RATIONALITY PROPERTIES OF LINEAR ALGEBRAIC GROUPS AND GALOIS COHOMOLOGY

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The goal of this series if lecture is firstly to present basic results for reductive algebraic groups for a non algebraically closed field k. Secondly, we are interested in the Galois cohomology of linear algebraic groups and also in invariants of such groups, e.g. weak approximation and R-equivalence. The program is as follows.

- The variety of tori, unirationality of reductive groups.
- Steinberg's theorem, Raghunathan's theorem.
- Parabolic subgroups, Borel-Tits theorem
- Anisotropic groups, classification of semisimple groups
- Classification of semisimple groups II
- Kneser's theorem, Bruhat-Tits theory
- R-equivalence, flasque resolutions of tori
- Norm principle, examples of non rational classical groups

The variety of tori, unirationality of reductive groups

1. Homogeneous spaces and Galois cohomology

Let k be a base field assumed to be of characteristic zero. Let k_s/k be a separable closure of k and denote by $\Gamma_k = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k.

Let G/k be a linear algebraic group. Let X be a (non empty) k-variety equipped with a right G-action. We say that X is homogeneous (resp. principal homogeneous) under G if $G(k_s)$ acts transitively (resp. simply transitively) on $X(k_s)$.

The G-principal homogeneous spaces are also called G-torsors.

1.1. **Examples.** Given a non degenerate quadratic form q, the projective quadric $\{q = 0\}$ is homogeneous under the orthogonal group $O(q)$. Given an integer $n \geq 1$ and $a \in k^\times$, the variety $\{x^n = a\}$ is principal homogeneous under $\mu_n = \mathrm{Spec}(\frac{k[t]}{(t^n-1)})$.

Given a k-subgroup $H \subset G$, recall that we can define the quotient $H\backslash G$. This is a quasi-projective variety and the quotient map $G \to H\backslash G$ is smooth and surjective.

1.2. **Lemma.** *Let X be a homogeneous space under G. Then the following are equivalent:*

(1) $X(k) \neq \emptyset$
(2) $X \cong H\backslash G$ *for some* $H \subset G$.

Note that the subgroup H is not uniquely defined; this will be discussed in Harari's lectures.

For a quotient $H\backslash G$ or equivalently $X := G/H$, this is not true in general that the map $\pi : G(k) \to X(k)$ is surjective. For example, $\mathbb{G}_m$ is the quotient of $\mathbb{G}_m$ by μ_n and the map $k^\times \to k^\times$, $x \mapsto x^n$, is not surjective. Here comes Galois cohomology into the picture. In this setting, we have the following exact sequence of pointed sets

$$1 \to H(k) \to G(k) \to X(k) \xrightarrow{\varphi} H^1(k, H) \to H^1(k, G).$$

The mapping φ is called the characteristic map. In term of cocycles, it is defined as follows. Given $x \in X(k)$, let $g \in G(k_s)$ such that $\pi(g) = x$. Then $z_\sigma := g^{-1}\,\sigma(g)$ is a 1-cocycle with value in $H(k_s)$. The class $[z]$ does not depend of the choice of g, and $\phi(x) = [z]$.

As we have seen, it is convenient to see elements of $X(k)$ as classes $[gH]$ with $g \in G(k_s)$ satisfying $g^{-1}\,\sigma(g) \in H(k_s)$ for all $\sigma \in \Gamma$.

There is an equivalent way to define the characteristic map. One knows that $H^1(k, H)$ classifies the principal homogeneous spaces under H. The point is that the preimage $\varphi^{-1}(x) \subset G$ is a principal homogeneous space under H, whose class is precisely $\phi(x)$.

1.3. **Proposition.** *The characteristic map induces a bijection*

$$G(k)\backslash X(k) \xrightarrow{\sim} \mathrm{Ker}\Big(H^1(k,H) \to H^1(k,G)\Big).$$

In other words, the $G(k)$-orbits on $X(k)$ are described in terms of Galois cohomology. This is specially interesting when $G = \mathrm{GL}_n$ since $H^1(k, \mathrm{GL}_n) = 1$ (Hilbert 90). If $X = \mathrm{GL}_n/H$, we have

$$\mathrm{GL}_n(k)\backslash X(k) \xrightarrow{\sim} H^1(k,H).$$

2. The variety of tori

By a reductive group G/k, we mean a connected linear algebraic group such that $G \times_k k_s$ is reductive. Equivalently, a linear algebraic group G is reductive if it is connected and has trivial unipotent radical. Firstly, we recall the two following key statements of the theory of reductive groups.

- All maximal k_s-tori are conjugated under $G(k_s)$.

Their common rank is called the rank of G_{k_s} and is denoted by $\mathrm{rank}(G_{k_s})$. Put $\mathrm{rank}(G) = \mathrm{rank}(G_{k_s})$, this is the absolute rank of G.

Ws say that k-subtorus T of G is maximal if $T \times_k k_s$ is a maximal k_s-torus.

2.1. **Lemma.** (1) *there exists a k-subtorus T which is is maximal.*
(2) *Let S be a k-subtorus of G. Then there exists a maximal k-torus of G containing S.*

The idea for the first fact is to use the Lie algebra $\mathfrak{g} = Lie(G)$ and the adjoint action. An element $X \in \mathfrak{g}$ is semisimple regular if its centraliser $C_G(X)$ is a maximal k-torus. Moreover, in that case, the Lie algebra of $C_G(X)$ is nothing but the infinitesimal centralizer $\mathfrak{C}_{\mathfrak{g}}(X)$.

Since the semisimple regular case is the generic one, it follows by Zariski density that it exists $X \in \mathfrak{g}$, X semisimple regular. Thus there exists a k-subtorus which is maximal [Bo, §18.1].

For the second statement, we can assume that G is semisimple. It goes by induction by considering the reductive subgroup $C_G(S)$ for a k-subtorus S of G.

Let $T \subset G$ be a maximal torus and denote by $N = N_G(T)$ the normalizer of T in G. We consider the quotient variety $X := G/N$.

2.2. **Lemma.** *There is a natural bijection between $X(k)$ and the set of maximal k-tori of G.*

Proof. Given $x = [gN] \in X(k)$, we remark that the k_s-subtorus $gTg^{-1} \subset G \times_k k_s$ does not depend of the choice of g and is stable under Galois. Therefore, $T_x := gTg^{-1}$ is equipped with a k-structure, it is a maximal k-subtorus of G. We defined a map from $X(k)$ to the set of maximal k-tori of G. The surjectivity comes from the conjugacy fact mentionned above. For the injectivity, we are given $x' = [g'N] \in X(k)$ such that $gTg^{-1} = g'Tg'^{-1}$. Then $gg'^{-1} \in N(k_s)$ and $x = x'$. $\square$

For this reason, X is called the variety of tori of G. The main result of this lecture is the following.

2.3. **Theorem.** *(Chevalley) The variety X of tori of G is k-rational, i.e. birationally isomorphic to an affine space.*

2.4. **Sketch of proof.** (after Borel-Springer [BSp, 7.9]) Let T be a maximal torus of G. We use the fact that the variety of tori $G/N_G(T)$ is also the variety of Cartan sub-algebras of $G = Lie(G)$. Let $t \in \mathfrak{t}$ be a semisimple regular element. Let U be the open subset of $\mathbb{A}(\mathfrak{g})$ of elements $x \in \mathfrak{g}$ such that $t+x$ is semisimple regular, i.e the Lie algebra $\mathfrak{h}_X = \mathfrak{C}_{\mathfrak{g}}(t+x)$ is a Cartan subalgebra. We fix a decomposition of k-vector spaces $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ and define the open (non empty) subvariety $V \subset U$ of the elements $x \in V$ such that $\mathfrak{m} \cap \mathfrak{h}_X = 0$. One shows then that the map $x \mapsto \mathfrak{h}_x$ is an open immersion of V in $G/N_G(T)$, the image being the open subvariety of $G/N_G(T)$ consisting of Cartan subalgebras $\mathfrak{h}$ such that $\mathfrak{h} \cap \mathfrak{m} = 0$.

3. Unirationality and weak approximation

3.1. **Theorem.** *(Grothendieck) The variety G is k-unirational, i.e there exists a rational dominant map $\mathbb{A}^N \to G$ from an affine space to G. In particular, $G(k)$ is Zariski dense in G. Furthermore, $G^{ss,reg}(k)$ is Zariski dense in G.*

The proof goes firstly by showing the result for tori and secondly by using the fibration use the fibration $G^{ss,reg} \to X$ sending a semisimple regular element to its centralizer.

It is obvious that one dimensional tori are k-rational varieties. It is also true that 2-dimensional tori are rational varieties (Chevalley).

3.2. **Corollary.** *Assume that* $\mathrm{rank}(G) \leq 2$. *Then G is a k-rational variety.*

The corollary applies in particular to groups of type G_2, namely automorphism groups of octonion algebras.

The variety of tori relies to weak approximation by the so-called fibration method. The following is a special case of a result by Kunyavskiĭ-Skorobogatov [KH].

3.3. **Theorem.** *Let $G/\mathbb{Q}$ be a reductive group. Then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ for the real topology.*

Again, one shows first the result for tori and use the fibration $G^{ss,reg} \to X$.

Steinberg's theorem and Raghunathan's theorem

4. Introduction

Let k be a base field assumed to be of characteristic zero. Let k_s/k be a separable closure of k and denote by $\Gamma_k = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k.

Let G/k be a (connected) reductive group. Our goal is to describe representants of the pointed set $H^1(k,G)$ coming from suitable subgroups. The first statement of that flavour is the following.

4.1. **Lemma.** *Let $T \subset G$ be a maximal torus. Then the map $H^1(k, N_G(T)) \to H^1(k,G)$ is surjective.*

It has been recently sharpened.

4.2. **Theorem.** [CGR] *Let $T \subset G$ be a maximal torus. Then there exists a finite k-subgroup $S \subset N_G(T)$ such that map $H^1(k,S) \to H^1(k,G)$ is surjective.*

In other words, all classes come from a fixed k-finite subgroup. For example, for PGL_n, one can take the semi-direct product $S = \Big((\mu_n)^n/\mu_n \Big) \rtimes S_n$.

5. Steinberg's theorem

We say that G is split if G admits a maximal k-torus which is k-split, i.e. isomorphic to $\mathbb{G}_m^{\mathrm{rank}(G)}$. We say that G is quasi-split if G admits a Borel subgroup B, i.e. such that $B \times_k k_s$ is a Borel subgroup of $G \times_k k_s$ which is by definition a maximal connected solvable subgroup of $G \times_k k_s$.

5.1. **Theorem.** *(Steinberg, 1965,* [St]*)*
Let G be a quasi-split reductive group. Then

$$H^1(k,G) = \bigcup_{T \subset G} \mathrm{Im}\Big(H^1(k,T) \to H^1(k,G) \Big)$$

where T runs over the maximal k-tori of G.

Few examples were known before : PGL_n, SO_{2n}. That solved Serre's conjecture I for fields of cohomological dimension ≤ 1, i.e. such that $H^2(k,A) = 0$ for any finite Galois module A. The basic example of such a field is the function field of a complex curve (Tsen's theorem [S1, III.3]).

5.2. **Corollary.** *Assume that $\mathrm{cd}(k) \leq 1$. Let G/k be a reductive group. Then G is quasi-split and $H^1(k,G) = 1$.*

For a field of cohomological dimension ≤ 1, the cohomology of tori vanishes. So the vanishing of $H^1(k,G)$ follows from Steinberg's theorem for G quasi-split. Since a group G is an inner form of a quasi-split reductive group (see next lecture), it follows that it is quasi-split as well.

With some additional work (i.e. Grothendieck's vanishing's theorem of non abelian H^2), one can compute the Galois cohomology of an arbitrary linear algebraic group.

5.3. **Theorem.** [S1, III.2.4, corollaire 3]
Assume that $\mathrm{cd}(k) \leq 1$. *Let* G/k *be a linear algebraic group and let* G^0 *be its neutral component. Then the quotient map* $G \to G/G^0$ *induces a bijection*

$$H^1(k,G) \xrightarrow{\sim} H^1(k,G/G^0).$$

6. Proof of Steinberg's theorem

6.1. **The adjoint quotient** [H, III]**.** The argument is of geometrical nature. Let G/k be a reductive group. We consider the ring of central functions

$$C(G) = k[G]^{ad(G)} = \Big\{ f \in k[G] \;\mid\; f(gxg^{-1}) = f(x) \;\forall g \in G(k) \Big\}.$$

The variety $G//G := \mathrm{Spec}(C(G))$ is the adjoint quotient of G . It turns out that the quotient map $G \to G//G$ separates the semisimple conjugacy classes.

Now let $T \subset G$ be a maximal torus. We denote by $W_T := N_G(T)/T$ its Weyl group which is a finite étale k-group. We have a natural map

$$T//N_G(T) = T//W_T \to G//G.$$

6.1. **Theorem.** *(Chevalley) The map* $T//W_T \to G//G$ *is an isomorphism.*

6.2. **Case of quasi-split semisimple simply connected group.** Assume that G is semisimple simply connected and quasi-split. Assume furthermore that T is a maximal torus of a Borel subgroup of G. Then $T//W_T$ is isomorphic to an affine space $\mathbb{A}^r_k$. In the split case, the isomorphism $G//G \to \mathbb{A}^r_k$ is provided by the fundamental characters $\chi_1, \cdots, \chi_r$ of G, i.e. the characters of the representations whose highest weight are the fundamental weights $\overline{\omega}_1, \cdots, \overline{\omega}_r$ (with respect to $T \subset B$). The map

$$\pi : G \to G//G \cong \mathbb{A}^r_k$$

is called the Steinberg's map of G.

6.2. **Example.** If $G = \mathrm{SL}_n$, the fundamental representations are the alternate powers of the standard one. Then Steinberg's map applies an element $g \in \mathrm{SL}_n(k)$ to the coefficients of the characteristic polynomial of g.

In that case, there is a splitting of π which is given by

$$(a_1, \cdots, a_{n-1}) \mapsto \begin{pmatrix} 0 & 0 & . & . & 1 \\ 1 & 0 & 0 & . & a_{n-1} \\ . & . & . & . & . \\ 0 & . & 1 & 0 & a_2 \\ 0 & . & 0 & 1 & a_1 \end{pmatrix}.$$

This fact is actually general out of type A_{odd}.

6.3. **Theorem.** [St]
If G has no type A_{odd}, the Steinberg's map $\pi : G \to \mathbb{A}^r_k$ admits a section $\mathcal{C} : \mathbb{A}^r_k \to G^{reg}$ where G^{reg} stands for the open subset of regular elements of G.

The proof provides an explicit section built out from the Weyl group and radicial subgroups of G. We are interested in its following consequence (which is true also for type A_{odd} by a more subtle argument).

6.4. **Corollary.** *Let $g \in G(k_s)$ be a regular semisimple element. Assume that the geometric conjugacy class $C(g)$ is rational, i.e. stable under Γ_k. Then $C(g)$ contains an element of $G(k)$.*

Proof. Our hypothesis implies that $\pi(g) \in \mathbb{A}^r(k)$. Put $g_0 = \mathcal{C}(\pi(g))$. Then g_0 is semisimple regular and belongs to the geometric conjugacy class of g. □

6.5. **Remark.** For suitable generalizations of this statement to other conjugacy classes and arbitrary reductive groups, see Kottwitz [Ko].

We can now complete the proof of Steinberg's theorem 5.1.

Proof. Let $\lambda : G^{sc} \to D(G)$ be the universal covering of the derived group of G. The groups G^{sc}, $D(G)$ and G are quasi-split and have the same adjoint group, i.e. we have the following commutative diagram

$$\begin{array}{ccccc} G^{sc} & \xrightarrow{\lambda} & D(G) & \subset & G \\ \downarrow & & \downarrow & & \downarrow \\ G^{sc}_{ad} & = & D(G)_{ad} & = & G_{ad}. \end{array}$$

In particular, G acts by inner automorphisms on G^{sc}.

We are given a class $[z] \in H^1(k, G)$ and we want to show that it comes from a maximal k-torus of G. We consider the twisted k-group ${}_zG^{sc}$ and pick a semisimple regular element $g_0 \in {}_zG^{sc}(k)$. We have

$${}_zG(k) = \Big\{ g \in G(k_s) \mid Ad(z_\sigma).\sigma(g) = g \Big\}.$$

It follows that g_0, seen in $G(k_s)$, defines a rational conjugacy class of G. By Corollary 6.4, there exists $h \in G^{sc}(k_s)$ such that $g_1 := Ad(h^{-1}) \,.\, g_0 \in G(k)$. From $Ad(z_\sigma).\sigma(g_0) = g_0$, we get that

$$Ad(z_\sigma) \,.\, \sigma(Ad(h) \,.\, g_1) = Ad(h) \,.\, g_1.$$

Pushing by $\lambda : G^{sc}(k) \to G(k)$ yields

$$Ad\Big(\lambda(h)^{-1}\, z_\sigma\, \sigma(\lambda(h))\Big) \,.\, \lambda(g_1) = \lambda(g_1).$$

The element $\lambda(g_1)$ is semisimple regular so its centralizer $T_1 := C_G(\lambda(g_1))$ is a maximal torus. Thus

$$[z_\sigma] = \big[\, \lambda(h)^{-1}\, z_\sigma\, \sigma(\lambda(h))\,\big] \in \mathrm{Im}\Big(H^1(k,T_1) \to H^1(k,G)\Big).$$

$\square$

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7. Raghunathan's theorem

Steinberg's theorem is indexed by the set of maximal tori of a quasi-split group G. Our goal is to describe this set with regards to this result. We denote by $T \subset G$ a maximal torus of a given Borel subgroup $B \subset G$ (T is the centralizer in G of a maximal k-split torus of G). Consider the variety $X = G/N$ of maximal tori, where $N := N_G(T)$. We have an exact sequence $1 \to T \to N \to W_T \to 1$ where W_T is the Weyl group; it is a finite k-group. By composing the characteristic map $X(k) \to H^1(k,N)$ with the map $H^1(k,N) \to H^1(k,W_T)$, we get a map

$$Type : X(k) \to H^1(k,W_T),$$

which is the *type* of a torus. When G is split, note that W is constant so the pointed set $H^1(k,W) = \mathrm{Hom}_{ct}(\Gamma_k, W)/int(W)$ classifies Galois W-algebras over k.

