}, \zeta)$ satisfies the following universal property:

For each affine R-scheme $\mathfrak{T}$ and for each $\eta \in F(R[\mathfrak{T}])$, there exists a unique morphism $u : \mathfrak{T} \to \mathfrak{X}$ such that $F(u^*)(\zeta) = \eta$.

We can also deal with R–functors in groups. A basic example is the R–functor $\underline{\mathrm{Aut}}(X)$ of automorphisms of a given R-scheme X, that is defined by $\underline{\mathrm{Aut}}(X)(S) = \mathrm{Aut}_S(X_S)$. for each R–ring S. We can add of course additional structures (groups,...).

2.1.1. **Example.** *Automorphisms of the additive group.* In characteristic zero, it is well-known that the R–functor $\underline{\mathrm{Aut}}_{gr}(\mathbb{G}_a)$ is representable by $\mathbb{G}_m$. That is for each k–ring R, the R-group schemes automorphisms of $\mathbb{G}_{a,R}$ are homotheties by $R^\times$ [DG, II.3.4.4]. In characteristic $p > 0$, the k–functor $\underline{\mathrm{Aut}}_{gr}(\mathbb{G}_a)$ is not representable by an algebraic k–group since for each k–ring R, $\underline{\mathrm{Aut}}_{gr}(\mathbb{G}_a)(R)$ consists in the morphisms $X \to a_0 X + a_1 X^p + \cdots + a_r X^{p^r}$ where $a_0 \in R^\times$ and the a_i are nilpotent elements of R.

2.2. **Definition.** An affine R–group scheme $\mathfrak{G}$ is a group object in the category of affine R-schemes. It means that $\mathfrak{G}/R$ is an affine scheme equipped with a section $\epsilon : \mathrm{Spec}(R) \to \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \to \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{tikzcd}
(\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} \arrow[r, "m\times id"] \arrow[d, "can"', "\cong"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[r, "m"] & \mathfrak{G} \\
\mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) \arrow[r, "id\times m"] & \mathbb{G} \times_R \mathfrak{G} \arrow[ur, "m"'] &
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "id\times\epsilon"] \arrow[dr, "p_2"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"'] & \mathrm{Spec}(R) \times \mathfrak{G} \arrow[l, "\epsilon\times id"'] \arrow[dl, "p_1"] \\
& \mathfrak{G} &
\end{tikzcd}$$

Symetry:

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "id\times\sigma"] \arrow[d, "s_{\mathfrak{G}}"'] & \mathfrak{G} \times_R \mathfrak{G}. \arrow[d, "m"'] \\
\mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathfrak{G}
\end{tikzcd}$$

We say that $\mathfrak{G}$ is commutative if furthermore the following diagram commutes

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "switch"] \arrow[d, "m"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"'] \\
\mathfrak{G} \arrow[r, phantom, "="] & \mathfrak{G}.
\end{tikzcd}$$

Let $R[\mathfrak{G}]$ be the coordinate ring of $\mathfrak{G}$. We call $\epsilon^* : R[\mathfrak{G}] \to R$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \to R[\mathfrak{G}]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. They satisfies the following rules:

Co-associativity:

$$\begin{tikzcd}
R[\mathfrak{G}] \arrow[r, "m^*"] \arrow[dr, "m^*"'] & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[r, "m^*\otimes id"] & (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]) \otimes_R R[\mathfrak{G}] \\
& R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \arrow[r, "id\otimes m^*"] & R[\mathfrak{G}] \otimes_R (R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]). \arrow[u, "can"', "\cong"]
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
R[\mathfrak{G}] \arrow[r, "id\otimes\epsilon^*"] & R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & R[\mathfrak{G}] \arrow[l, "\epsilon^*\times id"'] \\
& R[\mathfrak{G}] \arrow[ul, "id"] \arrow[u, "m^*"] \arrow[ur, "id"'] &
\end{tikzcd}$$

Cosymmetry:

$$\begin{array}{ccc} R[\mathfrak{G}] \otimes R[\mathfrak{G}] & \xrightarrow{\sigma^* \otimes id} & R[\mathfrak{G}]. \\ m^* \Big\uparrow & & s^*_{\mathfrak{G}} \Big\uparrow \\ R[\mathfrak{G}] & \xrightarrow{\epsilon^*} & R. \end{array}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R–algebra[1]. Given an affine R–scheme $\mathfrak{X}$, there is then a one to one correspondence between group structures on $\mathfrak{X}$ and Hopf R–algebra structures on $R[\mathfrak{X}]$.

If $\mathfrak{G}/R$ is an affine R–group scheme, then for each R–algebra S the abtract group $gG(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \to \mathfrak{G}(S)$. It means that the functor $h_{\mathfrak{G}}$ is actually a group functor.

2.2.1. **Lemma.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the Yoneda lemma induces a one to one correspondence between group structures on $\mathfrak{X}$ and group structures on $h_{\mathfrak{X}}$.*

In other words, defining a group law on $\mathfrak{X}$ is the same that do define compatible group laws on each $\mathfrak{G}(S)$ for S running over the R-algebras.

2.3. **Examples.**

2.3.1. *Constant group schemes.* Let Γ be a finite abstract group. We consider the R–scheme $\mathfrak{G} = \bigsqcup_{\gamma\in\Gamma} \mathrm{Spec}(R)$. Then the group structure on Γ induces a group scheme structure on $\mathbb{G}$ with multiplication

$$\mathfrak{G} \times_R \mathfrak{G} = \bigsqcup_{(\gamma,\gamma')\in\Gamma^2} \mathrm{Spec}(R) \to \mathfrak{G} = \bigsqcup_{\gamma\in\Gamma} \mathrm{Spec}(R)$$

applying the component (γ, γ') to $\gamma\gamma'$. There usual notation for such an object is Γ_R. This group scheme occurs as solution of the following universal problem.

2.3.2. *Vector groups.* Let N be a R–module. We consider the commutative group functors

$$V_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto \mathrm{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee,$$

$$W_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto N \otimes_R S.$$

2.3.1. **Lemma.** *The R–group functor V_N is representable by the affine R–scheme $\mathfrak{V}(N) = \mathrm{Spec}(S^*(N))$ which is then a commutative R–group scheme. Furthermore N is of finite presentation if and only if $\mathfrak{V}(N)$ is of finite presentation.*

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[1]This is Waterhouse definition [Wa, §I.4], other people talk about cocommutative coassociative Hopf algebra.

Proof. It follows readily of the universal property of the symmetric algebra

$$\mathrm{Hom}_{R'-mod}(N \otimes_R R', R') \xleftarrow{\sim} \mathrm{Hom}_{R-mod}(N, R') \xrightarrow{\sim} \mathrm{Hom}_{R-alg}(S^*(N), R')$$

for each R-algebra R'. □

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N. We note that $N = \mathfrak{V}(N)(R)$.

Its group law on the R–group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \to S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The counit is $\sigma^* : S^*(N) \to S^*(N)$, $X \mapsto -X$.

2.3.2. **Remarks.** (1) If $N = R$, we get the affine line over R. Given a map $f : N \to N'$ of R–modules, there is a natural map $f^* : \mathfrak{V}(N') \to \mathfrak{V}(N)$.

(2) If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that $\mathfrak{W}(N)$ is representable by an affine group scheme.

(3) If R is noetherian, Nitsure showed the converse holds [Ni04]. If N is finitely generated projective, then $\mathfrak{W}(N)$ is representable iff N is locally free.

2.3.3. **Lemma.** *The construction of (1) provides an antiequivalence of categories between the category of R-modules and that of vector group R-schemes.*

2.3.3. *Group of invertible elements, linear groups.* Let A/R be an algebra (unital, associative). We consider the R-functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times.$$

2.3.4. **Lemma.** *If A/R is finitely generated projective, then* $\mathrm{GL}_1(A)$ *is representable by an affine group scheme. Furthermore,* $\mathrm{GL}_1(A)$ *is of finite presentation.*

Proof. We shall use the norm map $N : A \to R$ defined by $a \mapsto \det(L_a)$ constructed by glueing. We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a. The same is true after tensoring by S, so that

$$\mathrm{GL}_1(A)(S) = \Big\{ a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \quad | \quad N(a) \in R^\times \Big\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{array}{ccc} \mathfrak{G} & \longrightarrow & \mathfrak{W}(A) \\ \Big\downarrow & & N \Big\downarrow \\ \mathbb{G}_{m,R} & \longrightarrow & \mathfrak{W}(R). \end{array}$$

□

Given a R–module N, we consider the R–group functor

$$S \mapsto \mathrm{GL}_1(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S).$$

So if N is finitely generated projective. then $\mathrm{GL}_1(N)$ is representable by an affine R–group scheme. Furthermore $\mathrm{GL}_1(N)$ is of finite presentation.

2.3.5. **Remark.** If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

2.3.4. *Diagonalizable group schemes.* Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R–algebra of A. As R-module, we have

$$R[A] = \bigoplus_{a \in A} R\, e_a$$

and the multiplication is given by $e_a\, e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R. We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R-algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \to R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$,

Antipode: $\sigma^* : R[A] \to R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \to R$, $\epsilon(e_a) = 1$.

2.3.6. **Definition.** *We denote by $\mathfrak{D}(A)/R$ (or $\widehat{A}$) the affine commutative group scheme* $\mathrm{Spec}(R[A])$. *It is called the diagonalizable R–group scheme of base A. An affine R–group scheme is diagonalizable if it is isomorphic to some $\mathfrak{D}(B)$.*

We denote by $\mathbb{G}_m = \mathfrak{D}(\mathbb{Z}) = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$.

2.3.7. **Proposition.** *Assume that R is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable R–group schemes.*

Proof. It is enough to contruct the inverse map $\mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{D}(B)) \to \mathrm{Hom}(A, B)$ for abelian groups A, B. We are given a group homomorphism $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R-gp}(\mathfrak{D}(B), \mathbb{G}_m) \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathbb{G}_m),$$

hence a map $B \to A$. $\square$

3. Basic results on algebraic groups

The first thing is that an affine algebraic group G can be embedded as closed k–subgroup to some GL_n [DG, II.2.1]. The reader must to be caution with the concept of "linear algebraic group" which means furthermore than G is smooth.

3.0.8. **Remark.** Gordeev and Popov proved a more precise embedding statement for G affine smooth over an infinite field k [GP]. There exists a linear representation V of G such that G embeds as closed k–subgroup of $\mathrm{GL}(V)$ as automorphism of some tensor $f \in V^* \otimes V^* \otimes V$. In particular, G is given by concrete equations of degrees ≤ 3 (note that explicit constructions have been given recently by Garibaldi/Guralnick [GG]). Then V is equipped with a (non associative, non unital) algebra structure $V \otimes V \to V$ and G arises as group automorphism group of that algebra.

3.1. **Reduced subschemes of affine algebraic groups.** Let G be an affine algebraic k-group and denote by G_{red} its underlying reduced scheme. We have an isomorphism $(G_{\mathrm{red}} \times_k G_{\mathrm{red}})_{\mathrm{red}} \xrightarrow{\sim} (G \times_k G)_{\mathrm{red}}$ [EGA1, 5.1.8].

If the base field k is perfect, we have $(G_{\mathrm{red}} \times_k G_{\mathrm{red}})_{\mathrm{red}} \xrightarrow{\sim} G_{\mathrm{red}} \times_k G_{\mathrm{red}}$ so that the group law $G \times G \to G$ induces a group law on G_{red}. Hence G_{red} is a closed subgroup k–scheme of G. But G_{red} is not normal in general in G, consider for example the $\mathbb{F}_3$–group $\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ in characteristic 3 (for the semi-direct product, cf. §4.0.1).

If the base field k is not perfect, we cannot do that and we have the following counterexample. Over $K = \mathbb{F}_p(t)$, we consider the K–subgroup G of $\mathbb{G}_{a,K}$ defined by the equation $x^{p^2} = t\, x^p$. Then $K[G] = K[x]/x^p(x^p - t) \cong K[x]/x^p \times k[x]/(x^p - t)$, hence a k–isomorphism $K[G_{\mathrm{red}}] \cong K \times K[x]/(x^p - t)$, we postpone the proof that G_{red} is not a k–subgroup.

3.2. **Smoothness.** We have the following nice criterion.

3.2.1. **Lemma.** *Let G be an affine algebraic k–group. Then the following are equivalent:*

(1) G is smooth;

(2) G is geometrically reduced;

(3) the ring $\mathcal{O}_{G,e} \otimes_k \overline{k}$ is reduced.

(4) G admits an non empty open subscheme U which is smooth.

We recall that for an affine algebraic k-variety X that we say that X is geometrically reduced if the algebra $\overline{k}[X]$ is reduced (that is $k[X]$ is separable in the Bourbaki sense [Bbk1, V.2]). If this condition is satisfied, it implies that the smooth locus of X is dense [GW, th. 6.20.(ii)] and also that the set $X(k_s)$ is dense in X (*ibid*, prop. 6.21).

Proof. Without lost of generality, one can assume than k is algebraically closed. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are trivial.

(3) $\Longrightarrow$ (4): We assume that the ring $\mathcal{O}_{G,e}$ is reduced. Let J be the nilradical of $k[G]$; it is a basic fact that the formation of nilradical commutes with localization. Hence $J \otimes \mathcal{O}_{G,e}$ is the nilradical of $\mathcal{O}_{G,e}$ so is zero and since the $k[G]$–module J is finitely generated, there exists an affine open neighbourhood U of e in G such that $k[U]$ is reduced. Then U is generically smooth, so that there exists a nom-empty open $U' \subset U \subset G$ which is smooth.

