Grothendieck's classification theorem on vector bundles on the Riemann sphere and more

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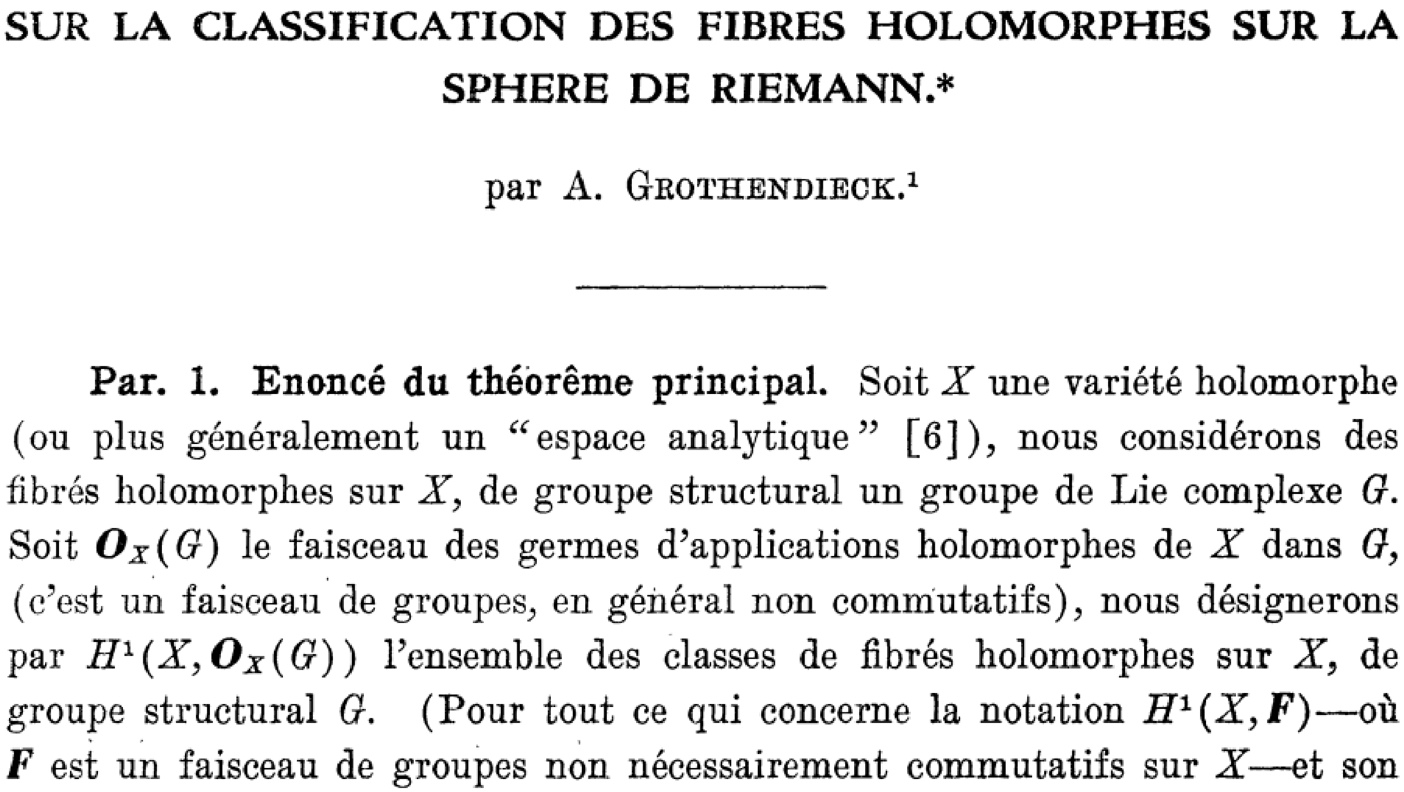
Low-dimensional Complex and Conformal Geometry