7.1. **Example.** For GL_n, the maximal tori are the induced tori $R_{E/k}(\mathbb{G}_m)$ for E/k running over the étale k-algebras (i.e. product of finite separable fields extensions) of dimension n. The type has value in $H^1(k,S_n)$ which classifies étale k-algebras of dimension n. It is an easy checking that $Type(R_{E/k}(\mathbb{G}_m)) = [E] \in H^1(k,S_n)$.

7.2. **Remark.** This invariant is finer that the isomorphism class of the torus. For example, for GL_n, the type has value in $H^1(k,S_n)$ but the map $H^1(k,S_n) \to H^1(k,\mathrm{GL}_n(\mathbb{Z}))$ is not injective in general.

This invariant is the right one with respect to Steinberg's theorem.

7.3. **Lemma.** *Let T_1, T_2 be two maximal k-tori of G having same type in $H^1(k,W)$. Then*

$$\mathrm{Im}\Big(H^1(k,T_1) \to H^1(k,G)\Big) = \mathrm{Im}\Big(H^1(k,T_2) \to H^1(k,G)\Big).$$

We can now state Raghunathan's theorem.

7.4. **Theorem.** *(2004,* [R]*) All types of tori occur, i.e. the type map $X(k) \to H^1(k,W_T)$ is surjective.*

So this result completes nicely the picture. Moreover, it gives a way to parametrize W-torsors by a k-rational variety.

The proof requires the following fact.

7.5. **Lemma.** *(Tits) Let S/k be a torus. Assume that k is not an algebraic extension of a finite field. Then there exists an element $s \in S(k)$ such that the group $s^{\mathbb{Z}}$ is Zariski dense in S.*

Proof of Theorem 7.4: We consider the case when Tits lemma holds. Since the type of a k-torus does not change by central extensions of groups, that boils down to the case of a semisimple simply connected group. We are given $[\xi] \in H^1(k, W_T)$ and we want to show that the twisted torus $T' = {}_\xi T$ can be embedded in G. It is convenient to fix a trivialisation map $\phi : T \times_k k_s \xrightarrow{\sim} T' \times_k k_s$ such that $\xi_\sigma = \phi^{-1} \sigma(\phi)$ for all $\sigma \in \Gamma_k$. By Lemma 7.5, there exists $t' \in T'(k)$ such that $(t')^{\mathbb{Z}}$ is Zariski dense in T'. Define $t := \phi^{-1}(t') \in T(k_s)$; this is a regular element of G.

7.6. **Claim.** *The conjugacy class $C(t')$ is rational.*

According to Chevalley's isomorphism $T//W_T \cong G//G$ (6.1), it is enough to check that the conjugates of t are $W_T(k_s)$-conjugates of t. But $\sigma(t) = \sigma(\phi^{-1}(t')) = \phi^{-\sigma}(t') = (\phi^{-\sigma}\,\phi)(t) = \xi_\sigma^{-1}(t)$, hence the claim.

Steinberg's result 6.4 provides an element $t_0 \in G(k)$ and $h \in G(k_s)$ such that $t = h^{-1}.t_0$. Then the Zariski closure of $(t_0)^{\mathbb{Z}}$ in G is a k-torus T_0 of G. It remains to check that $Type(T_0) = [\xi]$. Since $T_0 = hTh^{-1}$ the type of T_0 is given by the image in W_T of the cocycle $h^{-1}\sigma(h)$.

Given $\sigma \in \Gamma_k$, let n_σ be a lifting of $\xi_\sigma \in N(k_s)$. From $t_0 = \sigma(t_0)$, we have

$$\begin{aligned} h.t &= \sigma(h.t) \\ &= \sigma\big(h \,.\, \phi^{-1}(t')\big) \\ &= \sigma(h) \,.\, \phi^{-\sigma}(t') \qquad [\, t' \in T(k)\,] \\ &= \sigma(h) \,.\, n_\sigma^{-1} \,.\, t. \end{aligned}$$

It follows that $h^{-1}\sigma(h) \,.\, n_\sigma^{-1}$ belongs to $T(k_s)$. Hence $h^{-1}\sigma(h)$ and n_σ acts in the same way on $T = h^{-1} \,.\, T_0$. Thus $Type(T_0) = [\xi]$ as desired. $\square$

Parabolic subgroups, Borel-Tits theorem

8. Introduction

Let k be a base field assumed to be of characteristic zero. Let k_s/k be a separable closure of k and denote by $\Gamma_k = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k.

Following Borel-Tits [BoT], our goal is to show in that lecture and the next one how the classification of semisimple algebraic groups boils down to the case of the anisotropic ones. Before to give the precise definition here, let us present two important examples.

8.1. **Example. Wedderburn's theorem.** Let A be a central simple k-algebra. Then there exist a uniquely determined central division k-algebra D and an integer $r \geq 1$ such that $A \cong M_r(D)$.

8.2. **Example. Witt's theorem.** Let q be a non-degenerated quadratic form. Then there exists a uniquely determined anisotropic quadratic form q_0 and a non-negative integer $\nu(q)$ (the Witt index) such that $q \cong q_0 \perp \nu(q)\,\mathbb{H}$, where $\mathbb{H}$ stands for the hyperbolic plane, i.e. the quadratic form XY.

It turns out that both examples are special case of Borel-Tits' results. Those are obtained by the study of parabolic subgroups of reductive groups.

9. Basic facts on k-parabolic subgroups

9.1. **Definition.** Let G/k be a linear algebraic group. A k-subgroup P is a k-parabolic subgroup if the quotient variety G/P is projective.

As a k-variety $X \times_k k_s$ is projective (resp. quasi-projective) if and only if $X \times_k k_s$ is projective [EGA4, 9.1.5]. It follows that a k-subgroup P is parabolic if and only if $P \times_k k_s$ is parabolic.

Since a homogeneous space G/P is always quasi-projective, P is parabolic if and only if G/P is proper.

For any linear algebraic group G, the unipotent radical $R_u(P)$ is a split unipotent group and the neutral component of the quotient $G/R_u(P)$ is reductive. Moreover, the quotient map is split, and we can write $G = R_u(G) \rtimes L$ with $L \subset G$. Such a k-subgroup is called a Levi subgroup of G.

9.1. **Lemma.** *Let P be a k-parabolic subgroup of a linear algebraic group G.*

(1) *Let $P^\sharp$ be a k-subgroup of G satisfying $P \subset P^\sharp \subset G$. Then P is a k-parabolic subgroup of $P^\sharp$ and $P^\sharp$ is a k-parabolic subgroup of G.*
(2) *The k-parabolic subgroups Q of G which are included in P are the k-parabolic subgroups of P.*

Proof. (1) According to [DG, III.3.2.5], the morphism $P^\sharp/P \to G/P$ is a closed immersion. So $P^\sharp/P$ is projective. The map $G/P \to G/P^\sharp$ is surjective, so $G/P^\sharp$ is proper. Thus $P^\sharp$ is a k-parabolic subgroup of G.

(2) Let $Q \subset P$ be a k-parabolic subgroup of G. The fibers of the map $G/Q \to G/P$ are projective, so P/Q is projective. Hence Q is a k-parabolic subgroup of P. In the way around, assume that Q is a k-parabolic subgroup of P. Then the map $G/Q \to G/P$ has proper geometric fibers, so G/Q is proper. Thus Q is a k-parabolic subgroup of G. □

9.2. **Fixed point theorem.** Recall the following fundamental fact.

9.2. **Theorem.** *(Rosenlicht, see* [Sp, 14.1.7]*) Let H be an connected split solvable k-group (i.e. extension of a k-split torus by a split unipotent group) acting on a proper k-variety X satisfying $X(k) \neq \emptyset$. Then $X^H(k) \neq \emptyset$.*

In the geometric case, this is Borel's fixed point theorem [Bo, 10.4]. Recall also that connected unipotent groups are always k-split in characteristic zero [Sp, 14.3.9].

That enables us to investigate the behaviour of parabolic subgroups with respect to quotient mod $R_u(G)$.

9.3. **Corollary.** (1) *Let P be a k-parabolic subgroup of G. Then $R_uG \subset P$.*
(2) *The quotient morphism $\pi : G \to G/R_u(P)$ induces a one to one correspondence between the k-parabolic subgroups of G and those of $G/R_u(G)$.*

Proof. (1) We may assume that $k = k_s$. The unipotent group $R_u(G)$ acts on the proper k-variety G/P, se we can pick a point $[gP] \in (G/P)^{R_u(G)}(k)$ by Theorem 9.2. Hence $R_u(G) \subset gPg^{-1}$. Since $R_u(G)$ is a normal subgroup of G, we get $R_u(G) \subset P$ as desired.
(2) If P is a k-parabolic of G, it is clear that $\pi(P)$ is a k-parabolic subgroup of $G/R_u(G)$. From (1), we have moreover $P = \pi^{-1}(\pi(P))$. □

10. Split reductive groups, the geometric case

If $k = k_s$, all reductive groups are split. As several authors, we first review split reductive groups under the assumption $k = k_s$.

10.1. **Isomorphism theorem.** The major result is that reductive groups are classified by root datas arising from the adjoint representation of G.

10.1. **Theorem.** [Sp, 10.1] *Let $\Psi = (X, R, X^\vee, R^\vee)$ be a root datum. Then there exists a (connected) reductive group G over k with a maximal torus T such that the root datum $\Psi(G,T)$ is isomorphic to Ψ. Such a group G is unique up to isomorphism.*

10.2. **Bruhat's decomposition.** Let $(n_w)_{w\in W}$ be a set theoretic section of $N_G(T)(k) \to W$. We have the decomposition

$$G(k) = \bigsqcup_{w\in W} B(k)\, n_w\, B(k).$$

10.3. **Parabolic subgroups.** Recall that a split reductive k-group contains a maximal k-torus T and a k-Borel subgroup B. The pair (T,B) is called a Killing couple. All Killing couples are conjugated under $G(k)$.

10.2. **Lemma.** *A k–parabolic subgroup of G contains a Borel subgroup.*

Proof. Let P be a k-parabolic subgroup of G. By Theorem 9.2, $(G/P)^B(k) \neq \emptyset$. By picking a point $[gP] \in (G/P)^B(k) \neq \emptyset$, we get $g^{-1}Bg \subset P$. $\square$

So it is enough to list the k-parabolic subgroups containing B. Let $T \subset B$ be a maximal torus of B. Those are the "standard" parabolic subgroups $P_I = U_I \rtimes L_I$ defined as follows for any subset $I \subset \Delta$, where Δ is the basis of the root system $\Phi(G,T)$ defined by the Borel subgroup B. We denote by $[I]$ the subset of $\Phi(G,T)$ of roots which are linear combination of elements of I.

- $T_I = \Big(\bigcap_{\alpha\in I} \ker(\alpha) \Big)^0 \subset T$;
- $L_I = C_G(T_I) \subset G$;
- $U_I \subset U$ is the subgroup generated by the U_α, $\alpha \in \Phi(G,T)_+ \setminus [I]$.

The group $U_\alpha \cong \mathbb{G}_a$ is the root group attached to α and $\mathfrak{u}_\alpha \cong k$ is its Lie algebra. The Lie algebra of P_I is

$$\mathfrak{p}_I := \mathfrak{t} \oplus \bigoplus_{\alpha \in [I]} \mathfrak{u}_\alpha$$

where $[I] = \Phi(L_I,T) \subset \Phi(G,T)$ stands for the root system generated by $I \subset \Delta$. In particular $P_\Delta = B$ and $P_\emptyset = G$.

The P_I's give rise to distinct conjugacy classes. Therefore there is a one to one correspondence between the conjugacy classes of k-parabolic subgroups of G and subsets of Δ.

We may define as well the k-parabolics $P_I^- = U_I^- \rtimes L_I$ where U_I^- is the subgroup of G generated by the $U_{-\alpha}$ for $\alpha \in I$.

10.3. **Remark.** Parabolic groups are determined by their unipotent radical since we have $P_I = N_G(U_I)$.

10.4. **Remark.** L_I is the unique Levi subgroup of P_I which contains T (see [Sp, 8.4.4]).

11. More on parabolic groups

We come back to the case where the the ground field k is assumed only of characteristic zero.

11.1. **Intersection of parabolic subgroups.**

11.1. **Proposition.** *Let P, Q be two k-parabolic subgroups of G.*

(1) *$P \cap Q$ contains a maximal k-torus.*
(2) *$(P \cap Q).R_u(P)$ is a k-parabolic subgroup of G. It is equal to P if and only if Q contains a Levi subgroup of P.*

Proof. We can assume in the proof that k is separably closed.

(1) Lemma 10.2 boils down to show that the intersection of two Borel subgroups B and B' contains a maximal torus. We have $B' = gBg^{-1}$ for a suitable $g \in G(k)$. Let $T \subset B$ be a maximal k-torus of B. By Bruhat decomposition (10.2), there exists $n \in N_G(T)(k)$ such that $g \in B(k)nB(k)$. We write then $g = b_1 n b_2$ and get $b_1 T b_1^{-1} \subset B \cap B'$.

(2) Let (T, B) be a Killing couple. From (1), we may assume that $T \subset P \cap Q$ and that $B \subset P$. Then $P = P_I$ is a standard k-parabolic subgroup. We consider the root subsystem $\Phi(Q,T) \subset \Phi(G,T)$. Since Q is parabolic, we know that $\Phi(Q,T)$ is a parabolic root subsystem, i.e. $\Phi(G,T) = \Phi(Q,T) \cup -\Phi(Q,T)$. Thus any Borel subgroup B' of $P \cap Q$ containing T contains at least one of the U_α, $U_{-\alpha}$ for every $\alpha \in [I]$. Then the k-solvable group $B'.R_u(P)$ contains T and one of the U_α, $U_{-\alpha}$ for every $\alpha \in \Phi(G,T)$. Therefore it is a Borel subgroup and by Lemma 9.1, $(P \cap Q).R_u(P)$ is parabolic. The second assertion is obvious. □

11.2. **Opposite parabolic subgroups.**

11.2. **Definition.** Two k-parabolic subgroups of G are opposite if $P \cap Q$ is a common Levi subgroup of P and Q.

In the split case, the k_s-parabolic subgroup P_I^- is the only opposite k_s-parabolic subgroup of G which is opposite to P_I and contains L_I (or even T by Remark 10.4). We abstract that with the following lemma.

11.3. **Lemma.** *Let P be a k-parabolic subgroup of G. Then there is a one to one correspondence between the opposite k-parabolic subgroups of P, the Levi subgroups of P and the maximal tori of P.*

In particular, opposite parabolic subgroups exist.

11.4. **Lemma.** [BoT, 4.10] *Let P, P' be two k-parabolic subgroups. Then the following conditions are equivalent:*

(1) *P and P' are opposite;*
(2) *The geometric conjugacy classes of P and P' are opposite*[1] *and P, P' contain opposite Borel subgroups;*
(3) *$(P \cap P').R_u(P) = P$, $(P \cap P').R_u(P') = P'$, and P, P' contain opposite Borel subgroups.*
(4) *$P \cap R_u(P') = 1$ and $P' \cap R_u(P) = 1$.*

[1] i.e. there exists $g \in G(k_s)$ such that ${}^gP \times_k k_s$ is opposite to ${}^gP \times_k k_s$.

We shall use only the implication (1) $\Longrightarrow$ (4). Let us prove it. We can assume that $k = k_s$ and assume that $P = P_I$, $P' = P_I^-$ are opposite standard subgroups. Then $P_I \cap U_I^- = 0$.

12. Borel-Tits' theorem

We split the result in two statements.

12.1. **Theorem.** *Let G be a reductive groups and let P be a k-parabolic subgroup of G.*

(1) *The fibration $G \to G/P$ is locally trivial for the Zariski topology.*
(2) *The variety G/P is k-rational.*
(3) *The map $G(k) \to (G/P)(k)$ is surjective and the map $H^1(k, P) \to H^1(k, G)$ is injective.*

Proof. (1) Let P' be an opposite k-parabolic subgroup of P. By Lemma 11.4, we have $R_u(P') \cap P = 1$, so the map $R_u(P') \to G/P$ is an immersion. By dimension reasons, it is an open immersion. Since $G(k)$ is Zariski dense in G, we can cover G/P by the trivializing open subsets $gR_u(P')$ ($g \in G(k)$) and we conclude that the fibration $G \to G/P$ is trivial for the Zariski topology.
(2) We have seen that $R_u(P')$ is birationally isomorphic to G/P. As k-variety, $R_u(P')$ is isomorphic to an affine space. Thus G/P is a k-rational variety.
(3) It readily follows that the map $G(k) \to (G/P)(k)$ is surjective. By the long exact sequence of pointed sets

$$1 \to P(k) \to G(k) \to (G/P)(k) \to H^1(k, P) \to H^1(k, G),$$

we get that the map $H^1(k, P) \to H^1(k, G)$ has trivial kernel. For establishing the injectivity, we make use of the following "torsion trick". We have to show that the maps $H^1(k, {}_zP) \to H^1(k, {}_zG)$ have trivial kernel for all $[z] \in H^1(k, P)$. But this works since ${}_zP$ is a k-parabolic subgroup of ${}_zG$. $\square$

12.2. **Theorem.** *Let G be a reductive group and let P, Q be two k-parabolic subgroups of G.*

(1) *If P and Q are minimal k–parabolic subgroups, then P and Q are conjugated under $G(k)$.*
(2) *If $P \times_k k_s$ and $Q \times_k k_s$ are conjugated under $G(k_s)$, then P and Q are conjugated under $G(k)$.*

Proof. (1) We are given two minimal k-parabolic subgroups P, Q of G. We consider the action of $R_u(Q)$ on the proper variety G/P. By Theorem 9.2, $(G/P)^{R_u(Q)}(k) \neq \emptyset$ and we pick a k-point x. Furthermore, Theorem 12.1 shows that $x = [gP]$ with $g \in G(k)$. So $R_u(Q) \subset gPg^{-1}$. Up to replace P by its conjugate, we have then $R_u(Q) \subset P$. It follows that

$$(*) \qquad (P \cap Q).R_u(Q) \subset P \cap Q.$$

Proposition 11.1.2 tells us that the left handside is a k-parabolic subgroup. By minimality of P and Q, we conclude that $(P \cap Q).R_u(P) = P = Q$.