(4) $\Longrightarrow$ (1): We denote by U the smooth locus of G, it is an non empty open subset of G which is stable by left translations under $G(k)$. We have then $U = G(k)U = G$, thus G is smooth. $\square$

3.2.2. **Remarks.** (a) In characteristic zero, algebraic groups are smooth (Cartier, [DG, II.6.11]), see [Oo] for another proof.

(b) The algebraic subgroup $x^p = ty^p$ of $\mathbb{G}_{a,\mathbb{F}_p(t)}$ is reduced but not geometrically reduced. Hence reduceness is not enough to detect smoothness.

We can come back to the $K = \mathbb{F}_p(t)$–group G defined by the equation $x^{2p} = tx^p$. Then G_{red} is not smooth and $\mathcal{O}_{\mathbf{G}_{red},e} = K$, so G_{red} cannot be equipped with a K–group structure. In particular, G_{red} is not a k–subgroup of $\mathbb{G}_{a,K}$.

3.3. **Algebraic subgroups generated by images, derived group of a smooth group.**

3.3.1. **Proposition.** [SGA3, VI_B.7.1] *Let G/k be an affine algebraic k–group. Let $(f_i : V_i \to G)_{i\in I}$ be a family of k-morphisms where the V_i are geometrically reduced k–schemes. Then there exists a unique smallest k–closed subgroup $\Gamma_G((f_i)_{i\in I})$ of G such that each f_i factorizes within that k–group. Furthermore $\Gamma_G((f_i)_{i\in I})$ is smooth.*

3.3.2. **Remarks.** (a) The k–group $\Gamma_G((f_i)_{i\in I})$ is called the k–subgroup generated by the f_i. Its formation commutes with arbitrary base field extensions.

(b) If we are given a morphism of k–groups $f : H \to G$ where H is an affine smooth k–group. The above statement provides a closed k–group G' such that f factorizes within G'. By construction, it is the reduced subscheme of G whose topological space consists in the reunion of the images of $H^{2n} \to G$, $(h_1, h_2, \dots, h_{2n-1}, h_{2n}) \mapsto f(h_1)f(h_2)^{-1} \dots f(h_{2n-1})f(h_{2n})^{-1}$, hence $\Gamma_G(f)$ is the reduced subscheme of G with underlying topological space $f(H)$.

(c) If $(H_j)_{j\in J}$ are smooth k–subgroups of G, then $\Gamma_G\big(i_j : H_j \to G\big)$ is called the closed k–subgroup generated by the H_j. It is smooth. Furthermore, if the H_i are connected, then $\Gamma_G\big(i_j : H_j \to G\big)$ is connected [SGA3, VI_B.7.2.1].

3.3.3. **Corollary.** *Let G/k be an affine algebraic k–group. Then G admits a maximal smooth closed k-subgroup which is denoted by $G^\dagger$. It satisfies the following properties:*

(i) $G^\dagger$ is the maximal smooth closed k-subscheme of G and the maximal geometrically reduced closed k–subscheme of G.

(ii) We have $G^\dagger(k) = G(k)$; if k is separably closed, and $G^\dagger$ is the schematic closure of $G(k)$ in G;

(iii) If k is perfect, then $G_{\mathrm{red}} = G^\dagger$;

(iv) If K/k is a separable field extension, then $G^\dagger \times_k K = (G_K)^\dagger$.

Proof. We apply the above remark c) to the family of all closed k-subgroups of G.

(ii) Let $i : X \subset G$ be a geometrically reduced closed k–subscheme of G. Then $\Gamma_G(i)$ is a smooth closed k–subgroup of G, hence $X \subset \Gamma_G(i) \subset G^\dagger$;

(iii) If $g \in G(k)$, we have then $gG^\dagger \subset G^\dagger$, hence $g \in G^\dagger(k)$. Thus $G^\dagger(k) = G(k)$. If k is separably closed, $G^\dagger(k)$ is schematically dense in $G^\dagger$, so we conclude that $G^\dagger$ is the schematic closure of $G(k)$ in G.

(iv) Since $G^\dagger$ is smooth, it is reduced so we have $G^\dagger \subset G_{\mathrm{red}}$. If k is perfect, then G_{red} is geometrically reduced, hence $G_{\mathrm{red}} \subset G^\dagger$, thus $G_{\mathrm{red}} = G^\dagger$.

(v) See [CGP, lemma C.4.1].

□

3.3.4. **Remarks.** (a) Consider again the case $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ in characteristic 3. Then $G^\dagger = \mathbb{Z}/2\mathbb{Z}$ and is not normal in G.

(b) The formation of $G^\dagger$ does not commute with inseparable extensions. Put $k = \mathbb{F}_p(t)$ and consider the k–subgroup $x^p + ty^p = 0$ of $\mathbb{G}_{a,k}^2$. Since $t \not\in (k_s)^p$, $G(k_s) = 0$ hence $G^\dagger = 0$. We put $k' = k(\sqrt[p]{t}) = k(t')$. Then we write $x^p + ty^p = (x + t'y)^p = 0$ and $G_{k'}^\dagger$ is isomorphic to $\mathbb{G}_{a,k'}$.

(c) If G is connected, $G^\dagger$ can be disconnected, see [CGP, C.4.3].

If G is an affine smooth k–subgroup, we consider the bracket map $f : G \times_k G \to G$, $(g_1, g_2) \mapsto [g_1, g_2]$. The k–subgroup $\Gamma_G(f)$ of G is called the derived k–group of G and is denoted by DG. It is a normal k–subgroup of G and is characterized by the fact that $DG(\overline{k}) = [G(\overline{k}), G(\overline{k})]$.

How far as I know, there is no general way to define the derived group beyond the smooth case.

3.4. Morphisms, I.

3.4.1. **Proposition.** *Let $f : H \to G$ be a morphism of affine algebraic k–groups. We assume that G is reduced. Then the following are equivalent:*

(1) f is faithfully flat;

(2) f is surjective (that is $H(\overline{k}) \to G(\overline{k})$ is surjective);

(3) f is dominant.

Note that (3) rephrases by saying that the map $f^* : k[G] \to k[H]$ is injective, it is then the viewpoint of [Wa, §14.1]; also (1) $\Longleftrightarrow$ (3) is true for G affine non reduced.

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are obvious. Assume that f is dominant. Then the Chevalley generic flatness theorem [DG, I.3.3.7] shows that the flat locus $U \subset G$ is dense. Then $V = f^{-1}(U)$ is a dense open subscheme of H which is flat over G. Since $H_{\overline{k}}$ is covered by the hV for h running over $H(\overline{k})$, it follows that $H_{\overline{k}}$ is flat over $G_{\overline{k}}$. Equivalently, H is flat over G so the morphism f. Its topological image $f(H)$ is open in G hence is close as well. Since the image is dense, we conclude that f is surjective. Thus f is flat and surjective. $\square$

3.4.2. **Proposition.** *Let $f : H \to G$ be a morphism of affine algebraic k–groups. Then the following are equivalent:*

(1) f is a closed immersion;

(2) f is an immersion;

(3) f is a monomorphism, that is $\ker(f) = 1$.

Proof. We do only the case H smooth (see §9.5 for a sketch of the general case or [DG, II.5.5.1]). The three assertions are insensitive to the base change from k to $\overline{k}$, so we can assume than k is algebraically closed.

The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are obvious. For the implication (3) $\Longrightarrow$ (1), we consider the closed k–subgroup $I = \Gamma_G(f)$ defined by Proposition 3.3.1. It is smooth and according to Remark 3.3.2.(b), it is the schematic image of f. We may assume that G is smooth and that f is dominant. By the previous Proposition, f is faithfully flat. We want to show that f is an isomorphism by construction a section $s : G \to H$. Since f is a monomorphism, we have $id_H \circ p_1 = id_H \circ p_2 : H \times_G H \to H \overset{id}{\to} H$. Now f is a strict epimorphism ([M1, 2.17, 2.18]), so that there exists a morphism of k–schemes $s : G \to H$ making the factorization as follows

$$\begin{tikzcd} H \arrow[d, "f"'] \arrow[dr, "id_H"] & \\ G \arrow[r, "s"] & H. \end{tikzcd}$$

We conclude that f is an isomorphism. $\square$

3.4.3. **Remark.** One could conclude also by using the more general result that a finitely presented faithfully flat monomorphism is an open immersion [EGA4, 17.9.1] so that the previous proposition yields that f is an isomorphism.

Lecture II: Using functors

4. Sequences of group functors

We say that a sequence of R–group functors

$$1 \to F_1 \overset{u}{\to} F_2 \overset{v}{\to} F_3 \to 1$$

is exact if for each R–algebra S, the sequence of abstract groups

$$1 \to F_1(S) \overset{u(S)}{\to} F_2(S) \overset{v(S)}{\to} F_3(S) \to 1$$

is exact.

If $w : F \to F'$ is a map of R–group functors, we denote by $\ker(w)$ the R–group functor defined by $\ker(w)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S. If $w(S)$ is onto for each R–algebra S/R, it follows that

$$1 \to \ker(w) \to F \overset{w}{\to} F' \to 1$$

is an exact sequence of R–group functors.

4.0.4. **Lemma.** *Let $f : \mathfrak{G} \to \mathfrak{G}'$ be a morphism of R–group schemes.*
(1) Then the R–functor $\ker(f)$ is representable by a closed subgroup scheme of $\mathfrak{G}$.

(2) The sequence of R–functors $1 \to \ker(f) \to \mathfrak{G} \to \mathfrak{G}' \to 1$ is exact iff there exists a k–map $s : \mathfrak{G}' \to \mathfrak{G}$ such that $f \circ s = id_{\mathfrak{G}_3}$.

Proof. (1) Indeed the carthesian product

$$\begin{array}{ccc} \mathfrak{N} & \longrightarrow & \mathfrak{G} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(R) & \overset{\epsilon'}{\longrightarrow} & \mathfrak{G}' \end{array}$$

does the job.
(2) Assume that the sequence is exact and take $id_{\mathfrak{G}_3} \in \mathfrak{G}_3(R[\mathfrak{G}_3])$. It lifts to an element $s \in \mathfrak{G}_2(R[\mathfrak{G}_3]) = \mathrm{Hom}_R(\mathfrak{G}_3, \mathfrak{G}_2)$ which satisfies by construction $f \circ s = id_{\mathfrak{G}_3}$. The converse is obvious. □

4.0.5. **Lemma.** *Let $1 \to F_1 \overset{i}{\to} F_2 \overset{f}{\to} F_3 \to 1$ be an exact sequence of R–group functors. If F_1 and F_3 are representable by affine R–schemes (and then by affine R–group schemes), so is F_2.*

Proof. denote by $\mathfrak{G}_1$ (resp. $\mathfrak{G}_3$) the affine R–group scheme which represents F_1 (resp. F_3). Again we lift the identity $id_{\mathfrak{G}_3} \in \mathfrak{G}_3(R[\mathfrak{G}_3]) = F_3(R[\mathfrak{G}_3])$ to an element $s \in F_2(R[\mathfrak{G}_3])$ which satisfies by construction $f \circ s = id_{\mathfrak{G}_3}$. We consider the R–map $\rho : F_1 \times F_3 \to F_2$ defined for each R–ring S by $F_1(S) \times F_3(S) \to F_2(S)$, $(\alpha_1, \alpha_3) \mapsto i(\alpha_1)s(\alpha_3)$. We claim that ρ is an isomorphism of R–functors. For the injectivity of $\rho(S)$, assume we have $i(\alpha_1)s(\alpha_3) = i(\alpha_1')s(\alpha_3')$. By pushing by f, we get that $\alpha_3 = \alpha_3'$, so that $\alpha_1 = \alpha_1'$. For the surjectivity of $\rho(S)$, we are given $\alpha_2 \in F_2(S)$ and put

$\alpha_3 = f(\alpha_2)$. Then f applies $\alpha_2 s(\alpha_3)^{-1}$ to 1, so that by exactness, there exists $\alpha_1 \in F_1(S)$ such that $i(\alpha_1) = \alpha_2 s(\alpha_3)^{-1}$. This ρ is aan isomorphism of R–functors and we conclude that F_2 is representable by an affine R–scheme.

$\square$

We can define also the cokernel of a R–group functor. But it is very rarely representable. The simplest example is the Kummer morphism $f_n : \mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$, $x \mapsto x^n$ for $n \geq 2$ for $R = \mathbb{C}$, the field of complex numbers. Assume that there exists an affine $\mathbb{C}$–group scheme $\mathfrak{G}$ such that there is a four terms exact sequence of $\mathbb{C}$–functors

$$1 \to h_{\mu_n} \to h_{\mathbb{G}_m} \overset{h_{f_n}}{\to} h_{\mathbb{G}_m} \to h_{\mathfrak{G}} \to 1$$

We denote by T' the parameter for the first $\mathbb{G}_m$ and by $T = (T')^n$ the parameter of the second one. Then $T \in \mathbb{G}_m(R[T, T^{-1}])$ defines a non trivial element of $\mathfrak{G}(R[T, T^{-1}])$ which is trivial in $\mathfrak{G}(R[T', {T'}^{-1}])$ It is a contradiction.

4.0.1. *An example : the semi-direct product.* Let $\mathfrak{G}/R$ be an affine group scheme acting on another affine group scheme $\mathfrak{H}/R$, that is we are given a morphism of R–functors

$$\theta : h_{\mathfrak{G}} \to \mathrm{Aut}(h_{\mathfrak{H}}).$$

The semi-direct product $h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ is well defined as R–functor.

4.0.6. **Lemma.** *$h_{\mathfrak{Y}} \rtimes^\theta h_{\mathfrak{X}}$ is representable by an affine R-scheme.*

Proof. We consider the affine R-scheme $\mathfrak{X} = \mathfrak{H} \times_R \mathfrak{G}$. Then $h_{\mathfrak{X}} = h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ has a group structure so defines a group scheme structure on $\mathfrak{X}$. $\square$

4.0.2. *Extensions by a vector group scheme.* We come back firstly to the example of Witt vectors of length 2 given in the introduction.