The paper

- 

SUR LA CLASSIFICATION DES FIBRES HOLOMORPHES SUR LA SPHERE DE RIEMANN.*

par A. GROTHENDIECK.[1]

Par. 1. Enoncé du théorême principal. Soit X une variété holomorphe (ou plus généralement un "espace analytique" [6]), nous considérons des fibrés holomorphes sur X, de groupe structural un groupe de Lie complexe G. Soit $\boldsymbol{O}_X(G)$ le faisceau des germes d'applications holomorphes de X dans G, (c'est un faisceau de groupes, en général non commutatifs), nous désignerons par $H^1(X, \boldsymbol{O}_X(G))$ l'ensemble des classes de fibrés holomorphes sur X, de groupe structural G. (Pour tout ce qui concerne la notation $H^1(X, \boldsymbol{F})$—où $\boldsymbol{F}$ est un faisceau de groupes non nécessairement commutatifs sur X—et son

- American Journal of Mathematics **79** (1957), 121-138.

The Riemann sphere

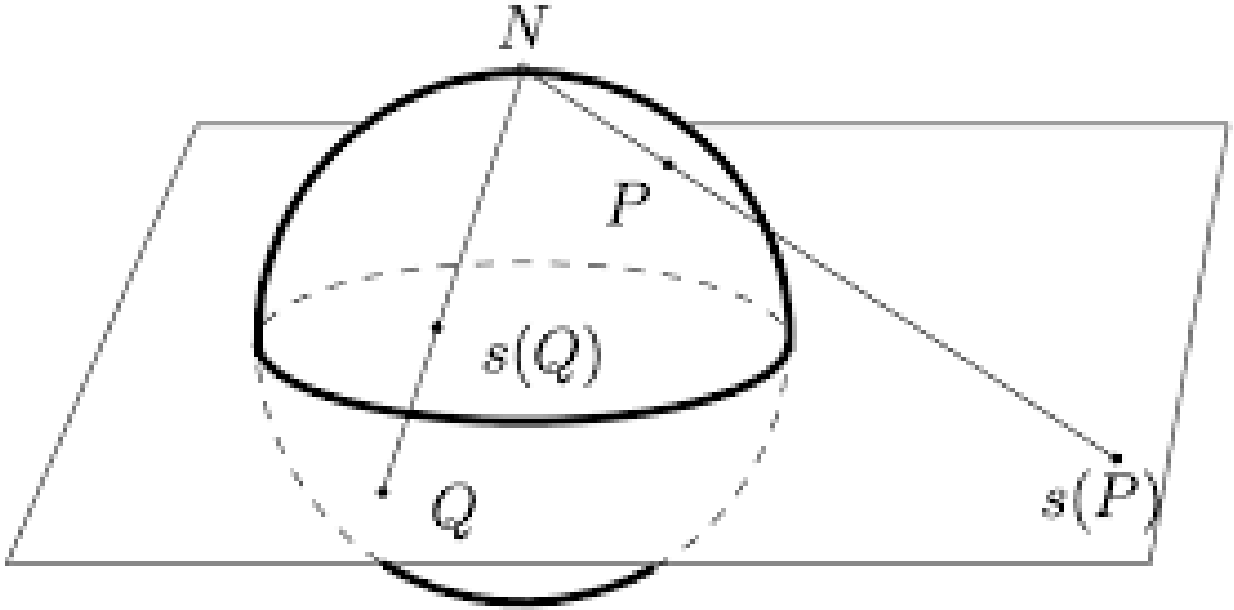
- As topological space, the Riemann sphere is the Alexandrov compactification $S = \mathbb{C} \sqcup \{\infty\}$ of $\mathbb{C}$.
- It means that the neighborhoods of ∞ are the $(\mathbb{C} \setminus K) \sqcup \{\infty\}$ where K runs over the compact subsets of $\mathbb{C}$.
- For an invertible matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we claim that the map

$$\begin{array}{rcl} f_A : \mathbb{C} \setminus \{\frac{-d}{c}\} & \longrightarrow & \mathbb{C} \\ z & \mapsto & \dfrac{az+b}{cz+d} \end{array}$$