(2) Up to conjugate P by a suitable element of $G(k)$, the same argument shows that we can assume that $R_u(Q) \subset P$ and the inclusion $(*)$ above holds. For showing that $P = Q$, we can extend the scalars to k_s. Since $(P \cap Q).R_u(Q)$ is parabolic, it contains a Killing couple (T, B). So $P = P_I$ and $Q = P_J$ are standard parabolic subgroups of G which are conjugated. Thus $P = Q$. $\square$

12.3. **Corollary.** *Let P be a minimal k-parabolic of G. Then the conjugacy class of P is auto-opposite.*

13. Come back to the examples

13.1. **Unicity in Wedderburn's theorem.** Let A be a central simple algebra of degree d. Then $A \otimes_k k_s \cong M_d(k_s)$, i.e. A is a k-form of the matrix algebra $M_d(k)$. It follows that the k-group $\mathrm{GL}_1(A)$ is a k-form of the linear group GL_d. If $A = M_r(D)$ with D a central division k-algebra, the k-group

$$P = \begin{pmatrix} D & * & . & . & * \\ 0 & D & * & . & * \\ . & . & . & . & . \\ 0 & . & 0 & D & * \\ 0 & . & 0 & 0 & D \end{pmatrix} \subset \mathrm{GL}_r(D) = \mathrm{GL}_1(A)$$

is a minimal k-parabolic subgroup of $\mathrm{GL}_1(A)$ and $P/R_u(P) = \mathrm{GL}_1(D)^r$. So D is encoded in P, which yields unicity in Wedderburn theorem.

13.2. **Witt theorem.** Let $q = q_0 \perp \nu\mathbb{H}$ with q_0 an anisotropic quadratic form. Let $P \subset SO(q)$ be the subgroup which normalises V_0, the underlying vector space of q_0. Then P is a minimal k-parabolic subgroup of $SO(q)$ and $P/R(P) = SO(q_0)$. Borel-Tits' theorem shows that the similartity class of q_0 is encoded in $SO(q)$, which is a weak version of Witt cancellation theorem.

Anisotropic groups, classification of semisimple groups

We continue the last lecture.

14. Maximal k-split tori, irreducible and anisotropic groups

Let G be a reductive group. We first add the following consequence of the fixed point theorem and of Borel-Tits theorem.

14.1. **Corollary.** *Let S be a k-split torus of G and let P be a k-parabolic subgroup. Then there exist $g \in G(k)$ such that $S \subset gPg^{-1}$.*

14.1. From tori to parabolics.

14.2. **Proposition.** [Bo, §20] *Let $S \subset G$ be a k-split torus. Then $C_G(S)$ is the Levi subgroup of a parabolic subgroup.*

The proof goes by inspection of the Galois action on the root system $\Phi(G \times_k k_s, T \times_k k_s)$ for a maximal k-torus T of G which contains S. The following result generalizes the case of (split) k_s-groups.

14.3. **Theorem.** *Maximal k-split tori of G are conjugate under $G(k)$.*

Proof. The proof goes by induction on the semisimple rank. Let S, S' be two maximal k-split tori. If G does not contain any proper k-parabolic subgroup, then $S = S' = 1$ by Proposition 14.2. If G contains a proper k-parabolic subgroup P, then we can assume that S and S' are maximal k-split tori of P by the preceding Corollary. Furthermore we can assume that S, S' are maximal tori of a Levi subgroup L of P. By induction, S and S' are conjugate under $L(k)$. □

14.2. From parabolic to tori.

Given a k-torus T, we denote by $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$ (resp. $\widehat{T} = \mathrm{Hom}(\mathbb{G}_m, T)$) the Galois module of characters (resp. cocharacters). We have a natural pairing $\widehat{T} \times \widehat{T} \to \mathbb{Z}$. We denote by T_d the k-split torus of character group $(\widehat{T}^0)^{\Gamma_k}$. Then T_d is the maximal k-split torus of T.

14.4. **Proposition.** [Bo, 20.6] *Let P be a proper k-parabolic subgroup of G, L be a Levi subgroup of G and let S be the connected center of L. Then :*

(i) $L = C_G(S_d)$;

(ii) P is minimal if and only if S_d is a maximal k-split torus of G.

As in the proof of Proposition 14.4, we can then define the relative root system $\Phi(G, S)$ with respect to a maximal k-split torus S of G. Then with mild modifications, the absolute theory works. In particular, we have standard parabolic subgroups and a Bruhat decomposition with respect to a minimal k-parabolic subgroup.

14.3. **Reducibility and isotropy.** We actually discuss already the notion of irreducibility.

14.5. **Definition.** Let H be a linear algebraic group. We say that H is reducible if H admits a proper k-parabolic subgroup.

We say that H is isotropic if H admits a non-trivial split k-subtorus.

The opposite notions are respectively irreducible and anisotropic. A non trivial split k-torus is isotropic but irreducible. Both notions are stable under central isogenies and also under extensions by a solvable group. From Propositions 14.2 and 14.4, we can derive the following characterisations.

14.6. **Corollary.** (1) *If G is reducible, then it is isotropic.*
(2) *If G is semisimple, reducible $\Longleftrightarrow$ isotropic.*
(3) *G is anisotropic if and only its adjoint group G_{ad} is anisotropic and its connected center $C(G)^0$ is anisotropic.*

15. Beginning of the classification

15.1. **Automorphism group of semisimple groups.** We fix a semisimple split group G/k and a Killing couple (T, B). We denote by G_{ad} the adjoint group of G and by T_{ad} the image of T in G_{ad}. We denote by Δ the base of the root system $\Phi(G, T)$ defined by B.

15.1. **Proposition.** [Sp, 2.12] *The functor* $\mathrm{Aut}(G)$ *is representable by a linear algebraic group. It has the following properties:*

(1) *The quotient* $\mathrm{Out}(G) := \mathrm{Aut}(G)/G_{ad}$ *is a finite constant group.*
(2) *We have an exact sequence of k-algebraic groups*

$$0 \to T_{ad} \to \mathrm{Aut}(G, B, T) \to \mathrm{Out}(G) \to 1.$$

(3) *We have* $\mathrm{Aut}\Big(G, B, T, (U_\alpha)_{\alpha\in\Delta}\Big) \xrightarrow{\sim} \mathrm{Out}(G)$*, so the map above is split.*
(4) *There is an injective map* $\mathrm{Out}(G) \hookrightarrow \mathrm{Aut}(\Delta)$*. It is an isomorphism if G is simply connected or adjoint.*

15.2. **Example.** Type A_{n-1}. Then there is an exact sequence $1 \to \mathrm{PGL}_n \to \mathrm{Aut}(\mathrm{SL}_n) \to \mathbb{Z}/2\mathbb{Z} \to 1$. From the theory of compact groups, we know that the Cartan involution $X \to {}^tX^{-1}$ provides a splitting of that exact sequence. But it not does normalize the Borel subgroup consisting of upper triangular matrices. So the splitting given by the Proposition is the conjugate of the Cartan involution by the antidiagonal standard matrix.

Then by descent, k-forms of G (up to isomorphism) correspond to the set $H^1(k, \mathrm{Aut}(G))$. This the reason why we cannot avoid non-connected groups.

15.2. **Quasi-split forms.**

15.3. **Definition.** A reductive group H is quasi-split if it admits a Borel subgroup Q, i.e. $Q \times_k k_s$ is a Borel subgroup of $H \times_k k_s$.

15.4. **Proposition.** (1) *Quasi-split k-forms of G are classified by* $H^1(k, \mathrm{Aut}(G, B, T)) \xrightarrow{\sim} H^1(k, \mathrm{Out}(G))$.

(2) *A k-form of G is an inner twisted form of a unique quasi-split k–form of G.*

Proof. (1) Let us denote $H^1_{qd}(k, \mathrm{Aut}(G)) \subset H^1(k, \mathrm{Aut}(G))$ the subset of quasi-split k-forms of G. The first step is to notice that

$$\mathrm{Im}\Big(H^1(k, \mathrm{Aut}(G, B, T)) \to H^1(k, \mathrm{Aut}(G))\Big) = H^1_{qd}(k, \mathrm{Aut}(G)).$$

If $z \in Z^1(k, \mathrm{Aut}(G, B, T))$, the twisted group ${}_zB$ is a Borel subgroup of ${}_zG$; hence ${}_zG$ is quasi-split. In the way around, we are given $z \in Z^1(k, \mathrm{Aut}(G))$ such that $G' :=_z G$ is quasi-split. It admits a Borel subgroup B', and let $T' \subset B'$ be a maximal torus of B'. Then $(T' \times_k k_s, B' \times_k k_s)$ is a Killing couple of $G \times_k k_s \overset{\phi}{\xrightarrow{\sim}} G' \times_j k_s G$. It follows that there exists $g \in G(k_s)$ such that $\phi^{-1}(T', B') = g(B, T)g^{-1}$. Since $z_\sigma = \phi^{-1}\, \sigma(\phi)$, one checks that $z' = g^{-1}\, z_\sigma\, \sigma(g)$ normalises (B, T); therefore $[z]$ comes from $H^1(k, \mathrm{Aut}(G, B, T))$. The second step is to establish that $H^1(k, \mathrm{Aut}(G, B, T)) \cong H^1(k, \mathrm{Out}(G))$. Let us de note by ρ the splitting of $\mathrm{Out}(G) \to \mathrm{Aut}(G, B, T)$ stated in Proposition 15.1.(3). The associated map $\rho_* : H^1(k, \mathrm{Out}(G)) \to H^1(k, \mathrm{Aut}(G, B, T))$ provides a splitting of $p_* : H^1(k, \mathrm{Aut}(G, B, T)) \cong H^1(k, \mathrm{Out}(G))$ which is indeed split surjective. For the injectivity, it is enough to prove that the $p_*^{-1}\big([\rho(a)]\big) = [a]$ for any $[a] \in H^1(k, \mathrm{Out}(G))$. We use the exact sequence

$$1 \to T/C(G) \to \mathrm{Aut}(G, B, T) \xrightarrow{p} \mathrm{Out}(G) \to 1.$$

Given such $[a]$, we twist this sequence by $\rho(a)$, i.e. we have

$$1 \to {}_{\rho(a)}(T/C(G)) \to {}_{\rho(a)}\,\mathrm{Aut}(G, B, T) \to {}_a\,\mathrm{Out}(G) \to 1.$$

By the twisting trick, there is a bijection between the fiber of p_* at $\rho([a]) \in H^1(k, \mathrm{Aut}(G, B, T))$ and

$$\mathrm{Im}\Big(H^1(k,_{\rho(a)}(T/C(G))) \to H^1(k,_{\rho(a)}\mathrm{Aut}(G, B, T))\Big).$$

The point is that the character group of $T/C(G)$ is the lattice of roots, i.e. $(\widehat{T/C(G)}) = \oplus_{\alpha \in \Delta} \mathbb{Z}\,\alpha$. The action of ρ on this lattice goes by permutation of the roots (15.1.(4)). It follows that the character group of ${}_{\rho(a)}(T/C(G))$ is a Γ_k-permutation module, hence ${}_\rho(a)(T/C(G))$ is a induced torus. By Shapiro's lemma and Hilbert 90, the Galois cohomology of induced tori vanishes, so $H^1(k, {}_{\rho(a)}(T/C(G))) = 0$ and $p_*{-1}(\rho([a])) = \{[\rho(a)]\}$ as desired.

Since the map $H^1(k, \mathrm{Aut}(G, B, T)) \cong H^1(k, \mathrm{Out}(G))$ factorises by $H^1(k, \mathrm{Aut}(G)) \cong H^1(k, \mathrm{Out}(G))$, it follows that $H^1(k, \mathrm{Aut}(G, B, T))$ injects in $H^1(k, \mathrm{Aut}(G))$. Thus $H^1(k, \mathrm{Aut}(G, B, T)) \cong H^1_{qd}(k, \mathrm{Aut}(G)) \cong H^1_{qd}(k, \mathrm{Aut}(G))$.

(2) Let $[z] \in H^1(k, \mathrm{Aut}(G))$ and put $a = p_*(z) \in Z^1(k, \mathrm{Out}(G))$. The same twisting trick shows that there exists $z' \in Z^1(k, {}_{\rho(a)}G)$ such that ${}_zG$ is isomorphic to ${}_{z'}\big({}_{\rho(a)}G\big)$. The unicity of the quasi-split group ${}_{\rho(a)}G$ is a consequence of (1).

□

If the underlying quasi-split form associated to a given k-form M of G is split, we say that M is an inner form. The morphism $\mathrm{Out}(G) \to \mathrm{Aut}(\Delta)$ gives rise to the map

$$H^1(k, \mathrm{Out}(G)) \to H^1(k, \mathrm{Aut}(\Delta)) = \mathrm{Hom}_{ct}(\Gamma_k, \mathrm{Aut}(\Delta))/\text{conjugacy} .$$

In other words, we attach to a k-form of G an action of the Galois group Γ_k on Δ up to conjugacy by $\mathrm{Aut}(\Delta)$. This is the so-called star action of Γ_k on Δ.

16. More Galois cohomology

16.1. **Witt-Tits decomposition.** The most general statement seems the following.

16.1. **Theorem.** *Let H be an affine group such that its neutral component H^0 is split reductive. We denote by Δ_H the Dynkin diagram of H.*

1. *Let $I \subset \Delta_H$. Denote by $H^1(k, N_H(P_I))_{irr}$ the classes of $[z]$ such that the twisted group ${}_zP_I$ is irreducible. Then the map $H^1(k, N_H(P_I))_{irr} \to H^1(k, H)$ is injective.*
2. *Let P_{I_1}, ... , P_{I_l} be representants of $H(k_s)$-conjugacy classes of the k–parabolic subgroups of H. Then we have*

$$\bigsqcup_{j=1,\dots,l} H^1(k, N_H(P_{I_j}))_{irr} \;\cong\; H^1(k, H).$$

The proof is based on Bruhat-Tits's paper [BT3, section 3].

Proof. (1) We are given cocycles $z, z' \in Z^1(k, \mathrm{N}_H(P_I))_{irr}$ having same image in $H^1(k, H)$. So there exists $h \in H(k_s)$ such that $z'_\sigma = h^{-1}\, z_\sigma\, \sigma(h)$ for all $\sigma \in \Gamma_k$. Then P_I defines in the same time a minimal k-parabolic subgroup of the twisted groups ${}_zH^0$ and ${}_{z'}H^0$. Since $\mathrm{Int}(h)$ is a k-isomorphism ${}_zH^0 \cong {}_{z'}H^0$, we see that P_I and hP_Ih^{-1} are two mimimal parabolic k-subgroups of ${}_zH^0$. By Borel-Tits' theorem, P_I and hP_Ih^{-1} are conjugate under ${}_zH^0(k)$. It follows that there exists $g \in {}_zH^0(k)$ such that $hP_Ih^{-1} = gP_Ig^{-1}$. Hence $n := g^{-1}h \in N_H(P_I)(k_s)$ and from the relation $z_\sigma\, \sigma(g)\, z_\sigma^{-1}$, we get

$$z'_\sigma = h^{-1}\, z_\sigma\, \sigma(h) = n^{-1} z_\sigma\, \sigma(n)$$

for all $\sigma \in \Gamma_k$. Thus $[z] = [z'] \in H^1(k, N_H(P_I))$.

(2)*First step : injectivity:* We are given $I, I' \subset \Delta$ occuring in the I_j's and cocycles $z \in Z^1(k, N_H(P_I))_{irr}$, $z' \in Z^1(k, N_H(P_{I'}))_{irr}$ having same image in $H^1(F, H)$. So there exists $h \in H(k_s)$ such that $z'_\sigma = h^{-1}\, z_\sigma\, \sigma(h)$ for all $\sigma \in \Gamma_k$. Then P_I (resp. $P_{I'}$) defines a minimal k-parabolic of the twisted groups

${}_zH^0$ (resp. ${}_{z'}H^0$). By the same argument as above, it follows that P_I and $hP_{I'}h^{-1}$ are conjugate under ${}_zH^0(k)$. Thus P_I and $P_{I'}$ are conjugate under $H(k_s)$ and $I = I'$. Statement (1) shows that $[z] = [z'] \in H^1(k, N_H(P_I))$.

Third step : surjectivity: Let $z \in Z^1(k, H)$. Let P be a a minimal k-parabolic subgroup of the twisted group ${}_zH^0$. There exists I occuring in the I_j's such that $P = hP_Ih^{-1}$ with $h \in H(k_s)$. It means that $z_\sigma\, \sigma(h)\, P_I\sigma(h)^{-1}\, z_\sigma^{-1} = hP_Ih^{-1}$ for all $\sigma \in \Gamma_k$, hence

$$h^{-1}\, z_\sigma\, \sigma(h)\, P_I\, \sigma(h)^{-1}\, z_\sigma^{-1}\, h = P_I.$$

It follows that $h^{-1}\, z_\sigma\, \sigma(h) \in N_H(P_I)(k_s)$. We conclude that $[z] \in \mathrm{Im}\Big(H^1(k, N_H(P_I))_{irr} \to H^1(k, H)\Big)$. $\square$

We can be slightly more precise with the following lemma.