4.0.7. **Example.** We work over the prime field $k = \mathbb{F}_p$ and equip the affine plane $\mathbf{A}_k^2$ with the following commutative group law $(x_0, x_1) + (y_0, y_1) = \Big(x_0 + y_0,\, S_1(x_0, y_0)\Big)$ where

$$S_1(x, y) = x_1 + y_1 \, - \sum_{i=1}^{p-1} \frac{i!(p-1)!}{(p-i)!} \, x^i y^{p-i}$$

can be interpreted as $\frac{(x+y)^p - x^p - y^p}{p}$. The projection map $W_2 \to \mathbb{G}_a$, $(x_0, y_0) \mapsto x_0$ is a group homomorphism and we have an exact sequence of k–group funtors

$$0 \to \mathbb{G}_a \overset{i}{\to} W_2 \to \mathbb{G}_a \to 0$$

where $i(t) = (0, t)$. It is no split.

Assume that we are given an action $\theta : \mathfrak{G} \to \mathrm{Aut}(V(M))$ where $V(M)$ is a vector R–group scheme attached to a f.g. projective R–module. In that case, the classification of group functors extensions of $\mathfrak{G}$ by $V(M)$ is quite similar with that of abstract groups. Let $0 \to V(M) \to E \to G \to 1$ be an extension of R–group functors. By lifting the identity $id \in \mathfrak{G}(R[G])$ to $E(R[G])$, we get a section $s : \mathfrak{G} \to E$ and an isomorphism of R–functors $E \xrightarrow{\sim} V(M) \times_R G$ so that E is representable by a R–group scheme. For each S/R, the section s defines a 2–cocycle $\mathfrak{G}(S) \times \mathfrak{G}(S) \to V(M)(S) = M \otimes_R S$. It defines then a class in the Hochschild cohomology group $H^2_0(\mathbf{G}, M)$ with respect of the action. As in the abstract case, it is straightforward to show that $H^2_0(\mathbf{G}, M)$ classifies indeed the isomorphism classes of of group functors extensions of $\mathfrak{G}$ by $V(M)$ with action θ, see [DG, II.3.2.3].

Let us give an important example.

4.0.8. **Example.** If $G = D_R(A)$ is a diagonalizable R–group scheme, Grothendieck has shown that Hochschild's cohomology vanishes in positive weight [SGA3, I.5.3.3]. It follows that group functors extensions of G by $V(M)$ are split.

5. Actions, transporters, centralizers, normalizers

5.1. **Actions.** Let F be a R–functor. We denote by $\underline{\mathrm{Aut}}(F)$ the R–functor in groups of automorphisms of X, that is defined by $\underline{\mathrm{Aut}}(X)(S) = \mathrm{Aut}_{S-functor}(F_S)$. An action of an affine group scheme G/R is a homomorphism of R–functors in groups $\theta : h_G \to \underline{\mathrm{Aut}}(F)$.

5.2. **Transporters, normalizers, centralizers.** If $F_1, F_2 \subset F$ are R–subfunctors, the *transporter* is

$$\underline{\mathrm{Transp}}(F_1, F_2)(S) = \Big\{ g \in G(S), \mid \ \theta(g)(F_1(S')) \subset F_2(S') \, \text{for all } S\text{–rings } S' \Big\}$$

and the *transporter strict* is

$$\underline{\mathrm{Transpst}}(F_1, F_2)(S) = \Big\{ g \in G(S), \mid \ \theta(g) \ \text{ induces a bijection } F_1(S')) \xrightarrow{\sim} F_2(S') \text{ for all } S\text{–rings } S' \Big\}$$

for each R–ring S. Both are R–subfunctors of h_G. If $F_1 = F_2$, $\underline{\mathrm{Transpst}}(F_1, F_1)$ is called also the normalizer (or stabilizer) of F_1 in F and is denoted by $\underline{\mathrm{Norm}}_G(F_1)$.

This is coherent with the usual terminology of group normalizers for the action of G on itself by inner automorphisms. Also the centralizer of F_1 is the R–subfunctor of G defined by

$$\underline{\mathrm{Cent}}_G(F_1)(S) = \Big\{ g \in G(S), \mid \ \theta(g)(f) = f \, \text{for all } S\text{–rings } S' \text{ and for all } f \in F_1(S')) \ \Big\}$$

5.2.1. **Theorem.** [DG, II.1.3.6] *Assume that an affine algebraic k–group acts on a separated k–scheme X of finite type. Let Y, Z be closed k–subschemes of X.*

(1) The k–functors $\underline{\mathrm{Transp}}(Y, Z)$ and $\underline{\mathrm{Transpst}}(Y, Z)$ are represented by closed k–subgroups of G.

(2) The k–functors $\underline{\mathrm{Cent}}_G(Y)$ and $\underline{\mathrm{Norm}}_G(Y)$ are represented by closed k–subgroups of G.

In particular, if H is a closed k-subgroup of G, then the centralizer and the normalizer $\underline{\mathrm{Cent}}_G(H)$ and $\underline{\mathrm{Norm}}_G(H)$ are represented by closed k–subgroups denoted respectively by $\mathrm{Cent}_G(H)$ and $\mathrm{Norm}_G(H)$. It is called the scheme-theoretical centralizer (resp. normalizer). Finally, $\mathrm{Cent}_G(G)$ is called the scheme-theoretical center of G and is denoted by $C(G)$.

5.2.2. **Example.** Assume than k is algebraically closed. In the book of Borel, centralizers and normalizers are considered in the setting of varieties. This centralizer (resp. normalizer) is the reduced subgroup of the scheme theoretical one. We present here an example where the objects are distincts. We assume that k is of characteristic 2 and consider the k–group $G = \mathrm{PGL}_2 = \mathrm{GL}_2/\mathbb{G}_m$ where GL_2 is equipped with the standard coordinates $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation $c = 0$ defines the standard Borel subgroup B of G. We denote by U its unipotent radical. The equation $c^2 = 0$ défines another k–subgroup J of G and we observe that $B = J_{\mathrm{red}}$. We claim that $J \subset N_G(U)$ and in particular that $B \subsetneqq N_G(U)$. In other words, the classical result "$P = N_G(U)$" ([Hu, §30.4, exercice 4]) holds only in the setting of reduced varieties.

We denote by $u_\pm : \mathbb{G}_a \to G$ the standard root groups of G and by α_2 the kernel of the groyp $\mathbb{G}_a \to \mathbb{G}_a$, $t \mapsto t^2$. According to [W93, prop. 4], the morphism of k-schemes

$$\alpha_2 \times_k B \to J,\ (x, b) \mapsto u_-(x)\, b$$

is an isomorphism. It remains then to check that $u_-(\alpha_2) \subseteq N_G(U)$. Let R be a k–algebra, let $x \in \alpha_2(R)$ and $b \in R$; we compute in $\mathrm{GL}_2(R)$

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 - bx & b \\ x - x(bx+1) & 1 + bx \end{pmatrix} = (1+bx)\begin{pmatrix} 1 & b(1+bx) \\ 0 & 1 \end{pmatrix}.$$

It yields that $u_-(\alpha_2) \subseteq N_G(U)$ thus $J \subseteq N_G(U)$.

6. Weil restriction

We are given the following equation $z^2 = 1 + 2i$ in $\mathbb{C}$. A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine schemes and for functors.

We are given a ring extension S/R or $j : R \to S$. Since a S-algebra is a R–algebra, a R-functor F defines a S-functor denoted by F_S and called the scalar extension of F to S. For each S–algebra S', we have $F_S(S') = F(S')$. If X is a R-scheme, we have $(h_X)_S = h_{X \times_R S}$.

Now we consider a S–functor E and define its Weil restriction to S/R denoted by $\prod_{S/R} E$ by

$$\Big(\prod_{S/R} E\Big)(R') = E(R' \otimes_R S)$$

for each R–algebra R'. We note the two following functorial facts.

(I) For a R-map or rings $u : S \to T$, we have a natural map

$$u_* : \prod_{S/R} E \to \prod_{T/R} E_T.$$

(II) For each R'/R, there is natural isomorphism of R'–functors

$$\Big(\prod_{S/R} E\Big)_{R'} \xrightarrow{\sim} \prod_{S\otimes_R R'/R'} E_{S\otimes_R R'}.$$

For other functorial properties, see appendix A.5 of [CGP].

At this stage, it is of interest to discuss the example of vector group functors. Let N be a R–module. We denote by j_*N the scalar restriction of N from S to R [Bbk1, §II.1.13]. The module j_*N is N equipped with the R–module structure induced by the map $j : R \to S$. It satifies the adjunction property $\mathrm{Hom}_R(M, j_*N) \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N)$ (*ibid*, §III.5.2).

6.0.3. **Lemma.** *(1)* $\prod_{S/R} V(N) \xrightarrow{\sim} V(j_*N)$.

(2) If N is f.g. projective and S/R is finite and locally free, then $\prod_{S/R} W(N)$ is representable by the vector group scheme $\mathfrak{W}(j_*N)$.

For a more general statement, see [SGA3, I.6.6]. , finite locally free is equivalent to finite flat of finite presentation [GW, §12.6]; in particular if R is noetherian, finite locally free is equivalent to finite flat.

Proof. (1) For each R-algebra R', we have

$$\Big(\prod_{S/R} W(N)\Big)(R') = W(N)(R' \otimes_R S) = N \otimes_S (R' \otimes_R S) = j_*N \otimes_R R' = W(j_*N)(R').$$

(2) The assumptions implies that j_*N is f.g. over R, hence $W(j_*N)$ is representable by the vector R–group scheme $\mathfrak{W}(j_*N)$. □

If F is a R-functor, we have for each R'/R a natural map

$$\eta_F(R') : F(R') \to F(R' \otimes_R S) = F_S(R' \otimes_R S) = \Big(\prod_{S/R} F_S\Big)(R');$$

it defines a natural mapping of R–functor $\eta_F : F \to \prod\limits_{S/R} F_S$. For each S–functor E, it permits to defines a map

$$\phi : \mathrm{Hom}_{S-functor}(F_S, E) \to \mathrm{Hom}_{R-functor}\big(F, \prod_{S/R} E\big)$$

by applying a S–functor map $g : F_S \to E$ to the composition

$$F \overset{\eta_F}{\to} \prod_{S/R} F_S \quad \overset{\prod\limits_{S/R} g}{\longrightarrow} \quad \prod_{S/R} E.$$

Proof. We apply the compatibility with $R' = S_2 = S$. The map $S \to S \otimes_R S_2$ is split by the codiagonal map $\nabla : S \otimes_R S_2 \to S, s_1 \otimes s_2 \to s_1 s_2$. Then we can consider the map

$$\theta_E : \Big(\prod_{S/R} E\Big)_{S_2} \quad \overset{\sim}{\longrightarrow} \quad \prod_{S \otimes_R S_2/S_2} E_{S \otimes_R S_2} \quad \overset{\nabla_*}{\to} \quad \prod_{S/S} E = E.$$

This map permits to construct the inverse map ψ of ϕ as follows

$$\psi(h) : \, F_S \overset{l_S}{\to} \Big(\prod_{S/R} E\Big)_{S_2} \overset{\theta_E}{\to} \quad E$$

for each $l \in \mathrm{Hom}_{R-functor}\big(F, \prod\limits_{S/R} E\big)$. By construction, the maps ϕ and ψ are inverse of each other. $\square$

In conclusion, the Weil restriction from S to R is then right adjoint to the functor of scalar extension from R to S.

6.0.4. **Proposition.** *Let $\mathfrak{Y}/S$ be an affine scheme of finite type (resp. of finite presentation). Assume than S is finite and locally free over R. Then the functor $\prod_{S/R} h_{\mathfrak{Y}}$ is representable by an affine scheme of finite type (resp. of finite presentation).*

Again, it is a special case of a much more general statement, see [BLR, §7.6].

Proof. Up to localize for the Zariski topology, we can assume that S is free over R, namely $S = \oplus_{i=1,\dots,d} R\,\omega_i$. We see $\mathfrak{Y}$ as a closed subscheme of some affine space $\mathbb{A}^n_S$, that is given by a system of equations $(P_\alpha)_{\alpha \in I}$ with $P_\alpha \in S[t_1, \dots, t_n]$. Then $\prod\limits_{S/R} h_{\mathfrak{Y}}$ is a subfunctor of $\prod\limits_{S/R} W(S^n) \overset{\sim}{\longrightarrow} W(j_*(S^n)) \overset{\sim}{\longrightarrow} W(R^{nd})$ by Lemma 6.0.3. For each I, we write

$$P_\alpha\Big(\sum_{i=1,..,d} y_{1,i}\omega_i, \sum_{i=1,..,d} y_{2,i}\omega_i, \dots, \sum_{i=1,..,d} y_{n,i}\Big) = Q_{\alpha,1}\,\omega_1 + \dots + Q_{\alpha,r}\,\omega_r$$

with $Q_{\alpha,i} \in R\big[y_{k,i}; i=1,..,d; k=1,...,n\big]$. Then for each R'/R, $\Big(\prod\limits_{S/R} h_{\mathfrak{Y}}\Big)(R')$ inside ${R'}^{nd}$ is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine R-scheme $\mathfrak{X}$ of finite type. Furthermore, if $\mathfrak{Y}$ is of finite presentation, we can take I finite so that $\mathfrak{X}$ is of finite presentation too. $\square$

In conclusion, if $\mathfrak{H}/S$ is an affine group scheme of finite type, then the R–group functor $\prod\limits_{S/R} h_{\mathfrak{H}}$ is representable by an R-affine group scheme of finite type. There are two basic examples of Weil restrictions.