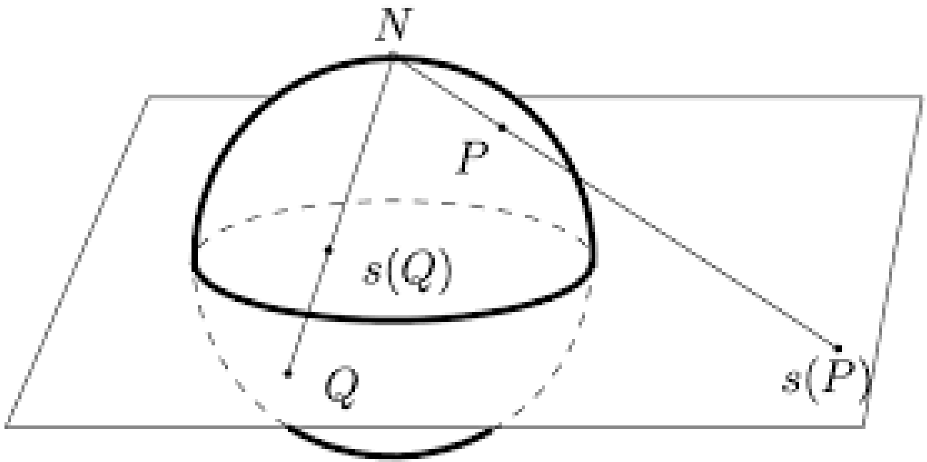
  extends (uniquely) to a homeomorphism $\widetilde{f}_A : S \xrightarrow{\sim} S$ by putting $\widetilde{f}_A(\frac{-d}{c}) = \infty$ and $\widetilde{f}_A(\infty) = \frac{a}{c}$.
- It turns out that we obtain a continuous action of the topological group $\mathrm{GL}_2(\mathbb{C})$ on S. Those are called Moebius transformations.

The Riemann sphere, II

- This is actually a continuous action of $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times$ on S. This action is 3-transitive.
- The first result is that the Riemann sphere S is homeomorphic to the 2-dimensional sphere $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid \quad x_0^2 + x_1^2 + x_3^2 = 1\}$.
- The map $s : S^2 \to S$ is the stereographic projection from the north pole $N = (0, 0, 1)$.



The Riemann sphere, III

- 

- We have to think that the plane is $\mathbb{C}$. The homeomorphism $S^2 \setminus \{N\} \to \mathbb{C}$ extends to a homeomorphism $S^2 \xrightarrow{\sim} S$ by putting $s(N) = \infty$.

- An avatar is that we have a continuous 3-transitive action of $\mathrm{PGL}_2(\mathbb{C})$ on S^2.

- We want to equip S with more structures and if possible in a $\mathrm{PGL}_2(\mathbb{C})$-equivariant way.

- What about the metric ?

Metrics on the Riemann sphere

- On S^2 we have the metric induced by the metric on $\mathbb{R}^3$, it is called the chordal metric. It does not seem to be of interest here.
- The other metric (the geographic one) is the right one.
- Its restriction on $\mathbb{C}$ (via the stererographic projection) is for $z'\,\overline{z} \neq 1$ and $z' \neq z$ is

  $$d(z, z') = 2\,\mathrm{Arctan}\Big(\frac{|\,z' - z\,|}{|\,1 + z'\,\overline{z}\,|}\Big).$$

  It is called the Fubini-Study metric.
- The related isometry group is $\mathrm{SO}_3(\mathbb{R})$ which is a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

The projective line

- A more conceptual way to deal with S is the projective line $P = \mathbb{P}(\mathbb{C}^2)$, that is, the space of (complex) lines of $\mathbb{C}^2$.
- As set we have $P = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by homothety.
- It comes with the quotient topology.
- We have a continuous injective map $i : \mathbb{C} \to P$, $z \mapsto [z : 1]$, that is, the class of the line generated by $ze_0 + e_1$.
- **Theorem.** By putting $i(\infty) = [1 : 0]$, the map i extends to a homeomorphism $S \xrightarrow{\sim} P$.
- Furthermore, this map is $\mathrm{PGL}_2(\mathbb{C})$-equivariant for the natural action of $\mathrm{PGL}_2(\mathbb{C})$ on P.