16.2. **Lemma.** *Let $M = R_u(M) \rtimes L$ be a linear algebraic group. Then we have natural bijections*

$$H^1(k, L) \cong H^1(k, M) \cong H^1(k, M/R_u(M)).$$

16.2. **Classification.** We specialise the decomposition to the case $H = \mathrm{Aut}(G)$. For $I \subset \Delta$, we needs to describe $N_G(P_I)$ and its Levi subgroups. Following [Sp, 16.3.9.(4)], we define the subgroup I-automorphisms of G by

$$\mathrm{Aut}_I(G) = \mathrm{Aut}(G, P_I, L_I).$$

There is then an exact sequence

$$1 \to L_I/C(G) \to \mathrm{Aut}_I(G) \to \mathrm{Out}_I(G) \to 1,$$

where $\mathrm{Out}_I(G) = \mathrm{Out}(G) \cap \mathrm{Aut}(\Delta, I)$. Note also that

$$\mathrm{Aut}(G, P_I) = U_I \rtimes \mathrm{Aut}_I(G).$$

The Witt-Tits decomposition takes then the following beautiful shape

$$\bigsqcup_{[I] \subset \Delta/\,\mathrm{Out}(G)} H^1(k, \mathrm{Aut}_I(G))_{irr} \quad \cong \quad H^1(k, \mathrm{Aut}(G)).$$

We see that the decomposition encodes the set $\mathrm{Out}(G).I$. At this point we can go back to our favorite examples.

16.3. **Example. Wedderburn theorem.** Let A be a central simple algebra of degree d and consider its class $[A] \in H^1(k, \mathrm{PGL}_d)$. Then there exists a unique index I such that $[A]$ comes from a unique $[z] \in H^1(k, L_I)_{irr}$. Such an L_I is of the shape $\Big(\mathrm{GL}_{m_1} \times \cdots \mathrm{GL}_{m_r}\Big)/\mathbb{G}_m$ with $m_1 + .. + m_r = d$. One can check by hands (or below with the notion of eligible indices) that all m_i are equal, so L_I is the quotient of $(\mathrm{GL}_m)^r$ by the diagonal $\mathbb{G}_m$. It follows that

$${}_zL_I \cong \mathrm{GL}_1(D)^r/\mathbb{G}_m$$

for some unique central simple algebra D. Since ${}_zL_I$ is irreducible, we conclude that D is a division algebra.

16.4. **Example. Witt theorem.** In even rank, this is the case of the orthogonal group $O(2n)$ of the hyperbolic quadratic form $\sum_{i=1,\dots,n} X_i Y_i$. The set $H^1(k, O(2n))$ classifies quadratic forms of rank $2n$. In the decomposition, one can keep only the indices whose Levi supplement is $O(2) \times \mathbb{G}_m^{n-1}$, $O(4) \times \mathbb{G}_m^{n-2}$, $\cdots$ $O(2n-2) \times \mathbb{G}_m$, $O(2n)$ (there are no contribution of the others). Then the decomposition reads

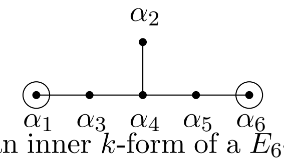
$$\bigsqcup_{i=1,\dots,n} H^1(k, O(2i))_{irr} \;\cong\; H^1(k, O(n)).$$

which is exactly Witt's theorem.

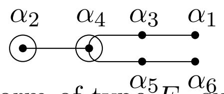
The Witt-Tits decomposition boils down the classification of arbitrary semisimple groups to irreducible one. In a certain extent, for $[z] \in H^1(k, \mathrm{Aut}_I(H))_{irr}$, the group ${}_zG$ can be recovered from its so-called anisotropic kernel ${}_zD(L_I)$ which is an anisotropic semisimple k-group [T1].

16.3. **Tits indices.** To a given k-form ${}_zG$ of G with $z \in Z^1(k, G)$, that its first invariant is its quasi-split form. The geometric conjugacy class of a minimal k-parabolic subgroup of ${}_zG$ provides the subset $I \subset \Delta$. This the Witt-Tits index of ${}_zG$.

Let us give an example about how we represent this data. The diagram



means that we deal with an inner k-form of a E_6-group such that the conjugacy class of minimal k-parabolic subgroups is $I = \{\alpha_1, \alpha_6\}$. The diagram



corresponds to an outer k-form of type E_6 such that the type of a minimal k-parabolic is $\{\alpha_2, \alpha_4\}$.

Since the orbit $\mathrm{Out}(G) .I \subset \Delta$ is encoded in the Witt-Tits decomposition, it follows that I is necessarily stable under $\mathrm{Out}(G)$ and even by $\mathrm{Aut}(\Delta)$ by considering the adjoint group of ${}_zG$; We have already seen in the previous lecture that I is always auto-opposite. There is one more condition related to action of the Weyl group.

16.5. **Definition.** We say that a subset $I \subset \Delta$ is eligible if it is self-opposite and if any $J \subset \Delta$ the following holds:

$$(*) \qquad J = w\, I \text{ for some } w \in W \text{ if and only } I = J.$$

Now we can state our observation.

16.6. **Proposition.** *Let $I \subset \Delta$ be the Witt-Tits indice of a twisted k-form of G. Then I is eligible.*

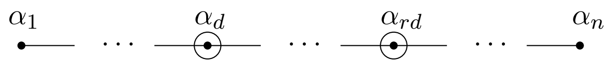
Proof. It is convenient to assume that G is adjoint. The Witt-Tits decomposition enables us to assume that our given k-form of G is ${}_zG$ where z is a cocycle with value in $\mathrm{Aut}_I(G)$. Consider a trivialization $\phi : G \times_k k_s \cong {}_zG \times_k k_s$ satisfying $\phi^{-1}\sigma(\phi) = z_\sigma$ for every $\sigma \in \Gamma_k$. The twisted k–group ${}_zP_I$ is a minimal parabolic k–subgroup of ${}_zG$, it is then of type I.

We have already seen that I is self-opposite. Let $J \subset \Delta$ such that $J = w\,I$ for some $w \in W$. Let $n_w \in N_G(T)(k)$ be a lift of w. Then $n_w.L_J = L_I$. In other words, L_I is a Levi k–subgroup of P_I and of $Q := n_w P_J n_w^{-1}$. Hence $\phi(Q)$ defines a parabolic k–subgroup of ${}_zG$, which has the same dimension than P_I. Borel-Tits's theorem implies that $\phi(P_I)$ and $\phi(Q)$ are ${}_zG(k)$-conjugated. This provides an element $g \in G(k_s)$ such that $\phi(Q) = \phi(gP_Ig^{-1})$. Thus $P_I \times_k k_s$ et and $Q \times_k k_s$ are $G(k_s)$-conjugated and $I = J$. $\square$

16.7. **Remark.** Geometrically speaking, eligible indexes are those of parabolic subgroups whose geometric conjugacy class is determined by the conjugacy class of their Levi subgroups.

The Tits indices (for all forms over all fields) were determined by Tits [T1]; the tables are also at the end of Springer's book [Sp].

We do not have tables for eligible indices, so we cannot compare both tables. Note that for the A_n-case, eligible indices are precisely the symmetric one, i.e. of the shape



We think interesting to compare both lists. One way could be by means of Tits indices of abelian subgroups of $\mathrm{Aut}(G)$, see section 3 of [GP].

Classification of semisimple groups II

Except in section 3, the ground field k is assumed for simplicity to be of characteristic zero.

17. Isogenies, Tits class

17.1. **Simply connected coverings.**

17.1. **Definition.** An isogeny $H_1 \to H_2$ is a surjective morphism of (connected) reductive with finite kernel.

We know that the kernel of an isogeny is a central subgroup of $H_1 \to H_2$. We fix once for all a semisimple split k-group G and a Killing couple (B,T). We denote by $G_{ad} = G/C(G)$ the adjoint group of G. We know that G admits a simply connected covering $\lambda : G^{sc} \to G$ defined by the following property : any isogeny $H \to G$ factorises uniquely by $\lambda : G^{sc} \to G$. The group $\ker(\lambda)$ is a split finite k-group of multiplicative type (i.e. a product of μ_m).

17.2. **Examples.** The morphism $\mathrm{SL}_n \to \mathrm{PGL}_n$ is the simply connected covering of PGL_n; its kernel is μ_n.

17.3. **Remark.** Geometrically speaking, this is true indeed that the group variety $G \times_k k_s$ is a simply connected variety. This can be seen from the analytic case by comparison theorems [SGA1]. If k is not separably closed, the variety G is not simply connected in the usual sense (but in the pointed sense).

The universal condition implies that $\mathrm{Aut}(G)$ acts on $\lambda : G^{sc} \to G$. So given a k-form ${}_zG$, we can twist everything by $z \in Z^1(k,G)$. So we get ${}_z\lambda : {}_zG^{sc} \to {}_zG$. By descent, one see that ${}_z\lambda$ is the universal covering of ${}_zG$ (check it !). Note that the fundamental group of ${}_zG$ is ${}_z\operatorname{Ker}(\lambda)$, which depends only of the quasi-split form of ${}_zG$.

17.2. **Boundary maps** [S1, I.5.7]**.** Let $f : \widetilde{H} \to H$ be a surjective morphism of reductive groups with central kernel A. We can then extend on the right the usual exact sequence. More precisely, there is a boundary map $\delta : H^1(k,H) \to H^2(k,A)$ such that the sequence

$$H^1(k,A) \to H^1(k,\widetilde{H}) \to H^1(k,H) \xrightarrow{\delta} H^2(k,A)$$

is an exact sequence of pointed sets. The boundary map is defined in terms of cocycles as follows. Let $[z] \in Z^1(k,H)$. Since the map $\widetilde{H}(k_s) \to H(k_s)$ is surjective, we take liftings $\widetilde{z}_\sigma$ of the z_σ's. Given $\sigma, \tau \in \Gamma_k$, the point is that $a_{\sigma,\tau} := \widetilde{z}_\sigma \, \sigma(z_\tau) \, \widetilde{z}_{\sigma\tau}^{-1}$ belongs to $A(k_s)$. One checks then this is a 2-cocycle whose cohomology class is independent of the lifting chosen. This class is $\delta([z]) = [a_{\sigma,\tau}] \in H^2(k,A)$.

17.4. **Example.** For the exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$, we get then a map $H^1(k, \mathrm{PGL}_n) \to \mathrm{Br}(k) = H^2(k, \mathbb{G}_m)$. This map associates to the class of a central k-algebra A of degree n its class $[A]$ in the Brauer group. A nice consequence of Hilbert 90 theorem is that this boundary map is injective [GS, §3].

17.3. **Tits class.** We shall define a natural invariant of a semisimple adjoint algebraic group in the H^2 of Galois cohomology of its fundamental group.

Given a semisimple adjoint group H, let $\widetilde{H} \to H$ be its simply connected covering and μ_H be its fundamental group. We have then a boundary map

$$\delta_H : H^1(k, H) \to H^2(k, \mu_H).$$

Now let $z \in Z^1(k, \mathrm{Aut}(G))$ and denote by G_0 the quasi-split form of G. By the previous lecture, we know that there exists $a \in Z^1(k, G_{0,ad})$ such that ${}_zG \cong {}_aG_0$. The first try to define such an invariant of the group ${}_zG$ is to apply the boundary

$$\delta_{G_0} : H^1(k, G_{0,ad}) \to H^2(k, \mu_{G_0})$$

to the class $[a] \in H^1(k, G_{0,ad})$. The problem is that this class is only well defined up to the action of the finite group $\mathrm{Out}(G_0)$. More precisely, by the torsion trick, the fiber at $[a]$ of $H^1(k, G_0) \to H^1(k, \mathrm{Aut}(G_0)) \cong H^1(k, \mathrm{Aut}(G))$ is $\ker\Big(H^1(k, {}_zG) \to H^1(k, \mathrm{Aut}({}_zG))\Big)$ which has no reason to be trivial.

The right construction has been pointed out by Knus, Merkurjev, Rost and Tignol [KMRT, page 426]. The idea is to see $G_0 \cong_b ({}_zG)$ as an inner twisted form of ${}_zG$ with $b \in Z^1(k, {}_zG_{ad})$ (b is the "opposite" cocycle of a). As before, the class $[b] \in H^1(k, {}_zG)$ is well defined up to $\mathrm{Out}(G_0)$. But $\mathrm{Aut}(G_0) \to \mathrm{Out}(G_0)$ is split surjective, so $\ker\Big(H^1(k, G_{0,ad}) \to H^1(k, \mathrm{Aut}(G_0))\Big) = 1$. Hence the class $[b] \in H^1(k, {}_zG_{ad})$ is uniquely defined. Therefore, we can define the Tits class

$$t({}_zG) = -\,\Delta_{G_{ad}}(b) \in H^2(k, \mu_{G_{ad}}).$$

The Tits class does not vary under isogenies. This fundamental invariant is related to representation theory. More precisely, the obtructions to define rationally geometrical representations can be explicited in terms of the Tits class [T2] [KMRT, page 427].

18. Serre's conjecture II

18.1. **Conjecture.** Let k be a field such that $\mathrm{cd}(k) \le 2$. Let H be a simply connected semisimple group. Then $H^1(k, H) = 1$.

The first important case is the case of the group $\mathrm{SL}_1(A)$ for a central simple algebra A/k. we know that $H^1(k, \mathrm{SL}_1(A)) = k^\times / \mathrm{Nrd}(A^\times)$.

18.2. **Theorem.** *(Merkurjev-Suslin* [MS]*) Assume that* $\mathrm{cd}(k) \le 2$. *Then* $\mathrm{Nrd}(A^\times) = k^\times$.

There is some converse statement: $\mathrm{cd}(k) \leq 2$ iff $\mathrm{Nrd}(D^\times) = L^\times$.for any central simple L-algebra D whose center L/k is a finite field extension.

For the other classical groups, G_2 and F_4, Serre's conjecture II has been proven by Bayer-Parimala [BP]. For other exceptional groups, it is still open despite recent progress [C][G3][COP][CGP]. Colliot-Thélène could come back to that in his serie of lectures. The conjecture has the following consequence on the classification of semisimple groups.

18.3. **Proposition.** *Assume that conjecture II holds for the field k.*

(1) *Let H be a semisimple group. Then the boundary map $H^1(k, H) \to H^2(k, \mu_H)$ is injective.*
(2) *The twisted k-forms of G are classified by their quasi-split forms and by their Tits classes. We mean that that two k-forms of G are isomorphic iff they have isomorphic quasi-split forms and same Tits classes.*

In the next lecture, we are interested in the case of p-adic fields which are of cohomological dimension 2 by local class field theory. More generally, we are interested in fields K which are complete for a discrete valuation with perfect residue field κ of cohomological dimension one. That includes fields of Laurent series $\kappa((t))$. Before local fields, we have to cover the case of finite fields.

19. Groups over finite fields

Let $k = \mathbf{F}_q$ be a finite field of characteristic $p > 0$. For an affine variety $X = \mathrm{Spec}(A)$, recall that the map $A \to A$, $x \mapsto x^q$, defines the Frobenius morphism $F : X \to X$ relatively to k.

19.1. **Lang's isogeny.** Let G/k be a linear algebraic connected k-group. Then $F : G \to G$ is a group morphism. We define the "Lang's isogeny" (which is not a group morphism) by

$$\mathcal{P} : G \to G, \ \ \mathcal{P}(g) = g^{-1}\, F(g),$$

Note that in the case of the additive group $\mathbf{G}_a$, this is nothing but the Artin–Schreier's covering. The right viewpoint is to consider the (right) action of G on itself by

$$s_a(g) := a \,.\, g = g^{-1}\, a\, F(g).$$

19.1. **Theorem.** *a) The group $G(k_s)$ acts transitively on itself by the action above.*

b) The morphism $\mathcal{P} : G \to G$ is a Galois covering of group $G(k)$.

Proof. a) We denote by $i : g \to g^{-1}$ the inversion morphism. Let $a \in G$. Then for any $X \in \mathrm{Lie}(G)\big(k(a)\big)$, one has

$$(ds_a)_e(X) = di_e(X)a + a.dF(X).$$

By means of a faithful representation, we see that $di_e = -id$. In the other hand, it is well known that $dF = 0$. Hence $(ds_a)_e = -id$. By homogeneity, it follows that $(ds_a)_g$ is invertible for all $g \in G$. The morphism s_a is then open. But this holds for any $a \in G$, so the orbit of a is also closed. Since G is connected, we conclude that there is a single orbit.

b) For avoiding any confusion, we denote by $\mathcal{P} : G_1 = G \to G$. Since $\mathcal{P}$ is étale and surjective, it is faithfully flat. For seeing that $\mathcal{P}$ is a $G(k)$-Galois covering, we first notice given $\sigma \in G(k)$ and $g \in G_1$ we have $\mathcal{P}(g^{-1}\sigma) = \sigma^{-1} g F(g^{-1})\, F(\sigma) = \mathcal{P}(\sigma)$. We deal then with a free action on $G(k)$ on the fibers of $\mathcal{P}$. It remains to see that the morphism

$$G_1 \times_k G(k) \xrightarrow{\gamma} G_1 \times_G G_1$$

defined by $(g, \sigma) \to (g, g.\sigma) = (g, \sigma^{-1} g)$ is an isomorphism. We know that γ is an isomorphism iff for any k–algebra A of finite type, we have a bijection $(G_1 \times G(k))(A) \approx (G_1 \times_G G_1)(A)$. Let A be such a k-algebra. We have

$$\begin{array}{rcl} \big(G_1 \times_G G_1)(A) & = & \{(g,h) \in G(A)^2 \quad | \quad \mathcal{P}(g) = \mathcal{P}(h)\} \\ & = & \{(g,h) \in G(A)^2 \quad | \quad (gh^{-1}) = F(gh^{-1})\} \\ & = & \{(g,h) \in G(A)^2 \quad | \quad h \in g.G(k)\} \\ & = & \big(G_1 \times G(k)\big)(A). \end{array}$$

This completes the proof that $\mathcal{P}$ is a $G(k)$-Galois covering. □

19.2. **Homogeneous spaces.** Lang's isogeny is the main ingredient for proving that homogeneous spaces have k-rational points.