(a) The case of a finite separable field extension k'/k (or more generally an étale k-algebra). Given an affine algebraic k'-group G'/k', we associate the affine algebraic k–group $G = \prod\limits_{k'/k} G'$ which is often denoted by $R_{k'/k}(G')$, see [Vo, §3. 12]. In that case, $R_{k'/k}(G') \times_k k_s \xrightarrow{\sim} R_{k'\otimes_k k_s/k_s}(G'_{k'\otimes_k k_s}) \xrightarrow{\sim} \prod\limits_{\gamma} G' \times^{\gamma}_{k'} k_s$ where γ runs over the k–embeddings of $k' \to k_s$. In particular, the dimension of G is $[k':k] \dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(b) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For example the quotient k-group $\Big(\prod\limits_{k[\epsilon]/k} \mathbb{G}_{m,k[\epsilon]}\Big)/\mathbb{G}_m$ is the additive k–group. Also if $p = \mathrm{char}(k) > 0$, $\prod\limits_{k[\epsilon]/k} \mu_{p,k[\epsilon]}$ is of dimension 1.

We list nice formal properties of Weil restriction taken from [CGP, A.5.2].

6.0.5. **Proposition.** *Let S be finite and locally free over R. (1) The formation of the Weil restriction commutes with fiber products: if $f: Y \to X$ and $g: Z \to X$ are S–morphisms then the natural map $\prod\limits_{S/R} \big(Y \times_X Z\big) \to \prod\limits_{S/R} Y \times_{\prod\limits_{S/R} X} \prod\limits_{S/R} Z$ is an isomorphism.*

(2) Let $f: Y \to X$ be a smooth map of affine S–schemes. Then $\prod\limits_{S/R} f : \prod\limits_{S/R} Y \to \prod\limits_{S/R} X$ is smooth and similarly for the property for being étale or an open immersion. In particular, if X is smooth over S, then $\prod\limits_{S/R} X$ is smooth over R.

(3) Let $f: Y \to X$ be a smooth S-map of affine S–schemes. If f is surjective, so is $\prod\limits_{S/R} f : \prod\limits_{S/R} Y \to \prod\limits_{S/R} X$.

6.0.6. **Remarks.** a) It is not true that an open cover U of X provides by Weil restriction an open cover of $\prod\limits_{S/R} X$. Take $R = k$, $S = k \times k$, $X = \mathbf{A}^1_{k\times k}$

and the cover $U_0 = \{t \neq 0\} \times_R S$ and $U_1 = \{t \neq 1\} \times_R S$. Then $\prod_{S/k}(X) \xrightarrow{\sim} \mathbf{A}^1_k \times_k \mathbf{A}^1_k$ and $\prod_{S/k}(U_0) = \mathbb{G}_m \times \mathbb{G}_m$ and $\prod_{S/k}(U_1) = \bigl(\mathbf{A}^1 \setminus \{1\} \times \mathbf{A}^1 \setminus \{1\}\bigr)$, hence the point $(0,1)$ is not in $\prod_{S/k}(U_0) \cup \prod_{S/k}(U_1)$ [CGP, A.5.3].

b) The assertion (3) is false without smoothness hypothesis. We assume that k is of positive characteristic $p > 0$. Put $k = R$ and $S = k[t]/t^2$ and $f : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$, $t \mapsto t^p$. It is surjective but $\prod\limits_{S/R} f : \prod\limits_{S/R} \mathbb{G}_{m,S} \to \prod\limits_{S/R} \mathbb{G}_{m,S}$ is not surjective since $Lie(x \mapsto x^p) : k \to k$ is trivial.

6.0.7. **Remark.** It is natural to ask whether the functor of scalar extension from R to S admits a left adjoint. It is the case and we denote by $\bigsqcup\limits_{S/R} E$ this left adjoint functor, see [DG, §I.1.6]. It is called the Grothendieck restriction.

If $\rho : S \to R$ is a R–ring section of j, it defines a structure R^ρ of S–ring. We have $\bigsqcup\limits_{S/R} E = \bigsqcup\limits_{\rho : S \to R} E(R^\rho)$. If $E = h_{\mathfrak{Y}}$ for a S–scheme $\mathfrak{Y}$, $\bigsqcup\limits_{S/R} \mathfrak{Y}$ is representable by the R–scheme $\mathfrak{Y}$. This is simply the following R-scheme $\mathfrak{Y} \to \mathrm{Spec}(S) \overset{j^*}{\to} \mathrm{Spec}(R)$.

6.1. **The case of a purely inseparable field extension.** Let k' be a purely inseparable extension of k. In some respect, this case behave well.

6.1.1. **Lemma.** [CGP, A.5.4.(2)] *Then* $\prod\limits_{k'/k}$ *commutes with the formation of disjoint unions. If* $(U'_i)_{i \in I}$ *is an affine cover of an affine* k'*–scheme* X'*, then* $\bigl(\prod\limits_{i \in I} U'_i\bigr)_{i \in I}$ *is an affine cover of* $\prod\limits_{i \in I} X'$.

The lemma permits by glueing to define Weil restriction for an arbitrary quasi-compact quasi-separated k–scheme. We have seen that $\prod_{k'/k}$ applies vector k'-groups (resp. affine smooth k'–group) to vector k-groups (resp. affine smooth k–group). It does not apply diagonalizable groups to diagonalizable groups. If $[k' : k] > 1$, then $R_{k'/k}(\mathbb{G}_m)$ is not diagonalizable (not even of multiplicative type) since $R_{k'/k}(\mathbb{G}_m) \times_k k'' \xrightarrow{\sim} R_{k' \otimes_k k''/k''}(\mathbb{G}_m)$ contains a non-trivial additive k–group provided $k' \subset k''$.

7. Tangent spaces and Lie algebras

7.1. **Tangent spaces.** We are given an affine R–scheme $\mathfrak{X} = \mathrm{Spec}(A)$. Given a point $x \in \mathfrak{X}(R)$, it defines an ideal $I(x) = \ker(A \overset{s_x}{\to} R)$ and defines an A–structure on R denoted R^x. We denote by $R[\epsilon] = R[t]/t^2$ the ring of R-dual numbers. We claim that we have a natural exact sequence of pointed set

$$\begin{array}{ccccccc} 1 \longrightarrow & \mathrm{Der}_A(A, R^x) & \xrightarrow{i_x} & \mathfrak{X}(R[\epsilon]) & \longrightarrow & \mathfrak{X}(R) \to 1 \\ & & & || & & \\ & & & \mathrm{Hom}_R\bigl(A, R[\epsilon]\bigr). & & \end{array}$$

where the base points are $x \in \mathfrak{X}(R) \subset \mathfrak{X}(R[\epsilon])$. The map i_x applies a derivation D to the map $f \mapsto s_x(f) + \epsilon\, D(f)$. It is a ring homomorphism since for $f, g \in A$ we have

$$\begin{aligned} i_x(fg) &= s_x(fg) + \epsilon\, D(fg) \\ &= s_x(f)\, s_x(g) + \epsilon\, D(f)\, s_x(g) + \epsilon s_x(f)\, D(g) \qquad \text{[derivation rule]} \\ &= (s_x(f) + \epsilon\, D(f))\,.\,(s_x(g) + \epsilon\, D(g)) \qquad [\epsilon^2 = 0]. \end{aligned}$$

Conversely, one sees that a map $u : A \to R[\epsilon]$, $f \mapsto u(f) = s_x(f) + \epsilon\, v(f)$ is a ring homomorphism iff $v \in \mathrm{Der}_A(A, R^x)$.

7.1.1. **Remark.** The geometric interpretation of $\mathrm{Der}_A(A, R^x)$ is the tangent space at x of the scheme $\mathfrak{X}/R$ (see [Sp, 4.1.3]). Note there is no need of smoothness assumption to deal with that.

We have a natural A-map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \to \mathrm{Der}_A(A, R^x);$$

it maps a A–map $l : I(x)/I^2(x) \to R$ to the derivation $D_l : A \to R$, $f \mapsto D_l(f) = l(f - f(x))$. This map is clearly injective but is split by mapping a derivation $D \in \mathrm{Der}_A(A, R^x)$ to its restriction on $I(x)$. Hence the map above is an isomorphism. Furthermore $I(x)/I^2(x)$ is a R^x–module hence the forgetful map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \xrightarrow{\sim} \mathrm{Hom}_{R-mod}(I(x)/I^2(x), R)$$

is an isomorphism. We conclude that we have the fundamental exact sequence of pointed sets

$$1 \longrightarrow \big(I(x)/I^2(x)\big)^\vee \xrightarrow{i_x} \mathfrak{X}(R[\epsilon]) \longrightarrow \mathfrak{X}(R) \to 1.$$

We record that the R–module structure on $I(x)/I(x)^2$ is induced by the change of variable $\epsilon \mapsto \lambda\, \epsilon$. This construction behaves well with fibred products.

7.1.2. **Lemma.** *Let $\mathfrak{Y} = \mathrm{Spec}(B)$ be an affine R–scheme and $y \in \mathfrak{Y}(R)$. The dual of the R–module map $v : I(x)/I^2(x) \oplus I(y)/I^2(y) \to I(x,y)/I^2(x,y)$ is an isomorphism and fits in the following commutative diagram*

$$\begin{tikzcd} 1 \arrow[r] & \big(I(x)/I^2(x)\big)^\vee \oplus \big(I(y)/I^2(y)\big)^\vee \arrow[r, "i_x \times i_y"] & \mathfrak{X}(R[\epsilon]) \times \mathfrak{Y}(R[\epsilon]) \arrow[r] & \mathfrak{X}(R) \times \mathfrak{Y}(R) \to 1 \\ 1 \arrow[r] & \big(I(x,y)/I^2(x,y)\big)^\vee \arrow[u, "v^\vee", "\cong"'] \arrow[r, "i_{(x,y)}"] & (\mathfrak{X} \times_R \mathfrak{Y})(R[\epsilon]) \arrow[u, equal] \arrow[r] & (\mathfrak{X} \times_R \mathfrak{Y})(R) \to 1. \arrow[u, equal] \end{tikzcd}$$

commutes.

We note that a R-module, $I(x)$ is a direct summand of $R[\mathfrak{X}]$. If we consider a R-ring S, il follows that $I(x) \otimes_R S$ is the kernel of $R[\mathfrak{X}] \overset{s_x \otimes id}{\to} S$.

In conclusion, we have then defined a (split) exact sequence of pointed R–functors

$$1 \longrightarrow \mathfrak{V}(I(x)/I(x)^2) \xrightarrow{i_x} \prod_{R[\epsilon]/R} \mathfrak{X}_{R[\epsilon]} \longrightarrow \mathfrak{X} \longrightarrow 1.$$

If $\mathfrak{X}/R$ is smooth of dimension d (see appendix), $I(x)/I(x)^2$ is locally free of rank d, so that $\mathfrak{V}(I(x)/I(x)^2) = \mathfrak{W}\big(I(x)/I(x)^2\big)$ is a nice vector R–group-scheme.

7.2. **Lie algebras.** Now $\mathfrak{G}/R$ is an affine group scheme. We denote by $\mathrm{Lie}(\mathfrak{G})(R)$ the tangent space at the origin $1 \in \mathfrak{G}(R)$. This is the dual of I/I^2 where $I \subset R[\mathfrak{G}]$ is the kernel of the augmentation ideal. We define the "Lie algebra of $\mathfrak{G}$" vector R–group scheme by

$$\mathrm{Lie}(\mathfrak{G}) = \mathfrak{V}(I/I^2).$$

It fits in the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Lie}(\mathfrak{G})(R) & \longrightarrow & \mathfrak{G}(R[\epsilon]) & \longrightarrow & \mathfrak{G}(R) \to 1 \\ & X & \mapsto & \exp(\epsilon X) & & \end{array}$$

which is a split exact of abstract groups where $\mathrm{Lie}(\mathfrak{G})(R)$ is equipped with the induced group law.

7.2.1. **Lemma.** *That induced group law is the additive law on* $\mathrm{Lie}(\mathfrak{G})(R)$*, namely* $\exp(\epsilon X + \epsilon Y) = \exp(\epsilon X) \,.\, \exp(\epsilon Y)$ *for each* $X, Y \in \mathrm{Lie}(\mathfrak{G})(R)$*.*

Proof. We apply Lemma 7.1.2 and use the product map $m : \mathfrak{G} \times_R \mathfrak{G} \to \mathfrak{G}$ to construct the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \big(I/I^2\big)^\vee \oplus \big(I/I^2\big)^\vee \arrow[r, "\exp \times \exp"] & \mathfrak{G}(R[\epsilon]) \times \mathfrak{G}(R[\epsilon]) \arrow[r] & \mathfrak{G}(R) \times \mathfrak{G}(R) \to 1 \\
1 \arrow[r] & \big(I(\mathfrak{G} \times_R \mathfrak{G})/I(\mathbb{G} \times_R \mathfrak{G})^2\big)^\vee \arrow[u, "v^\vee", "\cong"'] \arrow[r, "\exp_{\mathfrak{G} \times_R \mathfrak{G}}"] \arrow[d, "m_*"] & (\mathfrak{G} \times_R \mathfrak{G})(R[\epsilon]) \arrow[u, equal] \arrow[r] \arrow[d, "m"] & (\mathfrak{G} \times_R \mathfrak{G})(R) \to 1. \arrow[u, equal] \arrow[d, "m"] \\
1 \arrow[r] & \big(I/I^2\big)^\vee \arrow[r, "\exp"] & \mathfrak{G}(R[\epsilon]) \arrow[r] & \mathfrak{G}(R) \to 1.
\end{tikzcd}$$

Since the composite $\mathfrak{G} \overset{id \times \epsilon}{\to} \mathfrak{G} \times_R \mathfrak{G} \overset{m}{\to} \mathbb{G}$ is the identity, the composite map $\big(I/I^2\big)^\vee \overset{id \times 0}{\to} \big(I/I^2\big)^\vee \oplus \big(I/I^2\big)^\vee \to \big(I/I^2\big)^\vee$ is the identity. It is the same for the second summand, so we conclude that that the left vertical composite map is the addition. $\square$

7.2.2. **Remark.** The natural map $\mathrm{Lie}(\mathfrak{G})(R) \otimes_R S \to \mathrm{Lie}(\mathfrak{G})(S)$ is not bijective in general. It is the case if I/I^2 is a projective R–module of finite type, and in particular if $\mathfrak{G}$ is smooth over R. The condition "f.g. projective" is actually necessary for having this property in general, see [DG, II.4.4].