Riemann surface structure

- The map $i_0 : \mathbb{C} \to P$, $z \mapsto [z : 1]$ is a topological open embedding and so is $i_1 : \mathbb{C} \to P$, $z \mapsto [1 : z]$.
- We denote by U_0 (resp. U_1) the image of i_0 (resp. i_1). The pair (U_0, U_1) is called the standard open cover of P.
- The map i_0 induces a homeomorphism $\mathbb{C}^\times \cong U_0 \cap U_1$ and $i_1^{-1} \circ i_0 : \mathbb{C}^\times \to \mathbb{C}^\times$ reads $z \mapsto \frac{1}{z}$.
- Since it is an holomorphic function, it means that P is equipped with a structure of (compact) Riemann surface.

Riemann surface structure, II

- The projective line has a structure of Riemann surface and furthermore that structure is $\mathrm{PGL}_2(\mathbb{C})$-invariant.
- In particular, for each open subset $U \subset P$, we can deal with the ring of holomorphic functions $\mathscr{O}(U)$.
- The assignment $U \mapsto \mathscr{O}(U)$ defines the structural sheaf in rings $\mathscr{O}$ of P.
- We have $\mathbb{C} = \mathscr{O}(P)$ according to Liouville's theorem.

Motivations

- Of course there is a motivation to study vector bundles on P arising from Riemann surfaces, e.g. the Riemann-Roch theorem, the study of Galois ramified covers.
- Another motivation comes from the theory of complex differential equations. It is related with study of equations of the shape $dX/dz = A(z)\,X$.
- We need to embark in generalities for vector bundles and principal bundles firstly in the topological setting.

(Complex) vector bundles on topological spaces

- Let X be a topological space and let n be a nonnegative integer. A vector bundle V over X is a topological space V equipped with a continuous map $f : Y \to X$ such that X admits an open cover $(U_i)_{i \in I}$ and trivializations maps
$$\phi_i : U_i \times \mathbb{C}^n \xrightarrow{\sim} V \times_X U_i = f^{-1}(U_i)$$
such that on the overlaps
- $$\phi_i^{-1} \circ \phi_j : (U_i \cap U_j) \times \mathbb{C}^n \to (U_i \cap U_j) \times \mathbb{C}^n$$
is given by a continuous map $g_{i,j} : (U_i \cap U_j) \to \mathrm{GL}_n(\mathbb{C})$.
- The family $(g_{i,j})$ is called the 1-cocycle attached to the trivializations (ϕ_i).
- It means that V is obtained by glueing the $U_i \times \mathbb{C}^n$ with respect to $(g_{i,j})$.
- There is natural notion of morphisms and also of operations $(\bigoplus, \bigotimes, \Lambda^2$, etc ...)
- We say that V is trivial if it is isomorphic to $X \times \mathbb{C}^n$.

(Complex) vector bundles on topological spaces, II

- An important example is the tangent bundle of a differential manifold. For example the tangent bundle of the sphere S^1 (resp. S^3) is trivial but not that of S^2 (the hairy ball theorem).
- Assume that X is compact connected. Then for each vector bundle V of rank n over X, there exists a vector bundle V' of rank $n' >> 0$ such that $V \oplus V' \xrightarrow{\sim} X \times \mathbb{C}^n$.
- This statement leads to Swan's equivalence of categories between the category $\mathscr{V}(X)$ of vector bundles and that of finite projective modules over the ring $C(X)$ of complex continuous functions. It maps a vector bundle V to the module of sections $\Gamma_V = \mathrm{Hom}(\mathbb{C} \times X, V)$ of $V \to X$.
- A corollary is that there are no extension issues.