19.2. **Corollary.** [Bo, 16.5] *Assume that G acts on some non-empty k-variety V. Then $V(k) \neq \emptyset$.*

In particular $H^1(k, G) = 1$ (Lang's theorem [La]). We get then the same consequences as in Steinberg's theorem.

19.3. **Corollary.** *Let H/k be a reductive group. Then H is quasi-split and $H^1(k, H) = 1$.*

19.4. **Corollary.** [S1, III.2.4, corollaire 3] *Let H/k be a linear algebraic group and let H^0 be its neutral component. Then the quotient map $H \to H/H^0$ induces a bijection*

$$H^1(k, H) \xrightarrow{\sim} H^1(k, H/H^0).$$

We assume now that G is reductive. It is quasi-split, so admits a Killing couple. In the first lectures, we used several times the hypothesis made on the base field by picking a regular semisimple element in the Lie algebra of G. This can be achieved in the present setting granting to the following result by Chevalley.

19.5. **Theorem.** [SGA3, XIV.7] *There exists $X \in \mathrm{Lie}(G)$ which is semisimple regular.*

Kneser's theorem, Bruhat-Tits theory

20. Introduction

We discuss the following result.

20.1. **Theorem.** *(Kneser* [K1]*) Let K be a p-adic field. Let H/K be semisimple group.*

(1) *If H is simply connected, then $H^1(K,H)=1$.*
(2) *Let $\lambda : \widetilde{H} \to H$ be the universal covering of H and $\mu_H = \mathrm{Ker}(\lambda)$. Then the boundary map induces a bijection*

$$H^1(k,H) \overset{\delta}{\xrightarrow{\sim}} H^2(K,\mu_H).$$

So Serre's conjecture II holds for p-adic fields. In particular, $H^1(k,H)$ is equipped then with a structure of abelian group. By Tate's duality [S1, II.5], the right handside is isomorphic to $H^0(K,\widehat{\mu})^D$, where D stands for the Pontryagin dual and $\widehat{\mu} = \mathrm{Hom}(\mu, \mathbb{G}_m)$ is the Galois module of characters of μ. In particular, $H^1(K,H)$ is a finite set; this was known previously by Borel-Serre [BS] [S1, III.4.3].

The Tits indices of such groups are listed [T1][T3] and the main ingredient is the following.

20.2. **Theorem.** *Let H/K be an anisotropic group defined over a p-adic field K. Then H is of type $A \times \cdots A$.*

The proof of Kneser's theorem goes by case by case analysis. Note that the classical groups are the matter of Kneser's lectures [K2]. Bruhat and Tits gave an uniform way to prove the first part. We discuss here three things:

(1) Proof of part (2) of Kneser's theorem.
(2) Kneser's proof for the group $\mathrm{Spin}(q)$.
(3) Bruhat-Tits proof for a split semisimple simply connected group.

21. Proof of the second part of Kneser's theorem

The proof goes by subtori. We need three lemmas.

21.1. **Lemma.** *Let F be a field of characteristic zero and let T/F be a torus. Then the natural map*

$$\varinjlim_n H^2(F, {}_nT) \to H^2(F,T)$$

is an isomorphism.

Then Tate's duality has its counterpart for tori.

21.2. **Lemma.** *Let T/K be a torus defined over the p-adic field K.*

(1) $H^2(K,T) \xrightarrow{\sim} \varprojlim_n H^0(k, \widehat{nT})/n$.
(2) *If T is anisotropic, then $H^2(K,T) = 0$.*

Proof. (1) From Lemma 21.1, we have

$$\begin{aligned} H^2(K,T)^D &\xrightarrow{\sim} \varprojlim_n H^2(K, {}_nT)^D \\ &\xrightarrow{\sim} \varprojlim_n H^0(k, \widehat{{}_nT}) \qquad \text{[Tate duality]}. \end{aligned}$$

The exact sequence $0 \to \widehat{T} \overset{\times n}{\to} \widehat{T} \to \widehat{{}_nT} \to 0$ gives a map $H^0(k, \widehat{nT})/n \hookrightarrow H^0(k, \widehat{{}_nT})$ and the same kind of argument as in Lemma 21.1 shows that

$$\varprojlim_n H^0(k, \widehat{nT})/n \xrightarrow{\sim} \varprojlim_n H^0(k, \widehat{{}_nT}).$$

(2) If T is anisotropic, then $\widehat{T}(k) = 0$. Hence $H^2(k,T) = 0$ by (1). $\square$

21.3. **Lemma.** *Let H be a quasi-split semisimple group defined over the p-adic field K. Then H admits a maximal K-torus T which is anisotropic.*

Proof. Let H_0 be the split form of H.
Step 1: $H = H_0$: Let (T_0, B_0) be a Killing couple of H_0. Denote by W_0 the Weyl group of T_0 and by $X_0 = G_0/N_{H_0}(T_0)$ the variety of tori of G_0. Raghunathan's theorem states that the type map $X_0(K) \to H^1(K, W_0)$ is surjective. But $H^1(K, W_0) = \mathrm{Hom}_{ct}(\Gamma_K, W_0)/int(W_0)$ and we have a natural surjective morphism $\Gamma_K \to \widehat{\mathbb{Z}}$. We define then a morphism $\phi_0 : \Gamma_K \to W_0$ by sending the Frobenius (i.e. the topological generator of $\widehat{\mathbb{Z}}$ on the Coxeter element w_0 of W_0. Since w_0 has no non trivial fixed point on $\widehat{T} \otimes_{\mathbb{Z}} \mathbb{R}$, it follows that the twisted torus ${}_{\phi_0}T_0$ is anisotropic. Thus ${}_{\phi_0}T_0$ embeds in H_0.
Step 2: the general case: Then $H = {}_aH_0$ for a cocycle $a \in Z^1(k, \mathrm{Aut}(H_0, B_0, T_0))$. The point is that w_0 is fixed under $\mathrm{Aut}(H_0, B_0, T_0)$, so we can twist the torus ${}_{\phi_0}T_0$ by a. This new torus, say T, is a fortiori anisotropic and can be embedded in H according to Raghunathan's theorem. $\square$

We can now proceed to the proof of Theorem 20.1.(2).

Proof. The first statement (1) of Theorem 20.1 and the torsion trick show that the boundary map δ_H is injective. We focus then on the surjectivity.

Step 1: Reduction to the adjoint case: We consider the following commutative diagram

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
 & 1 \arrow[d] & & \nu \arrow[d] & \\
1 \arrow[r] & \mu_H \arrow[r] \arrow[d] & H^{sc} \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_{H^{sc}} \arrow[r] \arrow[d] & H^{sc} \arrow[r] & H_{ad} \arrow[r] \arrow[d] & 1 \\
 & \nu \arrow[d] & & 1 & \\
 & 1 & & &
\end{tikzcd}$$

By taking cohomology, we get the following commutative diagram of pointed sets

$$\begin{tikzcd}
H^1(K,\nu) \arrow[r, phantom, "="] \arrow[d] & H^1(K,\nu). \arrow[d, "\Delta"'] \\
H^1(K,H) \arrow[r, "\delta_H"] \arrow[d] & H^2(k,\mu_H) \arrow[d] \\
H^1(K,H^{ad}) \arrow[r] \arrow[d] & H^2(k,\mu_{H^{sc}}) \arrow[d] \\
H^2(K,\nu) \arrow[r, phantom, "="] & H^2(K,\nu).
\end{tikzcd}$$

If the boundary map for H_{ad} is surjective, it follows that $\mathrm{Im}(\delta_H).\Delta(H^1(K,\nu)) = H^2(k,\mu_H)$. But there is a natural action of $H^1(K,\nu)$ on $H^1(K,H)$, thus $\mathrm{Im}(\delta_H) = H^2(k,\mu_H)$.

Step 2: Reduction to the quasi-split case: The adjoint group H is an inner form of its quasi split form H_0. Then $H = {}_a(H_0)$ for a cocycle $a \in Z^1(k,H_0)$. Denote by μ the fundamental group of H_0 and H. The following commutative diagram

$$\begin{tikzcd}
H^1(k,H_0) \arrow[r, "\delta_{H_0}"] & H^2(k,\mu) \\
H^1(k,H) \arrow[r, "\delta_H"] \arrow[u, "\theta_z", "\wr"'] & H^2(k,\mu) \arrow[u, "?+\delta({[z]})", "\wr"']
\end{tikzcd}$$

commutes [S1, I.5.7]. So the surjectivity of δ_{H_0} is equivalent to the surjectivity of δ_H.

Step 3: By the reductions, we deals with a quasi-split adjoint group H/K. Lemma 21.3 provides a maximal K-torus T^{sc} of H^{sc} which is anisotropic. Define $T = T^{sc}/\mu_H$. The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_H & \longrightarrow & T^{sc} & \longrightarrow & T & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_H & \longrightarrow & H^{sc} & \longrightarrow & H & \longrightarrow & 1 \end{array}$$

induces the following commutative diagram of pointed sets

$$\begin{array}{ccccc} H^1(K,T) & \longrightarrow & H^2(K,\mu_H) & \longrightarrow & H^2(K,T^{sc}) \\ \downarrow & & \| & & \\ H^1(K,H) & \xrightarrow{\delta_H} & H^2(K,\mu_H) & & \end{array}$$

But $H^2(K,T^{sc}) = 0$ by Lemma 21.2. So we conclude that the boundary map δ_H is surjective. □

21.4. **Remark.** The proof shows that any semisimple K-group H has a maximal K-split torus which is anisotropic. In other words, Lemma 21.3 extends to all groups and the proof of that fact is done in an uniform way (compare [PR, Theorem 6.21]).

21.5. **Remark.** The proof shows also that the map $H^1(K,T) \to H^1(K,H)$ is surjective for any maximal K-torus which is anisotropic. This fact is true also in the real case.

22. Galois cohomology of $\mathrm{Spin}(q)$

We recall firstly facts from the quadratic form theory [Sc] for a base field k such that $\mathrm{Char}(k) \neq 2$.

22.1. **Witt group.** We denote by $W(k)$ the Witt group of k, i.e. the abelian group with the following presentation:

Generators: isometry classes of (regular) quadratic k-forms,

Relations: $[q \perp q'] - [q] - [q']$, $[\mathbb{H}]$.

22.1. **Examples.** $W(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$ and $W(\mathbb{R}) = \mathbb{Z}$.

The Witt theorem shows that there is a correspondence between $W(k)$ and the isometry classes of anisotropic quadratic forms. The tensor product of quadratic forms provides a ring structure on $W(k)$. The dimension modulo 2 induces a ring homomorphism

$$W(k) \to \mathbb{Z}/2\mathbb{Z}$$

whose kernel $I(k)$ is the fundamental ideal. We get then a sequence

$$\cdots \quad I^{n+1}(k) \subset I^n(k) \subset \quad \cdots \quad I^2(k) \subset I(k) \subset W(k).$$

Observe that the tensor product induces a structure of $k^\times$-modules on the $I^n(K)$ (and even of $k^\times/(k^\times)^2$-modules). Recall the notation $\langle a_1, ..., a_n\rangle$ for the quadratic forms $\sum a_i x_i^2$ and denote by $\langle\langle a_1, ..., a_n\rangle\rangle = \langle 1, -a_1\langle \otimes \cdots \otimes \langle 1, -a_n\rangle$ for n-fold Pfister forms. The $k^\times$-module $I^n(k)$ is generated by Pfister forms. Recall the following facts.

• The discriminant induces an isomorphism $I(k)/I^2(K) \xrightarrow{\sim} k^\times/(k^\times)^2$;

• The group $W(k)$ has the following presentation:

Generators: $k^\times$,

Relations: a^2, $ab + \big(ab(a+b)\big) - a - b$;

• The Pfister forms are multiplicative (i.e. the product of two values is a value); isotropic Pfister forms are hyperbolic [Sc, §4].

Indeed the quadratic forms $\langle a, b\rangle$ and $\langle ab, ab(a+b)\rangle$ are isomorphic; the amazing fact is there are no other relations.

22.2. **Remark.** It is a deep theorem by Arason-Pfister that $\bigcap\limits_n I^n(k) = 0$, see [Sc, §4].

22.2. **Residues, quadratic forms over local fields.** Let K be a complete field for a discrete valuation. Denote by O its valuation ring and by κ its residue field (of characteristic $\neq 2$). Let π be an uniformizing parameter of K. A quadratic K-form q can be written as

$$q = \langle u_1, ..., u_r\rangle + \pi\langle v_1, ..., v_s\rangle$$

where the u_i's and the v_j's are units of O. By reducing mod π, we get then two quadratic κ-forms $q_0 = \langle \overline{u}_1, ..., \overline{u}_r\rangle$, $q_1 = \langle \overline{v}_1, ..., \overline{v}_s\rangle$. It turns out that the Witt class of q_0 and q_1 are well defined. This can be checked using the presentation of $W(K)$. We get then a map

$$\begin{array}{ccc} W(K) & \xrightarrow{s_\pi \oplus \partial} & W(\kappa) \oplus W(\kappa) \\ [q] & \mapsto & ([q_0],\, [q_1]) \end{array}$$

The map s_π is called the specialization map, it depends of the choice of π. The map ∂ is the residue map, it is canonically defined.

22.3. **Theorem.** *(Springer, see* [M, §5]*)*

(1) *The map*

$$W(K) \xrightarrow{s_\pi \oplus \partial} W(\kappa) \oplus W(\kappa)$$

is an isomorphism.

(2) *If q, q_0, q_1 are as above, q is anisotropic K-form if and only of q_0 and q_1 are anisotropic κ-forms.*

Note that for working in a canonical way, it is necessary to deal with the exact sequence

$$0 \to W(k) \to W(K) \xrightarrow{\partial} W(\kappa) \to 0.$$

If K is a p-adic field, since $W(\kappa)$ is a group of order 4 (its structure depends of $\left(\frac{-1}{p}\right)$), the group $W(K)$ is of order 8 and is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^2$ or $(\mathbb{Z}/2\mathbb{Z})^4$. Since rank three quadratic forms over κ of isotropic, Theorem 22.3.(2) shows that any quadratic K-form of rank ≥ 5 is isotropic.

22.3. **Galois cohomology of** $\mathrm{SO}(q)$**.** Let q be a regular a quadratic form, as consequence of descent, we have seen that $H^1(k, \mathrm{O}(q)$ classifies the (regular) quadratic forms of rank $\mathrm{rank}(q)$. One can see also that directly by using Hilbert 90 as follows. Let $\mathrm{O}(q) \subset \mathrm{GL}(V)$ be the standard representation of $\mathrm{O}(q)$. Given $z \in Z^1(k, \mathrm{O}(q))$, we have $z_\sigma = g^{-1}\,\sigma(g)$ for some $g \in \mathrm{GL}(V)(k_s)$. One sees that the quadratic form $q' = q \circ g^{-1}$ is defined over k and its isometry class is function of $[z] \in H^1(k, \mathrm{O}(q))$.

22.4. **Lemma.** *Let q be a regular quadratic form of even rank $2n$. Then $H^1(k, \mathrm{SO}(q))$ injects in $H^1(k, \mathrm{O}(q))$ and corresponds to the isometry classes of (regular) quadratic forms of rank $2n$ of discriminant $\mathrm{disc}(q)$.*

Proof. We write $q = \langle a_1, \cdots, a_n \rangle$ in diagonal form. The point is that the exact sequence $1 \to \mathrm{SO}(q) \to \mathrm{O}(q) \to \mu_2 \to 1$ is split, so the map $H^1(k, \mathrm{SO}(q)) \to H^1(k, \mathrm{O}(q))$ has trivial kernel. Since this is true for all twists, this map is indeed injective by the usual torsion trick. About its image, we know that the map $H^1(k, \mathrm{O}(q)) \to H^1(k, \mu_2) = k^\times/(k^\times)^2$ sends a class $[q']$ to $\mathrm{disc}(q')/\mathrm{disc}(q)\,.\,(k^\times)^2$, hence the statement. □

Following [S1, III, annexe], we investigate the case of $\mathrm{Spin}(q)$ which is the simply connected covering of $\mathrm{SO}(q)$ ($\dim(q) = 2n$). We have an exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}(q) \to \mathrm{SO}(q) \to 1.$$

As usual, it induces the exact sequence of pointed sets

$$1 \to \mu_2 \to \mathrm{Spin}(q)(k) \to \mathrm{SO}(q)(k) \xrightarrow{\varphi}$$

$$k^\times/(k^\times)^2 \to H^1(k, \mathrm{Spin}(q)) \to H^1(k, \mathrm{SO}(q)) \xrightarrow{\delta} {}_2\mathrm{Br}(k).$$

The characteristic map is called the spinor norm $\mathrm{Ns} : \mathrm{SO}(q)(k) \xrightarrow{\varphi} k^\times/(k^\times)^2$. It is easy to describe by the Cartan-Dieudonné theorem which states that $\mathrm{O}(q)(k)$ is generated by reflexions τ_v for anisotropic v. Recall that $\tau_v(v) = v$ and τ_v is the identity on the orthogonal space of v with respect to q. The group $\mathrm{SO}(q)(k)$ is then generated by even product of reflexions ($2n$ are enough); we have

$$\mathrm{Ns}\Big(\prod_{i=1}^{2n} \tau_{v_i}\Big) = \prod_i q(v_i) \quad \in \quad k^\times/(k^\times)^2.$$

22.5. **Remark.** If q is isotropic, then the spinor norm $\mathrm{Ns} : \mathrm{SO}(q)(k) \xrightarrow{\varphi} k^\times/(k^\times)^2$ is surjective.