7.2.3. **Example.** Let M be a R-module and consider the R-vector group scheme $\mathfrak{V}(M)$. For each S/R, we have

$$\mathfrak{V}(M)(S[\epsilon]) = \mathrm{Hom}_{S[\epsilon]}(M \otimes_R S[\epsilon], S[\epsilon]) = \mathrm{Hom}_R(M, S[\epsilon]) = \mathfrak{V}(M)^2(S),$$

hence a R-isomorphism $\mathfrak{V}(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{V}(M))(R)$.

The exact sequence defines an action of $\mathfrak{G}(R)$ on $\mathrm{Lie}(\mathfrak{G})(R)$ and actually a representation $\mathrm{Ad} : \mathfrak{G} \to \mathrm{Aut}(\mathrm{Lie}(\mathfrak{G})) = \mathrm{Aut}(\mathfrak{V}(I/I^2))$. It is called the adjoint representation and denoted by Ad.

7.2.4. **Remark.** If I/I^2 is f.g. projective then $\mathfrak{V}(I/I^2) = \mathfrak{W}((I/I^2)^v ee)$ and then $(I/I^2)^v ee$ is a $R - \mathbb{G}$-module.

Denoting by $s : \mathfrak{G}(R) \to \mathfrak{G}(R[\epsilon])$ the section, we have

$$\mathrm{Ad}(g)\, \exp(\epsilon X) = s(g)\, \exp(\epsilon X)\, s(g^{-1}) \in \mathfrak{G}(R[\epsilon]).$$

If $f : \mathfrak{G} \to \mathfrak{H}$ is a morphism of affine R-group schemes, we have a map $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{H})$ of R-vector groups and the commutativity property $f\big(\exp(\epsilon X)\big) = \exp\Big(\epsilon \,.\, \mathrm{Lie}(f)(X)\Big)$.

7.2.5. **Lemma.** *Let M be a f.g. projective R-module and put $\mathfrak{G} = \mathrm{GL}(M)$. Then $\mathrm{End}_R(M) = \mathrm{Lie}(\mathfrak{G})(R)$ and the adjoint action is*

$$\mathrm{Ad}(g) \,.\, X = g\, X\, g^{-1}.$$

Proof. The R–group scheme $\mathfrak{G}$ is open in $W(\mathrm{End}_R(M))$ so that the tangent space at 1 in $\mathfrak{G}$ is the same than in $W(\mathrm{End}_R(M))$. By example 7.2.3, we get then an R–isomorphism $\mathrm{End}_R(M) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{G})(R)$. We perform now the computation of $\mathrm{Ad}(g)\, \exp(\epsilon X)$ in $\mathfrak{G}(R[\epsilon]) \subset \mathrm{End}_R(M) \otimes_R R[\epsilon]$. We have $\mathrm{Ad}(g)\, \exp(\epsilon X) = g\,(Id + \epsilon X)\, g^{-1} = Id + \epsilon g X g^{(1} = \exp(\epsilon\, g\, X\, g^{-1})$. □

More generally, we can define the Lie algebra R–functor of a group R–functor F by putting

$$\mathrm{Lie}(F)(S) = \ker\Big(F(S[\epsilon]) \to F(S)\Big).$$

It is a subgroup equipped with a map $S \times \mathrm{Lie}(F)(S) \to \mathrm{Lie}(F)(S)$ coming from the base change $\epsilon \mapsto \lambda\epsilon$. Also there is an adjoint action of the R–functor F on $\mathrm{Lie}(F)$. In that generality, we are actually mainly interested in the following examples.

7.2.6. **Lemma.** *(1) Let M be a R–module. Then $W(M) \xrightarrow{\sim} \mathrm{Lie}(W(M))$ and $\mathrm{End}_S(M \otimes_R S) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{GL}(W(M))(S)$ for each S/R.*

(2) Let N be a R–module. Then $\mathrm{End}_S\big(V(N)(S)\big) \xrightarrow{\sim} \mathrm{Lie}\big(\mathrm{GL}(V(N)\big)(S)$ for each S/R.

Proof. (1) The first thing is similar as example 7.2.3. For each S/R, we have indeed a split exact sequence of abstract groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{End}_S(M \otimes_R S) & \longrightarrow & \mathrm{GL}(\mathfrak{W}(M))(R[\epsilon]) & \longrightarrow & \mathrm{GL}(M)(R) & \longrightarrow & 1. \\ & & f & \mapsto & Id + \epsilon f & & & & \end{array}$$

(2) The proof is the same. $\square$

We come back to the case of the affine R-group $\mathfrak{G}$. We see the adjoint representation as a morphism of R–group functors

$$\mathrm{Ad} : \mathfrak{G} \to \mathrm{GL}(\mathrm{Lie}(\mathfrak{G})) = \mathrm{GL}\big(\mathfrak{V}(I/I^2)\big)$$

By applying the Lie functor, it induces then a morphism of vector R-group schemes

$$\mathrm{ad} : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}\Big(\mathrm{GL}\big(\mathfrak{V}(I/I^2)\big)\Big).$$

For each S/R, we have then a S–map

$$\mathrm{ad}(S) : \mathrm{Lie}(\mathfrak{G})(S) \to \mathrm{Lie}\Big(\mathrm{GL}\big(\mathfrak{V}(I/I^2)\big)\Big)(S) = \mathrm{End}_S(\mathrm{Lie}(\mathfrak{G})(S)).$$

For each $X, Y \in \mathrm{Lie}(\mathfrak{G})(S)$, we denote by

$$[X, Y] = \mathrm{ad}(S)(X).\ Y \in \mathrm{Lie}(\mathfrak{G})(S)$$

the Lie bracket of X and Y.

7.2.7. **Lemma.** *(1) Let $f : \mathfrak{G} \to \mathfrak{H}$ be a morphism of affine R–group schemes. For each $X, Y \in \mathrm{Lie}(\mathfrak{G})(R)$, we have*

$$\mathrm{Lie}(f)\,.\,[X, Y] = [\mathrm{Lie}(f)\,.\,X,\ \mathrm{Lie}(f)\,.\,Y\,] \in \mathrm{Lie}(\mathfrak{G})(R).$$

(2) In the case $\mathfrak{G} = \mathrm{GL}(M)$ with M f.g. projective, the Lie bracket $\mathrm{End}_R(M) \times \mathrm{End}_R(M) \to \mathrm{End}_R(M)$ reads $[X, Y] = XY - YX$.

Proof. (1) This readily follows from the fact that the map $\mathrm{Lie}(f) : \mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{H})$ is a $\mathfrak{G}$-module morphism where $\mathrm{Lie}(\mathfrak{H})$ where $\mathfrak{G}$ acts on $\mathrm{Lie}(\mathfrak{H})$ by $\mathrm{Ad}_{\mathfrak{H}} \circ f$.

(2) We consider the adjoint representation $\mathrm{Ad}(R) : \mathrm{GL}(M)(R) \to \mathrm{GL}(\mathrm{End}_R(M))(R)$ known to be $\mathrm{Ad}(g).X = gXg^{-1}$. We consider $\mathrm{Ad}(R[\epsilon]) : \mathrm{GL}(M)(R[\epsilon]) \to \mathrm{GL}(\mathrm{End}_R(M))(R[\epsilon])$; for $X, Y \in \mathrm{End}_R(M)$ we compute inside $(\mathrm{End}_R(M))(R[\epsilon])$ using Lemma 7.2.5

$$\begin{aligned} \mathrm{Ad}(R[\epsilon])(\exp(\epsilon X))\,.\,Y &= (1 + \epsilon X)Y(1 + \epsilon X)^{-1} \\ &= (1 + \epsilon X)Y(1 - \epsilon X) \\ &= Y + \epsilon(XY - YX). \end{aligned}$$

We conclude that $[X, Y] = XY - YX$. $\square$

7.2.8. **Proposition.** *The Lie bracket defines a Lie R-algebra structure on the R–module $\mathrm{Lie}(\mathfrak{G})(R)$, that is*

(i) the bracket is R–bilinear and alternating;

(ii) (Jacobi identity) For each $X, Y, Z \in \mathrm{Lie}(\mathfrak{G})(R)$, we have

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

We give here a short non orthodox proof specific to affine group schemes; for a more general setting, see [DG, II.4.4.3] and [SGA3, Exp. II].

Proof. We consider here only the case where $\mathfrak{G}$ admits a faithful representation in $\mathrm{GL}(R^n)$. Then the R–map $\mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathrm{GL}(M))$ is a monomorphism. From Lemma 7.2.7, it is then enough to check it for the linear group GL_n. That case is straightforward, we have $\mathrm{Lie}(\mathrm{GL}_n)(R) = M_n(R)$ and the bracket is $[X, Y] = XY - XY$(lemma 7.2.7). □

7.2.9. **Remark.** If $j : R \to S$ is a finite locally free morphism and $\mathfrak{H}/S$ a group scheme over S, it is a natural question to determine the Lie algebra of $\mathfrak{G}$. It is done in [CGP, A.7.6]. and we have $\mathrm{Lie}(\mathfrak{G}) = j_*\mathrm{Lie}(\mathfrak{H})$, that is $\mathrm{Lie}(\mathfrak{G})(R') = \mathrm{Lie}(\mathfrak{H})(S \otimes_R R')$ for each R'/R.

7.2.10. **Example.** If k is a field of characteristic $p > 0$, $\mathrm{Lie}(\mu_p)(k) = k$ with trivial Lie structure.

Lecture III: Descent techniques, quotients

8. Descent

We start with a classical descent issue.

8.1. **Embedded descent, field of definition.** We are given an affine k-scheme X_0 and are interested to descend closed subschemes defined over a field extension K/k. We say that Z descends to k if there exists a closed k–scheme Z_0 of X_0 such that $Z_0 \times_k K = Z$. If X_0 is an affine space, that means that we can find equations defining Z inside X_K. The first remark is that if Z descends to k, then Z_0 is unique. Assume that we have Z_0, Z_0', then $I_{Z_0} \otimes_k K = I_{Z_0'} \otimes_k K = I_Z$, so that $I_{Z_0} = I_{Z_0'}$ since K/k is faithfully flat.

8.1.1. *The Galois case.* Assume that K/k is Galois (finite or infinite). Then the following are equivalent:

(i) Z descends over k;

(ii) for each $\gamma \in \mathrm{Gal}(K/k)$, $I_Z = \gamma(I_Z)$;

It is obvious than $(i) \Longrightarrow (ii)$. To prove than $(ii) \Longrightarrow (i)$, we can apply Speiser's lemma on Galois descent for vector spaces [GS, 2.3.8]. We put $I_0 = (I_Z)^{\mathrm{Gal}(K/k)}$, it is an ideal of $k[X]$ which satisfies $I_0 \otimes_k K \xrightarrow{\sim} I_Z$.

This shows that a k_s–subscheme Z of X_{k_s} has a field of definition k_Z defined by $\mathrm{Gal}(k_Z/k) = \Big\{ \gamma \in \mathrm{Gal}(k_s/k) \mid \gamma(I_Z) = I_Z \Big\}$.

8.1.2. *Field of definition (cf.* [EGA4, §4.8]*).* Consider a k–vector space V_0 and a K–vector subspace W of $V = V_{0,K}$. Among all k–subfields $F \subseteq K$ such that W arises by scalar extension of a vector F–space of $V_0 \otimes_k F$, we claim there is one such F that is contained in all others. To see that such minimal field F exists, choose a k–basis $(e_i)_{i \in I}$ of V_0 a subset $B = (e_i)_{i \in J}$ projecting to a K–basis of V/W. Then F is generated over k by the coefficients of the vectors $(e_i \, mod \, W)_{i \not\in J}$ relative to the K–basis B of V/W (as follows of [Bbk1, II.8.6, prop. 7.(i)].

We call F the field of definition of the K–subspace W of $V = V_0 \otimes_k K$. This defines a unique F–subspace W' of $V_0 \times_k F$ such that $W' \otimes_F K = W$.

Coming back to Z inside $X_{0,K}$, the field of definition F of I_Z is called the field of definition of Z. It defines a F–vector space I' of $F[X]$ which is an ideal so defines a closed F–subscheme Z'. So Z descends to F and Z descends to k if and only if $k = F$.

8.1.3. *Descent for vector spaces in the inseparable case of height one.* Assuming $p > 1$, we consider a finite algebraic field extension K/k of height ≤ 1, that is satisfying $K^p \subseteq k$. In this case, descent for vector spaces rephrases in some differential calculus. If V is a K–vector space, a connection is a map $\nabla : V \to \Omega^1_{K/k} \otimes_K V$ which satisfies $\nabla(a\, v) = a\nabla(V) + da \otimes v$ for all $a \in K$, $v \in V$, it is called a p-connection if it satisfies some extra property. Then there is an equivalence of categories between the category of vector spaces and the category of K—vector spaces equipped with p–connections [GS, 9.3.6]. In one way, we associate to a k–vector space V_0 the K–vector space $K \otimes_k V$ and the connection $\nabla(a \otimes v) = da \otimes v$. In the way around, we associate to a pair (V, ∇) the k–vector space $V^\nabla = \Big\{ v \in V \mid \nabla(v) = 0 \Big\}$.