The boundary map δ is more complicated and it is related with the Hasse-Witt invariant of quadratic forms. Given $[q'] \in H^1(k, \mathrm{SO}(q)$, we have $[q'] - [q] \in I^2(k)$. We have then a map $H^1(k, \mathrm{SO}(q)) \subset I^2(k)$. In the other hand, we have the Milnor's morphism [M, §4]

$$e_2 : I^2(k)/I^3(k) \to {}_2\,\mathrm{Br}(k), \ \ \langle\langle a, b \rangle\rangle \mapsto [(a,b)],$$

where (a,b) stands for the quaternion algebra $X^2 = a, Y^2 = b, XY + YX = 0$. It turns out that the diagram

$$\begin{array}{ccc} H^1(k, \mathrm{SO}(q)) & \longrightarrow & I^2(k)/I^3(k) \\ {\scriptstyle\delta}\big\downarrow & & {\scriptstyle e_2}\big\downarrow \\ {}_2\,\mathrm{Br}(k) & = & {}_2\,\mathrm{Br}(k) \end{array}$$

commutes. But e_2 is injective by Merkurjev's theorem (and even bijective). So by diagram chase, we get that the image of $H^1(k, \mathrm{Spin}(q))$ in $H^1(k, \mathrm{SO}(q))$ is the set of isometry classes of quadratic form q' of rank $2n$ such that $[q'] - [q] \in I^3(k) \subset W(k)$.

22.4. **Kneser's theorem for** $\mathrm{Spin}(\mathbf{q})$**.** Let K be a p-adic field with $p \leq 3$. Let q be a quadratic K-form of rank ≤ 5. We proceed to the proof of the vanishing of $H^1(K, \mathrm{Spin}(q))$. By remark 22.5, the spinor norm for q is surjective, so it is enough to show that the map

$$H^1(K, \mathrm{Spin}(q)) \to H^1(K, \mathrm{SO}(q))$$

is trivial. Its image is included in $[q] \oplus I^3(K)$. But the $K^\times$ module $I^3(K)$ is generated by three Pfister forms $\langle\langle a, b, c \rangle\rangle$. Since $8 \geq 5$, three Pfister forms are isotropic and then hyperbolic.So $I^3(K) = 0$ and we conclude that $H^1(K, \mathrm{Spin}(q)) = 1$.

22.6. **Remark.** The fact that e_2 is bijective for local fields is actually quite easy by appealing to the classification of quadratic forms for such fields. So there is no need of Merkurjev's theorem here.

23. Bruhat-Tits buildings and Galois cohomology

Let K be a complete field for a discrete valuation, O its ring of integer. We assume that the residue field κ of O is perfect. We denote by $\widetilde{K}$ the maximal unramified extension of K and by $\widetilde{O}$ its valuation ring.

23.1. **Parahoric subgroups.** Let $G/\operatorname{Spec}(\mathbb{Z})$ be a semisimple simply connected Chevalley group scheme. It is equipped with the following data

(1) a maximal split torus $T/\operatorname{Spec}(\mathbb{Z})$ of G,
(2) an irreducible reduced root system $\Phi = \Phi(T, G) \subset \widehat{T} \otimes_{\mathbb{Z}} \mathbb{R}$ (where $\widehat{T} = \mathrm{Hom}_{\mathbb{Z}-gr}(T, \mathbb{G}_{m,\mathbb{Z}})$ stands for the character group of T) equipped with a base Δ which defines Φ^+,

(3) a family of morphisms $(U_\alpha : \mathbb{G}_{a,\mathbb{Z}} \to G)_{\alpha\in\Phi}$ and a Borel subgroup $B/\operatorname{Spec}(\mathbb{Z})$ of G such that for any ordering $\Phi^+ = (\alpha_i)_{i=1,..,q}$, the product on G induces an isomorphism of $\mathbb{Z}$–schemes

$$T \times \prod_{i=1,..,q} \mathbb{G}_a \quad \xrightarrow{id\times \prod\limits_{i=1,..,q} U_{\alpha_i}} \quad B.$$

We denote by α_0 the opposite of the maximal root of Φ, $\widehat{T}^0 = \operatorname{Hom}_{\mathbb{Z}-gr}(\mathbb{G}_{m,\mathbb{Z}}, T)$, $\Phi^\vee = (\alpha^\vee)_{\alpha\in\Phi} \subset \widehat{T}^0$ the dual root sytem and by $(\overline{\omega}_\alpha)_{\alpha\in\Delta}$ the fundamental weights, i.e. the elements of $\widehat{T} \otimes_\mathbb{Z} \mathbb{R}$ satisfying $\langle \omega_\alpha, \beta^\vee\rangle = \beta^\vee(\overline{\omega}_\alpha) = \delta_{\alpha,\beta}$. We denote by $V = \widehat{T}^0 \otimes_\mathbb{Z} \mathbb{R}$ and its dual $V' = \widehat{T} \otimes_\mathbb{Z} \mathbb{R}$. Recall that an affine root $a = (\alpha, n)$ with $\alpha \in \Phi$, $n \in \mathbb{Z}$ is an affine function $V \to \mathbb{R}$

$$a(v) = (\alpha, v) + n,$$

We consider the following finite set of affine roots

$$\Delta_e = \{(\alpha, 0)\}_{\alpha\in\Delta} \cup \{(\alpha_0, 1)\},$$

which consist of vertices of the completed Dynkin diagram associated to Δ. The elements Δ_e define the "alcôve"

$$C = \{\ v \in V \ \ | \ \ a(v) > 0 \ \ \forall a \in \Delta_e\}.$$

Ths standard parahoric subgroups of $G(K)$ are certain (abstract) bounded subgroups of $G(K)$. Let π be an uniformizing parameter of K. To any non-empty subset Ω of V, we can attach the subgroup P_Ω of $G(K)$ generated by

$$T(O), U_\alpha(\pi^{n_\Omega(\alpha)}O) \quad (\alpha \in \Phi),$$

with

$$n_\Omega(\alpha) = \operatorname{Sup}\Big\{ [(\alpha, v)],\ \ v \in \Omega \Big\}.$$

The second part of Bruhat-Tits [BT2] theory provides a group scheme $\mathfrak{P}_\Omega/\operatorname{Spec}(O)$ satisfying the following properties :

(1) $\mathfrak{P}_\Omega/\operatorname{Spec}(O)$ is smooth,
(2) $\mathfrak{P}_\Omega(O) = P_\Omega$,
(3) $\mathfrak{P}_\Omega \times_O K = G_K$.
(4) The special fiber $\mathfrak{P}_\Omega \times_O \kappa$ is a connected κ-group.

In particular, we can define for a non-empty subset $\Theta \subset \Delta_e$ the parahoric group scheme $\mathfrak{P}_\Theta/\operatorname{Spec}(O)$ associated to

$$C(\Theta) = \{\ v \in V \ \ | \ \ a(v) > 0 \ \ \forall a \in \Theta\}.$$

The torus $T \times_\mathbb{Z} O$ is a maximal O–torus of $\mathfrak{P}_\Theta$ and the K-subgroups $(U_\alpha/K)_{\alpha\in\Phi}$ de G_K extend in subgroup schemes $\mathfrak{U}_{\Theta,\alpha} \subset \mathfrak{P}_\Theta$ (*loc. cit.*, §4.1). In particular,if $\Theta = \Delta_e \setminus \Delta$, we have $\mathfrak{P}_\Theta/\operatorname{Spec}(O) = G \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(O)$. The group $\mathfrak{P}_{\Delta_e}$ is a Iwahori subgroup (which plays here the role of a Borel subgroup); the parahoric subgroups $(\mathfrak{P}_\Theta/\operatorname{Spec}(O))_{\Theta\subset\Delta_e}$ are called the standard parahoric subgrioups of G_K; the abstract subgroups $(\mathfrak{P}_\Theta(O))_{\Theta\subset\Delta_e}$ are the parahoric subgroups of $G(K)$ containing the Iwahori subgroup $\mathfrak{P}_{\Delta_e}(O)$.

We denote by $\overline{\mathfrak{P}}_\Theta/\kappa$ the special fiber $\mathfrak{P}_\Theta$. It is connected and is generated by the k–torus $T\times_{\mathbb{Z}}\kappa$ and the $\overline{\mathfrak{U}}_{\Theta,\alpha}$ for $\alpha\in\Phi$. We denote $M_\Theta/k = (\overline{\mathfrak{P}}_\Theta)_{réd}/k$, i.e. the quotient of $\overline{\mathfrak{P}}_\Theta$ by its unipotent radical. It is isomorphic to the Levi subgroup of $\overline{\mathfrak{P}}_\Theta/k$ generated by T/k and the $\overline{\mathfrak{U}}_{\Theta,\alpha}$ for $\alpha\in\Theta$.

We defined then standard parahoric subgroups; there are $r+1$ ($r=\mathrm{rank}(G)$) maximal bounded subgroups. The other parahoric subgroups are the $G(K)$-conjugate of those groups.

23.2. **Buildings.** The Bruhat-Tits building $\mathcal{I}$ of G/K is the polysimplicial complex whose vertices are the maximal parahoric subgroups of $G(K)$ and a subset F of the set of vertices is the set of vertices of a simplex of $\mathcal{I}$ if and only if the intersection of the $P\in F$ is a parahoric subgroup. The group $G(K)$ acts on $\mathcal{I}$ and the stabilizers of $\mathcal{I}$ for this action are the parahoric subgroups of $G(K)$. The building $\mathcal{I}$ is a contractible topological space. The simplest example if for SL_2 on the field $\mathbf{F}_2((t))$; this is the Bruhat-Tits building. In this case, one has $P_0=\mathrm{SL}_2(\mathbf{F}_2[[t]])$ and

$$P_1=\begin{pmatrix} \mathbf{F}_2[[t]] & t\mathbf{F}_2[[t]] \\ t^{-1}\mathbf{F}_2[[t]] & \mathbf{F}_2[[t]] \end{pmatrix}\cap \mathrm{SL}_2(\mathbf{F}_2((t))).$$

The building $\widetilde{I}$ of the group $G/\widetilde{K}$ is equipped with the action of $G(\widetilde{K})\rtimes\Gamma_\kappa$.

23.3. **The decomposition.** The Bruhat-Tits decomposition is the following

$$\coprod_{\Theta\subset\Delta_e} H^1(\kappa,M_\Theta)_{irr}\xrightarrow{\sim} H^1(\Gamma_\kappa,G(\widetilde{K}))\xrightarrow{\sim} H^1(K,G).$$

The bijectivity of the right handside follows from Steinberg's theorem since $\mathrm{cd}(\widetilde{K})=1$. The map $H^1(k,M_\Theta)\to H^1(K,G)$ is defined by the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma_\kappa,\mathfrak{P}_\Theta(\widetilde{O})) & \longrightarrow & H^1(K,G) \\ \wr\downarrow & & \\ H^1(\kappa,\overline{\mathfrak{P}}_\Theta), & & \\ \wr\downarrow & & \\ H^1(\kappa,M_\Theta), & & \end{array}$$

where the first vertical map is an isomorphism according to Hensel's lemma. The second vertical map is an isomorphism as well because the unipotent radical of $\overline{\mathfrak{P}}_\Theta$ is a κ–split unipotent subgroup. As in the case of the injective map $H^1(\kappa,P)\to H^1(\kappa,G)$ for a parabolic subgroup P/κ of G/κ, the injectivity of the map $H^1(\kappa,M_\Theta)=H^1(\Gamma_\kappa,P_\Theta(\widetilde{O}))\to H^1(\Gamma_\kappa,G(\widetilde{K}))$ comes from a conjugacy statement. The injectivity of the sum is an immediate consequence of the parahoric-parabolic correspondence. That is a one to one correspondence between parahoric subgroups of $\mathfrak{P}_\Theta$ and parabolic subgroups of M_Θ. The surjectivity is indeed the central link between Galois cohomology and buildings. Let z_σ be a 1-cocycle of Γ_κ with value in $G(\widetilde{K})$. We define the twisted action du group Γ_κ on the building $\widetilde{\mathcal{I}}$ by

$$\sigma * x = z_\sigma \, . \, \sigma(x) \quad (x \in I).$$

As Γ_κ is a compact group and $\widetilde{\mathcal{I}}$ is a space with negative curvature (precisely $CAT(0)$), there exists a fixed point x de $\widetilde{\mathcal{I}}$ for the twisted action of Γ_κ. There exists then $\theta \in C$ and $g \in G(\widetilde{K})$ such that $x = g \, . \, \theta$. Up to replace z_σ by $z'_\sigma = g^{-1} \, z_\sigma \, \sigma(g)$, we can assume that $x = \theta$. Hence

$$z_\sigma . \theta = \theta \quad (\sigma \in \Gamma_\kappa),$$

which means that z has value in $P_\theta(\widetilde{O}) = P_\Theta(\widetilde{O})$ for a subset Θ of $\mathbf{\Delta}$.

23.1. **Remark.** We can recover actually the decomposition of the Witt group of K from Bruhat-Tits decomposition. In other words, this decomposition generalises to arbitrary semisimple groups the Springer decomposition of $W(K)$.

23.4. **Kneser's theorem in the split case.** We assume now that K is a p-adic field. By Langs' theorem, the cohomology sets $H^1(\kappa, M_\Theta)$ vanishes. We conclude then that $H^1(K, G) = 1$.

23.2. **Remark.** The general case is technically more complicated but the split case relates quite faithfully the proof of Kneser's theorem by Bruhat-Tits [BT3, 3.15].

R-equivalence, flasque resolutions of tori

24. Introduction

As usual, the ground field is assumed for simplicity to be of characteristic zero. Given a k-variety X, Y. Manin defined the R-equivalence on the set of k-points $X(k)$ as the equivalence relation generated by the following elementary relation. Denote by $\mathcal{O}$ the semi-local ring of $\mathbf{A}^1_k$ at 0 and 1.

24.1. **Definition.** Two points x_0, $x_1 \in X(k)$ are elementary R-equivalent is there exists $x(t) \in X(\mathcal{O})$, such that $x(0) = x_0$ and $x(1) = x_1$.

We denote then by $X(k)/R$ the set of R-equivalence classes. This invariant measures somehow the defect for parametrizing rationally the k-points of X. The following properties follow readily from the definition.

(**1**) additivity : $(X \times_k Y)(k)/R \cong X(k)/R \times Y(k)/R$;

(**2**) "homotopy invariance" : $X(k)/R \xrightarrow{\sim} X(k(v))/R$.

The plan is to investigate R-equivalence for linear algebraic groups and to focus on the case of tori worked out by Colliot-Thélène-Sansuc [CTS1] [CTS2].

25. R-equivalence for linear algebraic groups

Let G/k be a connected linear algebraic group. First the R-equivalence on $G(k)$ is compatible with the group structure. More precisely, denote by $R(k, G) \subset G(k)$ the R-equivalence class of e. Then $R(k, G)$ is a normal subgroup and $G(k)/R(k, G) \cong G(k)/R$. Therefore $G(k)/R$ has a natural group structure. We can already ask the following optimistic open question based on known examples.

25.1. **Question.** Is $G(k)/R$ an abelian group ?

Notice the following fact.

25.2. **Lemma.** *Two points of $G(k)$ which are R-equivalent are elementary equivalent.*

Thus the elementary relation is an equivalence relation.

25.3. **Proposition.** *Let $U \subset G$ be an open subset. Then $U(k)/R \xrightarrow{\sim} G(k)/R$.*

Proof. By Grothendieck's theorem, G is an unirational k-variety. It means that there exists a (non-empty) subset V of an affine space and a dominant map $h : V \to G$. We can assume that $e \in h(V(k))$. Then $h(V(k))$ is Zariski dense in G and consists of elements R-equivalent to e. In particular $R(k, G)$ is Zariski dense in G, so $R(k, G).U = G$. Hence $U(k)/R \to G(k)/R$

is surjective. In the way around, we are given two elements u, $u' \in U(k)$ which are R-equivalent in G. By Lemma 25.2, there exists $g \in G(\mathcal{O})$ such that $g(0) = u$ and $g(1) = u'$. But we see that g belongs actually to $U(\mathcal{O})$, so we conclude that u and u' are R-equivalent in U. □

Recall that X is k-rational if X is birationally isomorphic to an affine space.

25.4. **Corollary.** *Let G_1 and G_2 be linear algebraic groups which are stably rationally equivalent. Then there is a bijection $G_1(k)/R \cong G_2(k)/R$. In particular, if G is k-rational, then $G(k)/R = 1$.*

We say that X a k-variety is stably k-rational if there exists $n \geq 0$ such that $X \times_k \mathbf{A}_k^n$ is k-rational. By the additivity property, we have

25.5. **Corollary.** *If G is k-rational, then $G(k)/R = 1$.*

26. R-equivalence on tori

We shall use that the category of k–tori is anti-equivalent to the category of Γ_k-lattices, i.e. the category of lattices equipped with a continuous action of Γ_k.

26.1. **Coflasque modules.** Let Γ be a finite group. We denote by $C(\Gamma)$ the following semigroup:

Generators : [M], M Γ-lattice;

Relations : $[P] = 0$, P permutation Γ-lattice.

In other words, two Γ–lattices M, N have same class in $C(\Gamma)$ if $M \oplus P \cong N \oplus Q$ with P, Q permutation Γ-lattice.

26.1. **Definition.** Let M be a Γ-lattice. We say that M is invertible if there exists a Γ-lattice N such that its class is invertible in $C(\Gamma)$.

In other words, invertible Γ-modules are direct summands of permutation modules.

26.2. **Definition.** Let M be a Γ-lattice. We say that M is coflasque if $H^1(\Gamma', M) = 0$ for all subgroups $\Gamma' \subset \Gamma$.

We say that M is flasque if the dual module M^0 is coflasque. By Shapiro's lemma, it follows that permutation lattices are flasque and coflasque. More generally, invertible Γ-lattices are flasque and coflasque.

26.3. **Remark.** This notion is stable by change of groups $f : \widetilde{\Gamma} \to \Gamma$: if M is a coflasque Γ-lattice, then it is a coflasque $\widetilde{\Gamma}$-lattice as well. If f is surjective, then the converse is true. Therefore this notion makes sense for profinite groups.