8.2. **Faithfully flat descent.** See [BLR, §6], [Br, §2] or [Vi, §4.3.1].

9. Flat sheaves

Our presentation is that of Demazure-Gabriel [DG, III] which involves only rings.

9.1. **Covers.** A fppf (flat for short) cover of the ring R is a ring S/R which is faithfully flat and of finite presentation[2] “fppf” stands for “fidèlement plat de présentation finie”.

9.1.1. **Remarks.** (1) If $1 = f_1 + \cdots + f_s$ is a partition of 1_R with $f_1, ...f_r \in R$, the ring $R_{f_1} \times \cdots \times R_{f_r}$ is a Zariski cover of R and a fortiori a flat cover.

(2) If S_1/R and S_2/R are flat covers of R, then $S_1 \otimes_R S_2$ is a flat cover of R.

(3) If S/R is a flat cover of S and S'/S is a flat cover of S, then S'/R is a flat cover of R.

(4) Finite locally free extensions S/R are flat covers, in particular finite étale surjective maps are flat covers.

9.2. **Definition.** We consider a R-functor $F : \{R - Alg\} \to Sets$. It is called additive if the natural map $F(S_1 \times S_2) \to F(S_1) \times F(S_2)$ is bijective for all R–rings S_1, S_2.

For each R–ring morphism $S \to S'$, we can consider the sequence

$$F(S) \longrightarrow F(S') \overset{d_{1,*}}{\underset{d_{2,*}}{\rightrightarrows}} F(S' \otimes_S S') \quad .$$

A functor of $F : \{R - Alg\} \to Sets$ is a *fppf sheaf* (or flat sheaf) for short if it is additive and if for each R–ring S and each flat cover S'/S, and the

[2] One may consider also not finitely presented covers, it is called *fpqc*, see [SGA3, IV] and [Vi].

sequence

$$\begin{tikzcd} F(S) \arrow[r] & F(S') \arrow[r, shift left, "d_{1,*}"] \arrow[r, shift right, "d_{2,*}"'] & F(S' \otimes_S S') \end{tikzcd}$$

is exact. It means that the restriction map $F(S) \to F(S')$ is injective and its image consists in the sections $\alpha \in F(S')$ satisfying $d_{1,*}(\alpha) = d_{2,*}(\alpha) \in F(S' \otimes_S S')$.

Given a R–module M and S'/S as above, the theorem of faithfully flat descent states that we have an exact sequence of S–modules

$$0 \to M \otimes_R S \to (M \otimes_R S) \otimes_S S' \overset{d_{1,*}-d_{2,*}}{\longrightarrow} (M \otimes_R S) \otimes_S S' \otimes_S S' \quad .$$

This rephases by saying that the vector group functor $V(M)/R$ (which is additive) is a flat sheaf over $\mathrm{Spec}(R)$. A special case is the exactness of the sequence

$$0 \to S \to S' \overset{d_{1,*}-d_{2,*}}{\longrightarrow} S' \otimes_S S'.$$

If N is a R-module, it follows that the sequence of R–modules

$$0 \to \mathrm{Hom}_R(N,S) \to \mathrm{Hom}_R(N,S') \overset{d_{1,*}-d_{2,*}}{\longrightarrow} \mathrm{Hom}_R(N, S' \otimes_S S')$$

is exact. This shows that the vector R–group scheme $\mathfrak{W}(M)$ is a flat sheaf. More generally we have

9.2.1. **Proposition.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the R–functor of points $h_{\mathfrak{X}}$ is a flat sheaf.*

Proof. The functor $h_{\mathfrak{X}}$ is additive. We are given a R–ring S and a flat cover S'/S. We write the sequence above with the R-module $R[\mathfrak{X}]$. It reads

$$0 \to \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S) \to \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S') \overset{d_{1,*}-d_{2,*}}{\to} \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S' \otimes_S S').$$

It follows that $\mathfrak{X}(S)$ injects in $\mathfrak{X}(S')$ and identifies with $\mathrm{Hom}_{R-rings}(R[\mathfrak{X}], S') \cap \mathrm{Hom}_{R-mod}(R[\mathfrak{X}], S)$. Hence the exact sequence

$$\begin{tikzcd} \mathfrak{X}(S) \arrow[r] & \mathfrak{X}(S') \arrow[r, shift left, "d_{1,*}"] \arrow[r, shift right, "d_{2,*}"'] & \mathfrak{X}(S' \otimes_S S') \end{tikzcd} \quad .$$

□

9.2.2. **Remark.** More generally, the proposition holds with a scheme $\mathfrak{X}/R$, see [Ro, 2.4.7] or [Vi, 2.5.4].

9.2.3. **Examples.** If E, F are flat sheaves over R, the R–functor $\mathrm{Hom}(E,F)$ of morphisms from E to F is a flat sheaf. Also the R–functor $\mathrm{Isom}(E,F)$ is a flat sheaf and as special case, the R–functor $\mathrm{Aut}(F)$ is a flat sheaf.

9.3. **Monomorphisms and epimorphisms.** A morphism $u : F \to E$ of flat sheaves over R is a monomorphism if $F(S) \to E(S)$ is injective for each S/R. It is an epimorphism if for each S/R and each element $e \in E(S)$, there exists a flat cover S'/S and an element $f' \in F(S')$ such that $e_{|S'} = u(f')$.

A morphism of flat sheaves which is a monomorphism and and an epimorphism is an isomorphism (exercise, solution [SGA3, IV.4.4]).

We say that a sequence of flat sheaves in groups over R $1 \to F_1 \to F_2 \to F_3 \to 1$ is exact if the map of sheaves $F_2 \to F_3$ is an epimorphism and if for each S/R the sequence of abstract groups $1 \to F_1(S) \to F_2(S) \to F_3(S)$ is exact.

9.3.1. **Examples.** (1) For each $n \geq 1$, the Kummer sequence $1 \to \mu_{n,R} \to \mathbb{G}_{m,R} \overset{f_n}{\to} \mathbb{G}_{m,R} \to 1$ is an exact sequence of flat sheaves where f_n is the n–power map. The only thing to check is the epimorphism property. Let S/R be a ring and $a \in \mathbb{G}_m(S) = S^\times$. We put $S' = S[X]/(X^n - a)$, it is finite free over S, hence is faithfully flat of finite presentation. Then $f_n(X) = a_{|S'}$ and we conclude that f_n is an epimorphism of flat sheaves.
(2) More generally, let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of f.g. abelian groups. Then the sequence of R–group schemes

$$1 \to \mathfrak{D}(A_3) \to \mathfrak{D}(A_2) \to \mathfrak{D}(A_1) \to 1$$

is exact.

9.4. **Sheafification.** Given a R-functor F, there is natural way to sheafify it in a flat functor $\widetilde{F}$. The first thing is to make the functor additive. For each Zariski cover $(S_j)_{j \in J}$ (J–finite), we have a map

$$F(S) \to \prod_{j \in J} F(S_j)$$

We define

$$F_{add}(S) = \prod_{j \in J} F(S_j)$$

where the limit is taken on Zariski covers of S. By construction, F_{add} is an additive functor and there is a natural map $F \to F_{add}$

Now, for each S/R, we consider the "set" $\mathrm{Cov}(S)$ of flat covers[3]. Also if $f : S_1 \to S_2$ is an arbitrary R-ring map, the tensor product defines a natural map $f_* : \mathrm{Cov}(S_1) \to \mathrm{Cov}(S_2)$. We define then

$$\widetilde{F}(S) = \varinjlim_{I \subset \mathrm{Cov}(S)} \ker\Big(\quad \prod_{i \in I} F_{add}(S_i) \quad \overset{d_{1,*}}{\underset{d_{2,*}}{\rightrightarrows}} \quad F_{add}(S_i \otimes_S S_j) \Big)$$

where the limit is taken on finite subsets I of $\mathrm{Cov}(S)$. It is a R-functor since each map $f : S_1 \to S_2$ defines $f_* : \widetilde{F}(S_1) \to \widetilde{F}(S_2)$. We have also a natural mapping $u_F : F \to F_{add} \to \widetilde{F}$.

[3]We do not enter in set-theoric considerations but the reader can check there is no problem there.

9.4.1. **Proposition.** *(1) For each R–functor F, the R–functor $\widetilde{F}$ is a flat sheave.*

(2) The functor $F \to \widetilde{F}$ is left adjoint to the forgetful functor applying a flat sheaf to its underlying R–functor. For each R–functor F and each flat sheaf E, the natural map

$$\mathrm{Hom}_{flat\,sheaves}(\widetilde{F}, E) \xrightarrow{\sim} \mathrm{Hom}_{R-functor}(F, E)$$

(applying a morphism $u : \widetilde{F} \to E$ to the composite $F \to \widetilde{F} \to E$) is bijective.

(1) follows essentially by construction [DG, III.1.8]. Note that in this reference, the two steps are gathered in one. For (2) one needs to define the inverse mapping. Observe that the sheafification of E is itself, so that the sheafification of $F \to E$ yields a natural morphism $\widetilde{F} \to E$.

Given a morphism of flat R-sheaves $f : E \to F$, we can sheafify the functors

$$S \mapsto E(S)/R_f(S), \;\; S \mapsto \mathrm{Im}\big(E(S) \to F(S)\big),$$

where $R_f(S)$ is the equivalence relation defined by $f(S)$. We denote by $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ their respective sheafifications. We have an induced mapping

$$f_* : \mathrm{Coim}(f) \to \mathrm{Im}(f)$$

between the coimage sheaf and the image sheaf. We say that f is strict when f_* is an isomorphism of flat sheaves.

9.4.2. **Lemma.** *If f is a monomorphism (resp. an epimorphism), then f is strict.*

In the first case, we have $E \xrightarrow{\sim} \mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$; in the second case, we have $\mathrm{coker}(f) \xrightarrow{\sim} \mathrm{Im}(f) \xrightarrow{\sim} F$, see [DG, III.1.2].

9.5. **Group actions, quotients sheaves.** Let G be a R–group flat sheaf and let F be a flat sheaf equipped with a right action of G. The quotient functor is $Q(S) = F(S)/G(S)$ and its sheafification is denoted by F/G. It is called the quotient sheaf[4].

When G and F are representable, the natural question is to investigate whether the quotient sheaf Q is representable. It is quite rarely the case. A first evidence to that is the following fact.

9.5.1. **Proposition.** *We are given an affine group scheme $\mathfrak{G}$ and a monomorphism $\mathfrak{G} \to \mathfrak{H}$ into an affine group scheme. Assume that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by a R–scheme $\mathfrak{X}$. We denote by $p : \mathfrak{H} \to \mathfrak{X}$ the quotient map and by $\epsilon_X = p(1_{\mathfrak{G}}) \in \mathfrak{X}(R)$.*

(1) The R–map $\mathfrak{H} \times_R \mathfrak{G} \to \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$ is an isomorphism.

[4]One can work in a larger setting, that of equivalence relations and groupoids, see [DG, §III.2].

(2) The diagram

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "i"] \arrow[d] & \mathfrak{H} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "\epsilon_{\mathfrak{X}}"] & \mathfrak{X} \end{tikzcd}$$

is carthesian.

(3) The map i is an immersion. It is a closed immersion iff $\mathfrak{X}/R$ is separated.

(4) p is an affine morphism.

(5) $\mathfrak{G}/R$ is flat iff p is flat.

(6) $\mathfrak{G}/R$ is smooth iff p is smooth.

The general statement is [SGA3, VI_B.9.2].

Proof. (1) The map $\mathfrak{H} \times_R \mathfrak{G} \to \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H}$ is a monomorphism. Let us show that it is an epimorphism of flat sheaves. We are given S/R and $(h_1, h_2) \in \mathfrak{H}(S)^2$ such that $p(h_1) = p(h_2)$. There exists a flat cover S'/S and $g \in \mathfrak{G}(S')$ such that $h_{1|S'} = h_{2|S'}\, g$. Hence $g \in \mathfrak{G}(S') \cap \mathfrak{H}(S)$. Since i is a monomorphism, we conclude by descent that $g \in \mathfrak{G}(S)$ whence (h_1, h_2) comes from (h_1, g).

(2) It follows that the following diagram

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "1_{\mathfrak{H}} \times id"] \arrow[d] & \mathfrak{H} \times_R \mathfrak{G} \cong \mathfrak{H} \times_{\mathfrak{X}} \mathfrak{H} \arrow[r] \arrow[d, "p_1"] & \mathfrak{H} \arrow[d, "p"] \\ \mathrm{Spec}(R) \arrow[r, "1_{\mathfrak{H}}"] & \mathfrak{H} \arrow[r, "p"] & \mathfrak{X} \end{tikzcd}$$

is carthesian as desired.

(3) If $\mathfrak{X}$ is separated, $\epsilon_{\mathfrak{X}}$ is a closed immersion and so is i.

(4) By faithfully flat descent with respect to $\mathfrak{H} \to \mathfrak{X}$, it follows from (1) that $\mathfrak{X}$ is affine over R [EGA4, IV_2.2.7.1.(xiii)].

(5) and (6) If p is flat (resp. smooth), so is i by base change. $\square$

One very known case of representatiblity result is the following.