26.4. **Lemma.** [CTS2, 0.6] *Let M be a Γ-lattice.*

(1) *M admits a coflasque resolution, that is an exact sequence of Γ-modules*
$$0 \to C \to P \to M \to 0$$
such that P is permutation and C is coflasque.

(2) *M admits a flasque resolution, that is an exact sequence of Γ-modules*
$$0 \to M \to P \to F \to 0$$
such that P is permutation and F is flasque.

(3) *The class of F in $C(\Gamma)$ depends only of M.*

We get then an additive map
$$\begin{array}{rcl} p : \{\ \Gamma\text{-lattices}\} & \longrightarrow & C(\Gamma) \\ M & \mapsto & [F]. \end{array}$$

26.2. **Flasque resolution of tori.** We are given a k-torus T, its character group $\widehat{T}$ is a Γ_k–lattice. The kernel of the action $\Gamma_k \to \mathrm{Aut}(\widehat{T})$ is of finite index, this is the Galois group of the minimal splitting field k_T/k. We denote by $\Gamma(T)$ its Galois group.

We say that T is coflasque (resp. flasque) if $\widehat{T}$ a flasque (resp. coflasque) Γ_k-lattice. Equivalently, $\widehat{T}$ a coflasque (resp. flasque) $\Gamma(T)$-lattice. By dualizing Proposition 26.4, we get a flasque resolution of the torus T, namely
$$1 \to S \xrightarrow{i} E \xrightarrow{f} T \to 1,$$
where E is an induced torus and S is a flasque torus.

26.5. **Theorem.** *The characteristic map $T(k) \to H^1(k,S)$ induces an isomorphism*
$$T(k)/R \xrightarrow{\sim} H^1(k,S).$$

If the theorem is true, we should have $H^1(k,S) \xrightarrow{\sim} H^1(k(t),S)$ by property (2) of the introduction. The proof goes by proving that fact before.

26.6. **Lemma.** *Let S/k be a flasque torus as above. Then*
$$H^1(k,S) \xrightarrow{\sim} H^1\big(\Gamma_k, S(\mathcal{O}_{k_s})\big) \xrightarrow{\sim} H^1\big(\Gamma_k, S(k_s(t))\big) \xrightarrow{\sim} H^1(k(t),S).$$

Proof. Tensorising the split sequence of Galois modules
$$1 \to k_s^\times \to k_s(t)^\times \to \bigoplus_{x \in \mathbf{A}^1(k_s)} \mathbb{Z} \to 0$$
by $\widehat{S}^0$ provides the split exact sequence of Γ_k- modules
$$1 \to S(k_s) \to S(k_s(t)) \to \bigoplus_{M \in \mathbf{A}^1_0} \mathrm{Coind}_k^{k(M)}(\widehat{S}^0) \to 0$$
Since S/k is flasque, $H^1\big(k, \mathrm{Coind}_k^{k(M)}(\widehat{S}^0)\big) = H^1(k(M), \widehat{S}^0) = 0$, so the long exact sequence of cohomology yields an isomorphism $H^1(k,S) \xrightarrow{\sim} H^1\big(\Gamma_k, S(k_s(t))\big)$. The last isomorphism is true for an arbitrary torus and

the midle one follows of the fact that $S(\mathcal{O}_{k_s})$ is a direct summand of $S(k_s(t))$. $\square$

We can now proceed to the proof of Theorem 26.5.

Proof. We have the exact sequence

$$E(k) \xrightarrow{f} T(k) \xrightarrow{\delta} H^1(k,S) \to H^1(k,E) = 1,$$

whose last term vanishes by Hilbert 90. We want to show that $f(E(k)) = R(k,T)$. One way is obvious: since E is a k-rational variety, we have $f(E(k)) \subset R(k,T)$. In the other hand, we have the exact sequence of Γ_k-modules

$$1 \to S(\mathcal{O}_{k_s}) \to E(\mathcal{O}_{k_s}) \to T(\mathcal{O}_{k_s}) \to 1.$$

We have then the following commutative diagram

$$\begin{tikzcd}
E(k) \arrow[r, "f"] \arrow[d] & T(k) \arrow[r, "\delta"] \arrow[d] & H^1(k,S) \arrow[r] \arrow[d, "\wr"] & 1 \\
E(\mathcal{O}) \arrow[r, "f"] & T(\mathcal{O}) \arrow[r, "\delta"] & H^1(\Gamma_k, S(\mathcal{O}_{k_s})) \arrow[r] & H^1(\Gamma_k, E(\mathcal{O}_{k_s})) = 1,
\end{tikzcd}$$

where the last term vanishes by the Lemma (note that the evaluation at 0 provides a splitting of all vertical maps). We are given $x \in R(k,T)$. It exists $x(t) \in T(\mathcal{O})$ such that $x(0) = e$ and $x(1) = x$. From the Lemma we get that $\delta(x(t)) = 0 \in H^1\big(\Gamma_k, S(\mathcal{O}_{k_s})\big)$. So by diagram chase, there exists $y \in E(\mathcal{O})$ lifting x. By specializing at 1, we get that $x = x(1) = f(y(1)) \in f(E(k))$ as desired. $\square$

27. Examples, local fields

27.1. **Case of a cyclic group.** The proof of the following fact is based on cyclotomic polynomials,

27.1. **Proposition.** *(Endo-Miyata* [EM], [CTS1, prop. 2]*) Assume that* Γ *is a cyclic group. Let* M *be* Γ*-lattice. Then the following are equivalent:*

1. *M is flasque;*
2. *M is coflasque;*
3. *M is invertible in $C(\Gamma)$.*

27.2. **Corollary.** *Let T/k be a torus split by a cyclic extension L/k. Then $T(k)/R = 1$.*

We can also give another proof of the following fact already seen in the first lecture.

27.3. **Corollary.** *Let $T/\mathbb{Q}$ be a torus. Then $T(\mathbb{Q})$ is dense in $T(\mathbb{R})$.*

27.2. **Norm tori.** Let L/k be a finite Galois extension of group Γ. The norm torus $R^1_{L/k}(\mathbb{G}_m)$ is the kernel of the norm map $R_{L/k}(\mathbb{G}_m) \to \mathbb{G}_m$. We have an exact sequence

$$1 \to R^1_{L/k}(\mathbb{G}_m) \to R_{L/k}(\mathbb{G}_m) \xrightarrow{N_{L/k}} \mathbb{G}_m \to 1.$$

27.4. **Proposition.** *For the norm torus $T = R^1_{L/k}(\mathbb{G}_m)$, we have*

$$T(k)/R \xrightarrow{\sim} H^{-1}(\Gamma, L^\times) = \mathrm{Ker}(L^\times \to k^\times)/I_\Gamma \,.\, L^\times.$$

In particular, it vanishes in the cyclic case; this is a version of Hilbert 90.

27.5. **Sketch of proof.** Define the map

$$\begin{array}{ccc} R_{L/k}(\mathbb{G}_m)^\Gamma & \xrightarrow{f} & R^1_{L/k}(\mathbb{G}_m) \\ (y_\sigma) & \mapsto & \prod\limits_{\sigma \in \Gamma} \sigma(y_\sigma)/y_\sigma. \end{array}$$

One shows that this map is surjective and its kernel is a flasque k-torus. Theorem 26.5 yields that $T(k)/R = H^{-1}(\Gamma, L^\times)$.

27.3. **Local fields.** Assume here that we deal with a p-adic field K. Tate's duality for tori [S1, II.5.8] states that the natural pairing

$$H^1(K, T) \times H^1(K, \widehat{T}) \to H^2(K, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

is a perfect duality of finite groups.

27.6. **Corollary.** *Let T/K be a K-torus and let $1 \to S \to E \to T \to 1$ be a flasque resolution. Then*

$$T(K)/R \xrightarrow{\sim} H^1(\Gamma, \widehat{S})^D.$$

In the case of norm tori, we have a nice formula.

27.7. **Example.** Let $T = R^1_{L/K}(\mathbb{G}_m)$ be the norm torus of a Galois extension L/K of group Γ. Then we have

$$T(K)/R \xrightarrow{\sim} H^3(\Gamma, \mathbb{Z})^D$$

Proof. We use the flasque resolution which arises in the proof of Proposition 27.4. Then we have an exact sequence of Γ-modules

$$0 \to \widehat{T} \to \widehat{E} \to \widehat{S} \to 0.$$

By Shapiro's lemma, we get an isomorphism

$$H^1(\Gamma, \widehat{S}) \xrightarrow{\sim} H^2(\Gamma, \widehat{T}).$$

In the other hand, from the sequence $0 \to \mathbb{Z} \to \mathbb{Z}[\Gamma] \to \widehat{T} \to 0$, we get an isomorphism $H^2(\Gamma, \widehat{T}) \xrightarrow{\sim} H^3(\Gamma, \mathbb{Z})$. □

We know that for bicyclic groups $\Gamma = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, we have $H^2(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H^3(\Gamma, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. This provides an example of torus T/K such that $T(K)/R \neq 1$.

28. R-trivial tori, Voskresenskiĭ's conjecture

28.1. **R-trivial tori.** Let T/k be a k-torus.

28.1. **Theorem.** *The following are equivalent:*

(*i*) *T is R-trivial, i.e. $T(F)/R = 1$ for all extensions F/k;*

(*ii*) *There exists a k–torus T' such that $T \times_k T'$ is a k-rational variety;*

(*iii*) *$p(\widehat{T})$ is invertible in $C(\Gamma)$.*

The proof of (*ii*) $\Longrightarrow$ (*iii*) requires some geometry. It shall be part of Colliot-Thélène's lectures.

Proof. (*i*) $\Longrightarrow$ (*ii*) : Denote by $\xi : \mathrm{Spec}(k(T)) \to T$ the generic point of the torus T. Then

$$\xi \in \mathrm{Im}\Big(E(k(T)) \xrightarrow{f} T(k(T))\Big).$$

This means that there exists an open subset U of T such that $f^{-1}(U) \cong U \times_k S$. Hence $T \times_k S$ is k-birational to the k-rational variety E. Thus $T \times_k S$ is k-rational.

(*iii*) $\Longrightarrow$ (*i*) : We assume that there exists a k-torus T' such that $T \times_k T'$ is a k-rational variety. Then $T(k)/R \times T'(k)/R = 1$, so $T(k)/R = 1$. The same holds for any extension F/k. □

28.2. **Stably k-rational tori.** Given a k-torus T, we have the following characterisation of stably k-rational tori.

28.2. **Theorem.** *The following are equivalent:*

(*i*) *T is the quotient of two induced tori;*

(*ii*) *T is a stably rational k-variety;*

(*iii*) *$p(\widehat{T}) = 0 \in C(\Gamma)$.*

The proof of (*ii*) $\Longrightarrow$ (*iii*) is the same than for Theorem 28.1, so it requires as well some geometry.

28.3. **Sketch of proof.** (*i*) $\Longrightarrow$ (*ii*) : Assume that there is an exact sequence $1 \to E_1 \to E_2 \to T \to 1$ where E_1, E_2 are quasi-trivial tori. By Hilbert 90, T is R-trivial and the same argument as in the proof of Theorem 28.1 shows that $T \times_k E_1$ is birationally k-isomorphic to E_2. Since induced tori are k-rational varieties, we conclude that T is stably k-rational.

(*iii*) $\Longrightarrow$ (*i*) : Let $1 \to S \to E \to T \to 1$ be a flasque resolution of T. Our hypothesis is that there exist quasi-trivial tori E_1, E_2 such that $E_2 = S \times E_1$. Replacing S by $S \times E_1$ and E by $E \times E_1$, we conclude that T is the quotient of induced tori.

We can now state Voskresenskiĭ's conjecture.

28.4. **Conjecture.** If T is stably k-rational, it is k-rational.

There are few evidences for this conjecture. Even the case of tori split by cyclic extensions is not known [V].

Norm principle, examples of non rational classical groups

As usual, the ground field k is assumed for simplicity to be of characteristic zero.

29. Introduction

Our goal is to present certain results for R-equivalence on semisimple groups. The first interesting case is $G = \mathrm{SL}_1(D)$ for a central simple division algebra D. Clearly the commutator subgroup $[D^\times, D^\times] \subset G(k)$ consists of elements of $R(k,G)$ (i.e. R-equivalent to e). This is a result by Voskeresenkiĭ that we have $[D^\times, D^\times] = G(k)$. In other words, there is an isomorphism

$$\mathrm{SL}_1(D)(k)/[D^\times, D^\times] \xrightarrow{\sim} \mathrm{SL}_1(D)(k)/R.$$

The left handside is linked with K-theory, it is denoted by $SK_1(D) = \ker\bigl(K_1(D) \to K_1(k)\bigr)$. The proof goes by showing that this group does not change under transcendental extensions of k [V, 18.2]. Platonov found examples of D such that $SK_1(D) \neq 0$ [P].

We come back at the end to this case which is a simply connected one. The plan is to discuss mainly the cases of $\mathrm{PSO}(q)$ and $\mathrm{Spin}(q)$ and to relate them with the general case involving norm principles.

30. Beginning for the reductive case

30.1. **Reduction to the anisotropic case.** Our first goal is to reduce the investigations to the case of anisotropic semisimple groups. The first reduction is obvious.

30.1. **Lemma.** *If $G = R_u(G) \rtimes L$, then $L(k)/R \xrightarrow{\sim} G(k)/R$.*

So we can assume from now on that G is reductive. More interesting is the following reduction.

30.2. **Proposition.** [G4, 1.2] *Let P be a minimal k-parabolic subgroup. Let $L = C_G(S)$ be a Levi subgroup of P where S is a maximal k-split torus of G. Then G is birationally isomorphic to L/S and we have group isomorphisms*

$$(L/S)(F)/R \xleftarrow{\sim} L(F)/R \xrightarrow{\sim} G(F)/R$$

for all fields F/k.

30.3. **Remark.** Note that L/S is anisotropic. If G is semisimple and an inner form, note that L/S is semisimple. If moreover G is adjoint, L/S is adjoint as well.

30.4. **Corollary.** *Assume that G is a quasi-split reductive group. Let $S \subset G$ be a maximal k-split torus and denote by $T = C_G(S)$ its centralizer.*

1. *Then $T(k)/R \xrightarrow{\sim} G(k)/R$;*
2. *If G is semisimple simply connected of adjoint, then $G(k)/R = 1$.*

Assertion (2) follows from the fact that the torus T is an induced torus (equivalently $\widehat{T}$ is a permutation module) in the simply connected/adjoint case.

30.2. ***R*-equivalence under certain group extensions.** Let $1 \to G \to \widetilde{G} \xrightarrow{\lambda} T \to 1$ be an extension of the reductive group G by a k-torus T. Our purpose is to compare $G(k)/R$ and $\widetilde{G}(k)/R$. There is something to do here since we have the following exact sequence

$$1 \to G(k) \to \widetilde{G}(k) \to T(k) \xrightarrow{\varphi} H^1(k, G) \to H^1(k, \widetilde{G}) \to H^1(k, T).$$

We observe first that R-equivalence can be defined for an arbitrary functor

$$F : \Big\{\text{semilocal } k\text{-algebra}\Big\} \to \text{Sets}, \quad A \mapsto F(A),$$

so not only functors which are representable by a k-variety. We say that two elements f_0, f_1 are elementary R-equivalent if there exists $f \in F(\mathcal{O})$ such that $f(0) = f_0$ and $f(1) = f_1$. The R-equivalence is the equivalence relation generated by this relation; this defined $F(k)/R$. The relevant functor here is

$$T_\lambda : \Big\{\text{semilocal } k\text{-algebra}\Big\} \to \text{Abelian groups}, \quad A \mapsto T_\lambda(A) = \mathrm{Coker}\big(\widetilde{G}(A) \to T(A)\big).$$

By definition, we have an exact sequence

$$1 \to G(A) \to \widetilde{G}(A) \to G(A) \to T_\lambda(A) \to 1$$

Note that $T_\lambda(k) = \ker\Big(T(k) \xrightarrow{\varphi} H^1(k, G)\Big)$. This permits to formulate the following formal fact.

30.5. **Lemma.** [G1, II.1.3] *We have a natural exact sequence of groups*

$$G(k)/R \to \widetilde{G}(k)/R \to T_\lambda(k)/R \to 1.$$

30.6. **Examples.** If A is a central simple algebra of degree n, for the exact sequence $1 \to \mathrm{SL}_1(A) \to \mathrm{GL}_1(A) \to \mathbb{G}_m \to 1$, we have $T_\lambda(k) = \mathrm{Nrd}(A^\times) \subset k^\times$.

Let q be a quadratic form of even rank n. Recall first that the Clifford group $\Gamma^+(q)$ is defined by the following commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{Spin}(q) \arrow[r] \arrow[d] & \mathrm{SO}(q) \to 1 \arrow[d, equal] \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "\times 2"'] & \Gamma^+(q) \arrow[r] \arrow[d, "Ns"'] & \mathrm{SO}(q) \to 1 \\
 & \mathbb{G}_m \arrow[r, equal] \arrow[d] & \mathbb{G}_m \arrow[d] & \\
 & 1 & 1 &
\end{tikzcd}$$

Since the morphism $Ns : \Gamma^+(q)(k) \to k^\times$ lifts the spinor norm $\mathrm{SO}(q)(k) \to k^\times/(k^\times)^2$, it is also called the spinor norm. Its image $T_\lambda(k)$ is the subgroup of $k^\times$ generated by even products of non trivial values of q.

The things go actually in the way around. One constructs first the group $\Gamma^+(q)$ and then $\mathrm{Spin}(q)$. Recall the definition of the Clifford algebra $C(q,V) = Sym(V)/v \otimes v - q(v)$ of the quadratic form (q,V). Then $C^+(q,V)$ is the subalgebra consisting of classes of even products $v_1 \otimes ... \otimes v_{2r}$. The Clifford group $\Gamma(q)$ is the group of invertible elements of $C^+(q,V)$ preserving V [Sc, §9]. This group surjects on $\mathrm{O}(q)$ and $\Gamma^+(q)$ is the neutral component of $\Gamma(q)$.