9.5.2. **Theorem.** *Let k be a field. Let H/k be an affine algebraic group and G/k be a closed subgroup. Then the quotient sheaf H/G is representable by a k–scheme of finite type X. Furthermore, if H is normal in G, G/H is affine and canonically equipped with a group struture and we have an exact sequence of k–groups $1 \to H \to G \to G/H \to 1$.*

9.5.3. **Remark.** A natural question is to determine the rational points of H/G. The first thing is to remark that $G(\overline{k})$ acts transitively on $(G/H)(\overline{k})$, so that we have a bijection $G(\overline{k})/H(\overline{k}) \xrightarrow{\sim} (G/H)(\overline{k})$. Then

$$(G/H)(k) = \Big\{ [g] \in G(\overline{k})/H(\overline{k}) \mid d_{1,*}(g) H(\overline{k} \otimes_k \overline{k}) = d_{2,*}(g) H(\overline{k} \otimes_k \overline{k}) \Big\}$$

where the identity holds inside $G(\overline{k} \otimes_k \overline{k})$. If k is perfect, it is nothing but the Galois fact $(G/H)(k) = \big((G(k_s)/H(k_s)\big)^{\mathrm{Gal}(k_s/k)}$.

The first part of Theorem 9.5.2 uses the following important representability statement.

9.5.4. **Proposition.** [DG, III.3.5.2] *Let G acts on a quasi-projective k–scheme X. Let $x \in X(k)$ and denote by $G_x = \mathrm{Cent}_G(x)$ the stabilizer of x.*

(1) The quotient G/G_x is representable by a quasi-projective k-variety.

(2) The orbit map induces an immersion $G/G_x \to X$.

The homeomorphic image of the immersion $G/G_x \to X$ is called the orbit of x with respect to G.

9.5.5. **Remark.** The above can be suitably generalized over rings, see [SGA3, XVI.2], by means of the theorem of Grothendieck-Murre.

Sketch of proof: We assume firstly that G is smooth, that is absolutely reduced. By faithfully flat descent, one can assume that k is algebraically closed.

(1) We know denote by X_0 the reduced subscheme of the schematic image of f_x. Since G is smooth, it is (absolutely) reduced and acts then on X_0. We know that the $X_0 \setminus G.x$ consists in orbits of smaller dimensions so that $G.x$ is an open subset of X_0. We denote it by U_x. We claim that the map $h_x : G \to U_x$ is faithfully flat. The theorem of generic flatness [DG, I.3.3.7] shows that the flat locus of h_x is not empty. By homogeneity, it is U_x, hence h_x is faithfully flat. Let us show now that it implies that U_x represents the orbit of x. The morphism $h_x : G \to U_x$ gives rise to a morphism of k–sheaf $h_x^\dagger : G/G_x \to U_x$. Since the map $h_x : G \to U_x$ is faithfully flat, the morphism h_x is an epimorphism of flat sheaves[5].

$h_x^\dagger$ is a monomorphism. Let S be a R–ring and let $y_1, y_2 \in (G/G_x)(S)$ having same image u in $U_x(S)$. There exists a flat cover S'/S such that y_1 (resp. y_2) comes from some $g_1 \in G(S')$ (resp. g_2). Then $g_1 \,.\, x = g_2 \,.\, x \in U_x(S')$ so that $g_2^{-1}\, g_1 \in G_x(S')$. Thus $u_1 = u_2 \in (G/G_x)(S)$.

(2) By construction, U_x is locally closed in X.

We come back to the general case. Let k'/k be a finite inseparable field extension such that the $\overline{k}$–group $(G_{\overline{k}})_{\mathrm{red}}$ is defined over k', that is there exists a k'–group G' such that $G' \times'_k \overline{k} = (G_{\overline{k}})_{\mathrm{red}}$. Then G' is geometrically reduced and is smooth over k'. From the first case, G'/G'_x is representable by a quasi-projective k'-scheme and the map $G'/G'_x \to X_{k'}$ is an immersion. But $G'_x = (G_x)_{k'} \cap G'$ and by [DG, lemme III.3.5.1], the representability of G'/G'_x is equivalent to that of $G_{k'}/G_{x,k'}$. By faithfully flat descent, we have

[5] We are given S/R and a point $u \in U_x(S)$. Then $h_x^{-1}(u) = \mathrm{Spec}(S')$ is a flat cover of S and there is a point $v \in G(S)$ mapping to u.

that G/G_x is representable by a quasi-projective k–scheme. To establish that $i : G/G_x \to X$ is an immersion, we can assume that k is algebraically closed. Since i is a monomorphism, we know that there exists a dense open subset U of X such that $i^{-1}(U) \to U$ is an immersion [DG, I.3.4.7] (or [GW, ex. 10.31]). But the $gi^{-1}(U)$ for $g \in G(k)$ cover G/G_x, so we conclude that i is an immersion. □

The fisrt part of Theorem 9.5.2 follows then of the fact that G admits a representation V such that there exists a point $x \in \mathbf{P}(V)(k)$ such that $G = H_x$ [DG, II.2.3.5]. Now, if H is normal, G/H is then a flat sheaf in R–groups then a R–functor in groups. What is much harder to establish is the fact that te quotient k–group G/H is affine, see [DG, II.3.5.6].

9.5.6. **Remark.** One interest of the Chevalley quotient is the fact it is universal. That is for each k–algebra R, $(H/G) \times_k R$ represents the quotient R–sheaf $(H \times_k R)/(G \times_k R)$. It can use as follows (see [CTS2, 6.12]). Assume we are given a closed immersion $\iota : \mathfrak{G} \to \mathfrak{H}$ of R-group schemes, a flat cover R'/R and a commutative square

$$\begin{array}{ccc} \mathfrak{G} \times_R R' & \xrightarrow{\iota_{R'}} & \mathfrak{H} \times_R R' \\ {\scriptstyle u}\downarrow{\scriptstyle \cong} & & {\scriptstyle v}\downarrow{\scriptstyle \cong} \\ G \times_k R' & \xrightarrow{i \times_k R'} & H \times_k R' \end{array}$$

where u, v are isomorphisms. We claim then that the quotient sheaf $\mathfrak{H}/\mathfrak{G}$ is representable by a R-scheme. According to Theorem 9.5.2, $\mathfrak{H}_{R'}/\mathfrak{G}_{R'}$ is representable by a quasi-projective R'-scheme which indeed descends to R.

9.5.7. **Examples.** (a) For each n, we have an exact sequence $1 \to \mu_n \to \mathbb{G}_m \overset{f_n}{\to} \mathbb{G}_m \to 1$ where $f_n(t) = t^n$.

(b) The projective linear group over k is $\mathrm{PGL}_n = \mathrm{GL}_n/\mathbb{G}_m$, that is the quotient of the linear group by its center.

9.6. **k-orbits.** We remind the definition of k–orbits [BLR, §10.2].

9.6.1. **Definition.** *Let G/k be an algebraic group acting on a k–scheme X of finite type. A k–orbit under G over X is a locally closed k–subscheme Y of X which is G-stable and such that there exists a finite field extension k'/k and a point $x \in X(k')$ such that $Y_{k'}$ is the orbit of x under $G_{k'}$.*

In particular, the G–orbit of a k-point of X defines a k–orbit under G over X.

9.6.2. **Lemma.** *Under the conditions of Definition 9.6.1, the following assertions hold:*

(1) In the case of a trivial action, a k–orbit over X is nothing but a k–point of X.

(2) Let Y be locally closed k–subscheme of X which is G-stable.

(a) The following are equivalent:

(i) Y is a k–orbit under G over X;

(ii) $Y_{\overline{k}}$ is a $G_{\overline{k}}$–orbit over $X_{\overline{k}}$;

(iii) $Y_{\overline{k}}$ is a $\overline{k}$–orbit under $G_{\overline{k}}$ over $X_{\overline{k}}$.

(b) Let L/k be a finite field extension. Then Y is a k–orbit on X under G if and only if Y_L is a L–orbit on X_L under G_L.

(3) Let $f : H \to G$ be a faithfully flat morphism of k–groups. Let Y be a locally closed k–subscheme of X. Then Y is a k–orbit on X under G if and only if Y is k–orbit on X under H (for the pull-back action of H on X).

9.6.3. **Lemma.** *Under the conditions of Definition 9.6.1, the following assertions hold:*

(1) Let Z be a k–orbit over X for G and let X^+ be another k–scheme of finite type equipped with an action of G. If $u : X \to X^+$ is a G–equivariant morphism, then there exists a unique k-orbit Z^+ under G over X^+ such that the map $Z \to X \to X^+$ factorizes within Z^+. Furthermore the map $Z \to Z^+$ is faithfully flat.

(2) Let $f : H \to G$ is a morphism of k–groups. Let Z be a k–orbit on X under H (for the pull-back action of H on X). Then Z is included in a unique k-orbit on X under G.

9.7. **Frobeniuseries (see** [SGA3, VII$_A$.4]**.** We assume here that $p > 1$. We denote by $F : k \to k$ the Frobenius and for each k-scheme X, we denote by $X^{(p)} = X \times_k^F$ the relative frobenius. For each k-algebra A, we have $X^{(p)}(A) = X(A^{(p)})$ where $A^{(p)} = A$ is equipped with the k–algebra structure $(c, a) \mapsto c^p a$. Since we have a k–morphism $A \to A^{(p)}$, $a \mapsto a^p$ for all k–algebra A, we get a natural k–morphism $Fr : X \to X^{(p)}$. If G is an affine algebraic k–group, $G^{(p)}$ is equipped with a natural structure of algebraic k–group and $Fr : G \to G^{(p)}$ is a homomorphism of k–groups. We can iterate the construction and define $Fr^n : G \to G^{(p^n)}$ for each ≥ 0.

We denote by ${}_{Fr^n}G$ the kernel of $Fr^n : G \to G^{(p^n)}$, it is called the Frobenius kernel of G. From the case of GL_d, it is easy to show that ${}_{Fr^n}G$ is a finite k–subgroup of G. In particular, we get a closed immersion $G/{}_{Fr^n}G \to G^{(p^n)}$.

9.7.1. **Proposition.** [SGA3, VII$_A$.8.3] *There exists an integer $n_0 \geq 0$ such that $G/{}_{Fr^n}G$ is smooth.*

In other words, an affine algebraic k–group is an extension of a smooth k–group by a finite k–group.

Lecture IV: Unipotent radicals, Levi subgroups

10. Solvable and unipotent groups

10.1. **Unipotent groups.**

10.1.1. **Definition.** *An affine algebraic k–group U is* unipotent *if $U_{\overline{k}}$ admits a finite composition serie over $\overline{k}$ with each successive quotient isomorphic to a $\overline{k}$–subgroup of $\mathbb{G}_{a,\overline{k}}$.*

In characteristic zero, $\mathbb{G}_{a,\overline{k}}$ has no proper $\overline{k}$–subgroup except $\{0\}$, so in this case, each successive quotient of the composition serie is isomorphic to a $\overline{k}$–subgroup of $\mathbb{G}_{a,\overline{k}}$. In characteristic $p>0$, the non-trivial proper k–subgroups of $\mathbb{G}_{a,\overline{k}}$ are given as the locus of a p-polynomial $a_0 + a_1 x^p + \cdots + a_r x^{p^r}$ [DG, IV.2.1]. For example α_p^n and $\mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 1$; since each such a $\overline{k}$–group admits a composition serie with each successive quotient isomorphic to α_p (resp. $\mathbb{Z}/p\mathbb{Z}$), we can assume that $U_{\overline{k}}$ has a composition serie with each successive quotient isomorphic to α_p, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{G}_{a,\overline{k}}$.

10.1.2. **Lemma.** *(1) An extension of an unipotent k–group by another unipotent k–group is unipotent.*

(2) A closed k-subgroup (resp. a quotient) of a k–unipotent k–subgroup is unipotent.

(3) Let U be a unipotent k–group. Then $\mathrm{Hom}_{k,gp}(U, \mathbb{G}_m) = 0$.

Proof. (1) follows of the definition and (2) is left to the reader.
(3) We can assume than k is algebraically closed. By induction on the length of a composition serie of U, we have only to consider the case $\mathbb{G}_a$ and if $p>1$, the cases of α_p, $\mathbb{Z}/p\mathbb{Z}$. Each case is easy by considering respective Hopf algebras. □

The commutative unipotent k–groups are very interesting objects. We have seen in the introduction the case of Witt vectors of length 2 and Tits example of a k–form G of $\mathbb{G}_a$, that is $G_{\overline{k}} \cong \mathbb{G}_{a,\overline{k}}$.

We have other nice characterizations of unipotent k–groups.

10.1.3. **Proposition.** *(1)* [SGA3, XVII.3.5] *(1) Let G be an affine algebraic k–group. The following are equivalent:*

1. *G is unipotent;*
2. *G can be embedded in some k–group of upper triangular matrices;*
3. *For each linear representation V of G, we have $H^0(G,V) \neq 0$.*

(2) [SGA3, XVII.4.11] *Let G be a smooth connected k–group. The following are equivalent:*

1. *G is unipotent;*
2. *G admits a central composition serie such that the successive quotients are k–forms of $(\mathbb{G}_a)^{n_i}$ (that is isomorphic over $\overline{k}$ to $(\mathbb{G}_{a,\overline{k}})^{n_i}$).*

In the other hand, we know that the k_s/k–forms of the group $\mathbb{G}_a^n$ are trivial [SGA3, XVII.4.1.5] whence the next fact.

10.1.4. **Corollary.** *Over a perfect field k, a smooth connected unipotent k–group is k–split, that is successive extensions by $\mathbb{G}_a$.*

A nice consequence of the characterization of unipotent k–groups by the fixed point property 10.1.3 is the following.

10.1.5. **Lemma.** *(Rosenlicht lemma)* [SGA3, XVII.5.7.3] *Let G be an unipotent k–group acting on a quasi–affine k–scheme of finite type. Then for each point $x \in X(k)$, the map $G/G_x \to G$ is a closed immersion.*

10.2. **Structure of commutative algebraic groups.**

10.2.1. **Definition.** *An affine algebraic k-group G is of multiplicative type if $G_{\overline{k}}$ is diagonalizable. It is called a k–torus if $G_{\overline{k}}$ is isomorphic to $\mathbb{G}^r_{m,\overline{k}}$.*

10.2.2. **Theorem.** *Let G be a commutative affine algebraic k–group.*

(1) [DG, IV.1.2.2] *The following are equivalent;*

(1) *G is of multiplicative type;*
(2) $\mathrm{Hom}_{\overline{k}-gp}(G_{\overline{k}}, \mathbb{G}_{a,\overline{k}}) = 0$*;*
(3) *For each unipotent $\overline{k}$–group H,* $\mathrm{Hom}_{\overline{k}-gp}(G_{\overline{k}}, H) = 0$*;*
(4) *For each unipotent k–group H,* $\mathrm{Hom}_{k-gp}(G, H) = 0$*;*
(5) $\mathrm{Hom}_{k-gp}(G, \mathbb{G}_a) = 0$.

(2) [DG, IV.3.3.1] *G admits a largest k–subgroup of multiplicative type G^m (that is diagonalizable over $\overline{k}$) and G/G^m is unipotent.*

(3) If k is perfect, the extension $0 \to G^m \to G \to G/G^m \to 0$ is split.

(4) [DG, V.1.2.5] *If k is perfect and G is unipotent, then there exists a morphism $f : \mathbb{G}_a^n \to \mathbb{G}_a^r$ such that G is isomorphic to* $\ker(f)$.

The way to pass from (2) to (3) is to define G^m as the intersection of the kernels if all homomorphisms from G to a commutative unipotent k–group. Note that (3) is also a consequence of the next fact that the geometric unipotent radical is defined over the perfect closure of the field k (see lemma 11.0.5).

10.2.3. **Exercise.** We consider a purely inseparable field extension k'/k and $G = \prod_{k'/k}(\mathbb{G}_{m,k'})$. Show that $\mathbb{G}_m = G^m$ and that the sequence $0 \to G^m \to G \to G/G^m \to 0$ is not split but splits after extension to k'.

10.3. **Solvable k–groups.**

10.3.1. **Definition.** *Let G be an affine algebraic k–group. The k–group G is solvable (resp. k–solvable) if G admits a composition serie whose successive quotients are commutative affine algebraic k–groups (resp. $\mathbb{G}_a$ or $\mathbb{G}_m$).*

Solvable and k–solvable groups behave well by extension. A k–subgroup an a quotient of a solvable k–group is solvable.

10.3.2. **Lemma.** *Let G be an affine algebraic k–group and let K/k be a field extension. Then G is solvable if and only if G_K is solvable.*

11. Unipotent radicals and Levi subgroups

11.0.3. **Lemma.** *Let G be an smooth connected affine algebraic k–group.*

(1) G admits a largest smooth connected unipotent normal k–subgroup $\mathcal{R}_{u,k}(G)$. Furthermore $\mathcal{R}_{u,k}\big(G/\mathcal{R}_{u,k}(G)\big) = 1$.

(2) G admits a largest k-split unipotent normal k–subgroup $\mathcal{R}_{us,k}(G)$. Furthermore $\mathcal{R}_{us,k}\big(G/\mathcal{R}_{us,k}(G)\big) = 1$.

Proof. (1) If U is a unique maximal smooth connected unipotent normal k–subgroup of G, then G/U has no non-trivial smooth connected unipotent normal since the pull-back of such a subgroup would contradict the maximality of U. This remark shows in particular the last point.

Unicity: We are given two maximal smooth connected unipotent normal k–subgroup U_1, U_2 of G. By Lemma 10.1.2, $U_2/(U_1 \cap U_2)$ is a unipotent k–subgroup of G/U_1, it is smooth connected so $U_2 = U_1 \cap U_2$. Hence $U_2 \subset U_1$ and by symmetry we conclude that $U_1 = U_2$.

Existence: It is obvious by noetherian reasons.

The proof of (2) is verbatim. □

11.0.4. **Lemma.** *Let G be an affine algebraic k–group.*

(1) G admits a unique maximal smooth connected normal solvable k–subgroup $\mathcal{R}_k(G)$. Furthermore, we have $\mathcal{R}_k\big(G/\mathcal{R}_k(G)\big) = 1$.

(1) G admits a unique maximal smooth connected normal k–solvable k–subgroup $\mathcal{R}_{s,k}(G)$. Furthermore, we have $\mathcal{R}_{s,k}\big(G/\mathcal{R}_{s,k}(G)\big) = 1$.

The proof is similar to that of 11.0.3. The group $\mathcal{R}_{u,k}(G)$ is called the *k–unipotent radical* of G and

11.0.5. **Proposition.** [CGP, 1.1.9] *The formation of $\mathcal{R}_{u,k}(G)$, $\mathcal{R}_{us,k}(G)$ and $\mathcal{R}_k(G)$ commutes with separable field extensions. In particular $\mathcal{R}_u(G_{\overline{k}}) = \mathcal{R}_{u,split}(G_{\overline{k}})$ (and $\mathcal{R}(G_{\overline{k}})$) descend to the perfect closure of k.*

11.0.6. **Remark.** It is not true that the formation of those radicals commute with inseparable extensions. We come back to our favorite example of $G = R_{k'/k}(\mathbb{G}_{m,k'})$ with k'/k a non-trivial purely inseparable extension. We claim that $\mathcal{R}_{u,k}(G) = 0$. Let U be an unipotent k-subgroup of G. By adjunction, we have $\mathrm{Hom}_{k-gp}(U, G) \xrightarrow{\sim} \mathrm{Hom}_{k'-gp}(U_{k'}, \mathbb{G}_{m,k'})$ which is zero by Lemma 10.1.2. In the other hand, the exercise 10.2.3 shows that the k'–unipotent radical is of dimension $[k' : k] - 1$ so is not defined over k.

11.0.7. **Definition.** *Let G be a smooth connected affine algebraic k–group. If $\mathcal{R}_u(G_{\overline{k}})$ (resp. $\mathcal{R}(G_{\overline{k}}) = 1$), we say that G is reductive (resp. semisimple).*

If $\mathcal{R}_{u,k}(G) = 1$, we say that G is pseudo-reductive.

11.0.8. **Remarks.** (a) Reductivity and semisimplicity are insensitive to field extensions.

(b) There is a notion of pseudo-semisimple k–group but which is not related to the radical. We say that a pseudo-reductive k–group is pseudo-semisimple is G is perfect, that is $DG = G$.

11.0.9. **Definition.** *Let G be a smooth algebraic k–group. A Levi subgroup L of G is a k–subgroup such that the $\overline{k}$–morphism $L_{\overline{k}} \to G_{\overline{k}} \to G_{\overline{k}}/\mathcal{R}_u(G_{\overline{k}})$ is an isomorphism.*

It implies that $G_{\overline{k}}$ is the semi-direct product of $L_{\overline{k}}$ by $\mathcal{R}_u(G_{\overline{k}})$.

If the geometric unipotent radical $\mathcal{R}_u(G_{\overline{k}})$ is defined over k, it is to say that $L \to G/\mathcal{R}_{u,k}(G)$ is an isomorphism.

In characteristic zero, Levi subgroups always exist (and are $G(k)$-conjugated), it is a theorem of Mostow [Mo], see [De] for a cohomological proof. In positive characteristic, even in the algebraically closed case, it is a delicate issue to decide whether Levi subgroup exist, see [MN1, MN2]. A nice case is when dealing with quotient which are commutative reductive, namely tori.

11.0.10. **Theorem.** [SGA3, XVII.3.11] *Let G be an extension of a k–torus T (i.e. a k–form of $\mathbb{G}_m^r$) by a smooth unipotent subgroup U. Then $G \xrightarrow{\sim} U \rtimes T$.*

Next is a nice useful case of existence of Levi subgroup (note that example of remark 11.0.6 is a special case).

11.0.11. **Proposition.** [CGP, A.5.16] *Let k'/k be a finite field purely inseparable extension. Let G be a reductive k–group and put $H = R_{k'/k}(G_{k'})$. Then $G \hookrightarrow H$ is a Levi subgroup.*

We sketch now a counterexample which is fully explained in [CGP, A.5.6]. The Greenberg functor permits to contruct an extension H of SL_2 over $\mathbb{F}_p$ by a vector $\mathbb{F}_p$–group U such that $H(\mathbb{F}_p) = \mathrm{SL}_2(\mathbb{Z}/p^2\mathbb{Z})$ and the map $H(\mathbb{F}_p) \to \mathrm{SL}_2(\mathbb{F}_p)$ is the reduction map. We claim that the sequence $0 \to U \to H \to \mathrm{SL}_2 \to 1$ is not split. The thing is that the mapping of abstract groups $\mathrm{SL}_2(\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_p)$ does not split.

12. List of structure results

We assume in this section that k is of characteristic $p > 0$.

12.1. Tits structure theorems for smooth solvable k–groups.

The statements are taken from [T] and [C2].

12.1.1. **Definition.** *We say that an affine algebraic k–group is k–wound if $G(k) = G\big(k[t^{\pm 1}]\big)$.*

We may ask also the stronger condition $G(k[[t]]) = G\big(k((t))\big)$ and we get the same family of solvable k–groups. The following statement clarifies the structure of solvable groups (and can be precised further).

12.1.2. **Theorem.** *Let G be a smooth connected solvable affine k-group.*

(1) Then $G/\mathcal{R}_{k,split}(G)$ is a k–wound smooth connected solvable k–group.

(2) Assume that G is k–wound. Then G is a central extension of a k–wound smooth connected unipotent subgroup by an anisotropic k–torus.

(3) Assume that G is unipotent and k–wound. Then G admits a composition serie $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ by normal subgroups such that G_{i+1}/G_i is smooth connected commutative unipotent and of p–torsion.

12.2. **On the field of definition of the solvable radical.**

12.2.1. **Proposition.** [CGP, 5.3.3] *Let G be a perfect smooth connected affine k–group. The maximal semisimple quotient of $G_{\overline{k}}$ and maximal semisimple adjoint quotient of $G_{\overline{k}}$ have the same field of definition over k. In particular, if the solvable radical of $G_{\overline{k}}$ is defined over k, then the maximal adjoint quotient of $G_{\overline{k}}$ is defined over k.*

13. Greenberg's functor

Many interesting group extensions arise from the Greenberg functor which has the flavour of the Weil restriction. We recall here basic facts, see the references [Gb], [M2, §III.4], [BLR], [B].

We are given a complete discrete valuation field K of valuation ring $O = O_K$ and of perfect residue field $k = O/\pi O$ of positive charactteristic p. Here $\pi \in O$ is a uniformizer. We denote by e_0 the absolute ramification index of O, i.e. $p = u\pi^{e_0}$ for a unit $u \in O$. We denote by O^{sh} the strict henselization of O, or in other words, its maximal unramified extension.

For each k-algebra Λ and $r \geq 0$, we denote by $W_r(\Lambda)$ the group of Witt vectors of length r and by $W(\Lambda) = \varprojlim W_r(\Lambda)$ the ring of Witt vectors (see [Se2, §II.6]). There exists a unique ring homomorphism $W(k) \to O$ commuting with the projection on $k = W_0(k)$ (*ibid*, II.5).

Let $\mathfrak{S}$ be an affine $W(k)$-scheme. Recall that for each $r \geq 0$, the functor k–$alg \to Sets$ given by

$$\Lambda \to \mathfrak{S}(W_r(\Lambda))$$

is representable by an affine k–scheme $\mathrm{Green}_r(\mathfrak{S})$. The projective limit

$$\mathrm{Green}\,(\mathfrak{S}) := \varprojlim_r \mathrm{Green}_r(\mathfrak{S})$$

is a scheme which satisfies $\mathrm{Green}(\mathfrak{S})(\Lambda) = \mathfrak{S}(W(\Lambda))$. If $\mathfrak{X}$ is an affine O–scheme, we deal also with the relative versions of the Greenberg functor

$$\underline{G}_r(\mathfrak{X}) := \mathrm{Green}_r\big(\prod_{O/W(k)} \mathfrak{X}\big), \quad \underline{G}(X) := \mathrm{Green}\,\big(\prod_{O/W(k)} \mathfrak{X}\big).$$

We have $\underline{G}_r(\mathfrak{X})(k) = \mathfrak{X}(O/p^r O)$ and $\underline{G}(\mathfrak{X})(k) = \mathfrak{X}(O)$. We also have $\underline{G}(\mathrm{Spec}(O)) = \mathrm{Spec}(k)$; if $\mathfrak{X}$ is a O–group scheme, then $\underline{G}(\mathfrak{X})$ and the $\underline{G}_r(\mathfrak{X})$ carry a natural k-group structure [B, 4.1].

Also if $\mathfrak{H}$ is a smooth affine group scheme defined over O, then we have a natural exact sequence of affine algebraic k–groups [CGP, A.6.3]

$$(*) \qquad 0 \to W(\mathrm{Lie}(\mathfrak{H}_k)^{(p)})) \to \underline{G}_2(\mathfrak{H}) \to \mathfrak{H}_k \to 1.$$

Furthermore the adjoint action of $\underline{G}_2(\mathfrak{H})$ on itself induces an representation of $\mathfrak{H}_k$ on $\mathrm{Lie}(\mathfrak{H}_k)$ which is nothing but the composite $\mathfrak{H}_k \to \mathfrak{H}_k^{(p)} \overset{Ad^{(p)}}{\to} \mathrm{GL}(\mathrm{Lie}(\mathfrak{H}_k))^{(p)}$.

If $\mathfrak{H}_k$ is reductive, then $W(\mathrm{Lie}(\mathfrak{H}_k^{(p)})$ is the unipotent radical of $\underline{G}_2(\mathfrak{H})$. It is known that if $\mathfrak{H}_k$ is non-commutative, then the sequence $(*)$ does not split [CGP, A.6.4].

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