31. The group $\mathrm{PSO}(q)$

Let q be a quadratic form of even rank $2n$. We have an exact sequence of k–groups

$$1 \to \mu_2 \to \mathrm{SO}(q) \to \mathrm{PSO}(q) \to 1.$$

It fits in the following commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{SO}(q) \arrow[r] \arrow[d] & \mathrm{PSO}(q) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "\times 2"'] & \mathrm{GO}^+(q) \arrow[r] \arrow[d, "\lambda"'] & \mathrm{PSO}(q) \arrow[r] & 1 \\
 & \mathbb{G}_m \arrow[r, phantom, "="] \arrow[d] & \mathbb{G}_m \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

where the group $\mathrm{GO}^+(q)$ is the neutral component of the group $\mathrm{GO}(q)$ of similitudes of (q,V), i.e.

$$\mathrm{GO}(q) = \Big\{(g,a) \mid q \circ g = a\, q\Big\} \subset \mathrm{GL}(V) \times \mathbb{G}_m.$$

The map λ, $(g,a) \mapsto a$, is called the multiplier map. So $T_\lambda(k) = G(q)$ is the group of multipliers (or similarity factors of q). Note that $\mathrm{GO}^+(q)$ and $\mathrm{PSO}(q)$ are stably k-rationally equivalent, so $\mathrm{GO}^+(q)(k)/R \xrightarrow{\sim} \mathrm{PSO}(q)(k)/R$. Since $\mathrm{SO}(q)$ is a k-rational variety (Cayley map), we have $\mathrm{GO}^+(q)(k)/R \xrightarrow{\sim} T_\lambda(k)/R$ by Lemma 30.5. In other words, we have to deal with R-equivalence for similarity factors of q.

31.1. **Theorem.** *(Milnor-Tate* [M, §5]*) The local residues induces an exact sequence*

$$0 \to W(k) \to W(k(t)) \to \bigoplus_\pi W(k_\pi) \to 0,$$

where runs over the monic irreducible polynomials of $k[t]$ *(as usual* $k_\pi = k[t]/\pi(t)$*).*

Given a polynomial $P(t)$, by applying residue maps to $[P(t)q] - [q] \in W(k(t))$, we get the following fact.

31.2. **Corollary.** *The group* $G(q)\big(k(t)\big)$ *is the subgroup of* $k(t)^\times$ *which is generated by* $(k(t)^\times)^2$*,* $G(q)$ *and the monic irreducible polynomials* π *such that* k_π *splits* q*.*

When specializing those polynomials at 1, we get elements of $N_{k_\pi/k}(k_\pi^\times)$. We proved essentially the following result.

31.3. **Proposition.**

$$\mathrm{GO}^+(q)/R \xrightarrow{\sim} \mathrm{PSO}(q)/R \xrightarrow{\sim} T_\lambda(k)/R \xrightarrow{\sim} G(q)/(k^\times)^2.hyp_q(k),$$

where $hyp_q(k)$ is the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ for L/k running over the finite splitting extensions of q.

For rank 6 quadratic forms, the question of stable rationality is fully understood by means of R-equivalence.

31.4. **Theorem.** *(Merkurjev* [Me4, theorem 3]*) Assume that* $\mathrm{rank}(q) = 6$.

(1) *If q has trivial signed discriminant, then* $\mathrm{PSO}(q)$ *is k-rational.*
(2) *Assume that q has non trivial signed discriminant $\delta \in k^\times/(k^\times)^2$ and put $L = k(\sqrt{\delta})$.*
    (a) *If the Clifford L-algebra $C(q)$ has index*[2] 1 *or* 2*, then* $\mathrm{PSO}(q)$ *is stably k-rational.*
    (b) *If the Clifford L-algebra $C(q)$ has index* 4*, then* $\mathrm{PSO}(q)$ *is not R-trivial and then not stably k-rational.*

The generic form $\langle t_1, t_2, t_3, t_4, t_5, t_6 \rangle$ provides then an example of a group $\mathrm{PSO}(q)$ which is not R-trivial disproving then Platonov's rationality conjecture for adjoint groups. The proof uses index reduction theory [Me2]. For more concrete (and quite elementary) such examples over fields of iterated Laurent series, see [G2]. For the other adjoint classical groups, see [BMT], [Me2] and [V, §19].

32. Norm principle

Recall first the following results for a quadratic form q and a finite field extension L/k.

- (Scharlau's norm principle [Sc, II.8.6]) $N_{L/k}(G(q_L)) \subset G(q)$;
- (Knebusch's norm principle [La, VII.5]) Denote by $D_q(k)$ the subgroup of $k^\times$ generated by the values of q. Then $N_{L/k}(D_q(L)) \subset D_q(k)$.

So we have examples of groups of the shape $T_\lambda(k) \subset T(k)$ which satisfies a norm principle.

32.1. **Question.** Given a morphism $\lambda : \widetilde{G} \to T$ to the k-torus T, is is true that

$$N_{L/k}\Big(T_\lambda(L)\Big) \subset T_\lambda(k) \subset T(k),$$

where $T_\lambda(k) = \mathrm{Im}\Big(G(k) \to T(k)\Big)$.

There are a lot of evidences but no general proof.

[2]If A/k is a central simple algebra, Wedderburn's theorem states that $A \cong M_r(D)$ for a unique central simple division algebra D. The index of A is $\mathrm{ind}_k(A) = \sqrt{\dim_k(D)}$.

32.1. **Link with R-equivalence.** If we come back to the case of the similarity factors $G(q)$, we remark that the R-trivial elements $hyp(q)$ satisfies obviously the norm principle $N_{L/k}(hyp(q_L)) \subset hyp(q)$ (the same holds for reduced norms of a central simple algebra). It turns out that this fact is general.

32.2. **Theorem.** [G1, II] [Me2] *Let $\lambda : \widetilde{G} \to T$ be a morphism to the k-torus T.*

(1) *With the notations above, we have*

$$N_{L/k}\Big(R(L, T_\lambda)\Big) \subset R(k, T_\lambda) \subset T(k).$$

(2) *If $\widetilde{G}(L)/R = 1$, then the norm principle holds.*

The second assertion follows immediately of (1). Since the Clifford group $\Gamma^+(q)$ is k-rational, Theorem 32.2 shows that the image of the spinor norm of a quadratic form q is stable under norm maps. If q represents 1, it yields another proof of Knebusch norm's principle; and the general case follows then easily [G1, II.4.3]

32.2. **Norm groups.** The group $hyp(q)$ is a special case of norm group.

32.3. **Definition.** Let X be a k-variety. The norm group $N_X(k)$ of X is the subgroup of $k^\times$ which is generated by the $N_{L/k}(L^\times)$ for L/k running over the finite extensions of k such that $X(L) \neq \emptyset$.

32.4. **Examples.** • If X is the variety of maximal isotropic subspaces of the projective quadric associated to the even dimensional quadratic form q, then $N_X(k) = hyp(q)$.

• If X is the projective quadric associated to a quadratic form q, Knebusch's norm principle permits to show that $N_X(k)$ is the image of the spinor norm.

• If X is the Severi-Brauer variety of central simple algebra A, then $N_X(k) = \mathrm{Nrd}(A^\times)$ [GS, §2, exercise 8].

For any functor $F : \big\{$étale k-algebras$\big\} \to \{$Abelian groups$\}$ equipped with norm maps $N_{L/k} : F(L) \to F(k)$ satisfying the usual composition rule, we can define its norm group $N_X(k, F)$ with respect to the k-variety X. This is the subgroup of $F(k)$ which is generated by the $N_{L/k}(F(L))$ for L/k running over the finite extensions of k such that $X(L) \neq \emptyset$.

That applies to a k-torus T defining then $N_X(k, T) \subset T(k)$. Note that $N_X(k) = N_X(\mathbb{G}_m)$. We are interested in the case of the variety X of Borel subgroups of our reductive group G. Recall its construction. If G_0 is the quasi-split form of G, we can assume that $G = {}_zG_0$ for $z \in Z^1(k, G_{0,ad})$ and we have $X = {}_z(G_0/B_0)$ where B_0 is a k-Borel subgroup of G_0. The variety X is homogeneous under G and $X(k)$ corresponds to the k-Borel subgroups of G. In particular, $X(k) \neq \emptyset$ if and only if G is quasi-split.

32.5. **Proposition.** *Assume that* $G = \mathrm{Ker}(\lambda)$ *is semisimple simply connected. Then*

$$N_X(k, R(\,.\,, T)) \subset R(k, T_\lambda) \subset T_\lambda(k) \subset T(k).$$

The hypothesis on G is used in the following lemma.

32.6. **Lemma.** *Assume that G is quasi-split (and semisimple simply connected). Then the map* $\widetilde{G}(A) \to T(A)$ *is surjective for any semi-local k-algebra A.*

Proof. Let $S \subset \widetilde{S}$ be respectively maximal k–split tori of G and $\widetilde{G}$. Then $C_G(S)$ (resp. $C_{\widetilde{G}}(\widetilde{S})$) is a maximal k-torus of G (resp. $\widetilde{G}$). Hence we have an exact sequence of tori

$$0 \to C_G(S) \to C_{\widetilde{G}}(\widetilde{S}) \xrightarrow{\lambda} T \to 1.$$

We have then an exact sequence for étale cohomology

$$C_{\widetilde{G}}(\widetilde{S})(A) \to T(A) \xrightarrow{\varphi_A} H^1_{\acute{e}t}(A, C_G(S)).$$

But $C_G(S)$ is an induced torus, so Hilbert 90 for étale cohomology shows that $H^1_{\acute{e}t}(A, C_G(S))$ vanishes [CTS2, 4.1]. Thus $C_{\widetilde{G}}(\widetilde{S})(A) \to T(A)$ is onto and a fortiori $\widetilde{G}(A) \to T(A)$ is surjective. □

We can now proceed to the proof of Proposition 32.5.

Proof. First case: G is quasi-split: In that case $X(k) \neq \emptyset$ and we have then to prove that

$$R(k, T_\lambda) = R(k, T).$$

But this follows from the Lemma.

General case: Let L/k be a finite field extension such that $X(L) \neq \emptyset$. So G_L is quasi-split. By Lemma 30.4.(2), $G(L)/R = 1$, so the norm principle holds for L/k, i.e. $N_{L/k}(R(L, T_\lambda)) \subset R(k, T_\lambda)$. By the first case, we have $R(L, T_\lambda) = R(L, T)$, so we get $N_{L/k}(R(L, T)) \subset R(k, T_\lambda)$. The proposition follows by gathering all such extensions L/k. □

32.3. **Case of local fields.** We assume here that everything is defined over a p-adic field K.

32.7. **Lemma.** *Let X/K be the variety of Borel subgroup of G. Then* $N_X(K) = K^\times$.

If $T = \mathbb{G}_m$, it follows of Proposition 32.5 that we have an isomorphism

$$G(K)/R \xrightarrow{\sim} \widetilde{G}(K)/R$$

In the other hand, we know that simply connected K–groups are stably K-rational [CP]. With some mild additional effort [BK, appendix], we get the following general result.

32.8. **Theorem.** *Let $\lambda : \widetilde{G} \to T$ be a surjective morphism whose kernel is simply connected. Then we have an isomorphism*

$$\widetilde{G}(K)/R \xrightarrow{\sim} T(K)/R.$$

In other words, that reduces the computation of this invariant to the case of tori.

33. Simply connected groups

33.1. $\mathbf{SL_1(A)}$**.** Let A/k be a central division k–algebra.

33.1. **Theorem.** *(Wang, see* [GS, §4.7]*) If* $\operatorname{ind}(A)$ *is squarefree, then* $SK_1(A) = 0$.

33.2. **Sketch of proof.** There is a standard reduction which boils down to the case of a central simple division algebra A of prime degree p. We are given an element $a \in A^\times$ of reduced norm 1 and we want to show that a is a product of commutators. Since $[A^\times, A^\times]$ is Zariski dense, we can assume that $a \notin k$. Let $L \subset A$ be the k-subalgebra generated by a; it is a degree p field extension of k. We have $\mathrm{Nrd}(a) = N_{L|k}(a) = 1$. The algebra $A_L := A \otimes_k L$ contains the subalgebra $L \otimes_k L$ which is not a division algebra, hence neither is A_L. Since $\deg_L(A_L) = p$, Wedderburn's theorem shows that $A \otimes_k L$ must be split. We now use that there is a norm map $SK_1(A_L) \to SK_1(A)$ coming from K-theory (e.g. [GS, §2.8]). By the split case we have $SK_1(A_L) = 0$, hence the composite map

$$SK_1(A) \to SK_1(A_L) \stackrel{N_{L|k}}{\to} SK_1(A_L)$$

is trivial. Hence $p\,SK_1(A) = 0$. Take a Galois closure $\widetilde{L}|k$ of L and denote by $K|k$ the fixed field of a p-Sylow subgroup in $\mathrm{Gal}(\widetilde{L}|k)$. Since $\mathrm{Gal}(\widetilde{L}|k)$ is a subgroup of the symmetric group S_p, the extension $\widetilde{L}|K$ is a cyclic Galois extension of degree p. The composite

$$SK_1(A) \to SK_1(A_K) \stackrel{N_{K|k}}{\to} SK_1(A)$$

is multiplication by $[K : k]$ which is prime to p. But we know that $pSK_1(A) = 0$, so the map $SK_1(A) \to SK_1(A_K)$ is injective. Up to replacing k by K and L by $\widetilde{L}$, we may thus assume that $L|k$ is cyclic of degree p. Let σ be a generator of $\mathrm{Gal}(L|k)$. According to the classical form of Hilbert's Theorem 90, there exists $c \in L^\times$ satisfying $a = c^{-1}\sigma(c)$. On the other hand, L is a subfield of A which has degree p over k, so we know that there exists $b \in A^\times$ with $b^{-1}cb = \sigma(c)$ (Skolem-Noether theorem). Hence $a = c^{-1}\sigma(c) = c^{-1}b^{-1}cb$ is a commutator in $A^\times$, and as such yields a trivial element in $SK_1(A)$.

33.3. **Conjecture.** (Suslin) If $\operatorname{ind}(A)$ is not squarefree, then $\mathrm{SL}_1(A)$ is not R-trivial.

Merkurjev has proven the conjecture when 4 divides $\operatorname{ind}(A)$ [Me2]. Its proof uses Rost theorem for SK_1 of biquaternion algebras, i.e. tensor product of two quaternions algebras.

33.4. **Theorem.** [Me1] *Let $A = (a_1, b_1) \otimes (a_2, b_2)$ be a quaternion algebra and consider its Albert form*[3] *$q = \langle a_1, a_2, -a_1a_2, -b_1, b_2, b_1b_2\rangle$. Denote by $k(q)$ the function field of the projective quadric defined by q. Then there is an exact sequence*

$$0 \to SK_1(A) \to H^4(k, \mathbb{Z}/2\mathbb{Z}) \to H^4(k(q), \mathbb{Z}/2\mathbb{Z}).$$

Note that Suslin defined for a central simple algebra A/k of degree n an invariant $SK_1(A) \to H^4(k, \mu_n^{\otimes 3})$ [Su]. This last invariant is trivial for biquaternion algebras. However it is expected that Suslin invariant is the double of an invariant which is Rost invariant in the case of biquaternion algebras (voir [G5]).

33.2. Spin **groups.** Theorem 33.4 deals already with Spin groups because we have the exceptional isomorphim $\mathrm{Spin}(q) \cong \mathrm{SL}_1(A)$ [KMRT, §16]. Let q be a quadratic form. Our goal is to present Chernousov-Merkurjev-Rost's theorem relating the group $\mathrm{Spin}(q)/R$ and the K-cohomology of the projective quadric X defined by q [CM]. We consider the following map

$$\partial : \bigoplus_{x \in X_{(1)}} K_2^M(k(x)) \to \bigoplus_{x \in X_{(0)}} k(x)^\times$$

made from Bass-Tate tame symbols [GS, §7.1]. Its cokernel is denoted by $A_0(X, K_1)$. Since X is proper, the norm map

$$N : \bigoplus_{x \in X_{(0)}} k(x)^\times \to k^\times$$

vanishes on $\mathrm{Im}(\partial)$. It defines a map $A_0(X, K_1) \to k^\times$ whose kernel is denoted by $\overline{A}_0(X, K_1)$.

Rost defined a natural homomorphism $\rho : \Gamma^+(q)(k) \to A_0(X, K_1)$ such that the following diagram

$$\begin{array}{ccc} \Gamma^+(q)(k) & \xrightarrow{\rho} & A_0(X, K_1) \\ {\scriptstyle Ns}\big\downarrow & & {\scriptstyle N}\big\downarrow \\ k^\times & = & k^\times \end{array}$$

commutes [Ro]. It induces then a homomorphism $\mathrm{Spin}(q)(k) \to \overline{A}_0(X, K_1)$.

33.5. **Theorem.** [CM] *The preceding map induces an isomorphism*

$$\mathrm{Spin}(q)(k)/R \xrightarrow{\sim} \overline{A}_0(X, K_1).$$

The vanishing of $\overline{A}_0(X, K_1)$ for certain quadrics is a crucial ingredient of Voevoedsky's proof of the Milnor's conjecture [Vo] [K]. So the theorem is applied within its following corollary (which involves only the surjectivity of the Rost invariant).

33.6. **Corollary.** *If* $\mathrm{Spin}(q)$ *is stably k-rational, then* $\overline{A}_0(X, K_1) = 0$.

[3] which is defined only up to similarity.

The group Spin(q) is k-rational for the forms of the special kind below [Me5, 6.4].

33.7. **Theorem.** *Assume that q is a Pfister neighbor (i.e. a subform of a Pfister form $\langle\langle a_1, ..., a_n\rangle\rangle$ of dimension greater than 2^{n-1}). Let p a quadratic form of rank at most 2. Then for the quadratic form $q \perp p$, $\overline{A}_0(X_{q\perp p}, K_1) = 0$.*

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