Principal bundles in Topology

- Let G be a topological group. A G-bundle Y over X is a topological space Y equipped with a continuous right action of G together with a continuous map $f : Y \to X$ which is G-invariant and such that X admits an open cover $(U_i)_{i \in I}$ and trivializations (which are G-equivariant)

$$\phi_i : U_i \times G \xrightarrow{\sim} Y \times_X U_i = f^{-1}(U_i).$$

- It follows that each

$$\phi_i^{-1} \circ \phi_j : (U_i \cap U_j) \times_X G \to (U_i \cap U_j) \times_X G$$

  is the left translation by a continuous function $g_{i,j} : U_i \cap U_j \to G$. Then $(g_{i,j})$ is the cocycle associated with the trivializations.

- A G-bundle is trivial if it is G-isomorphic to $X \times G$, equivalently the map $f : Y \to X$ admits a continuous section.

Functorialities

- Given a continuous homomorphism $u : G \to G'$ of topological groups, we can form $u_*(Y) = Y \wedge^G G'$ over X (where G acts on the right on Y and on the left on G' through u).
- Then $u_*(Y)$ is a G'-bundle.
- For a real example, the Moebius strip over the circle S^1 can be constructed with $G = \mathbb{Z}/2\mathbb{Z}$, $G' = \mathbb{R}^\times$, the 2-cover $p : S^1 \xrightarrow{\times 2} S^1$ and the map $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{R}^\times = \mathrm{GL}_1(\mathbb{R})$.
- We can restrict a G-bundle E over X to an open subset but we can actually consider the fiber product $E \times_X X'$ for any continuous map $f : X' \to X$.
- We get the pull-back G-bundle $f_*(E)$. The nice thing is that pull-back and push-out commute.

Vector bundles

- If $G = \mathrm{GL}_n(\mathbb{C})$, a principal G-bundle over X is the same thing than a complex vector bundle over X. In one way we associate to a $\mathrm{GL}_n(\mathbb{C})$-bundle Y the contracted product

$$Y \wedge^{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n = (Y \times \mathbb{C}^n)/\mathrm{GL}_n(\mathbb{C}).$$

- The converse map is the frame bundle construction $F(V)$. For each real vector bundle $V \to X$ of dimension n and for each open U of X which trivializes V, $F(V) \times_X U$ is the set of trivializations $\mathbb{C}^n \times U \xrightarrow{\sim} V \times_X U$ (or equivalently bases of $V \times_X U$).
- If $n = 1$, the frame construction is to take $V \setminus X$ over X, that is, V minus the zero section.

Example : the Hopf line bundle

- The Hopf line bundle is the topological space

  $$\mathrm{Hopf} = \left\{(l, v) \in P \times \mathbb{C}^2 \mid v \in l\right\} \subset P \times \mathbb{C}^2.$$

  The map $f : \mathrm{Hopf} \to P$ is the first projection.
- It comes with a morphism $\mathrm{Hopf} \to P \times \mathbb{C}^2$. The projection on the first factor provides a morphism $v : \mathrm{Hopf} \to P \times \mathbb{C}$.
- We have a homeomorphism $u_0 : U_0 \times \mathbb{C} \xrightarrow{\sim} f^{-1}(U_0)$, $([z : 1], \lambda) \mapsto \big((\lambda z, \lambda), [z : 1]\big)$ and similarly $u_1 : U_1 \times \mathbb{C} \xrightarrow{\sim} f^{-1}(U_1)$, $([1 : z], \lambda) \mapsto \big((\lambda, \lambda z), [1 : z]\big)$.
- The point is that $u_1^{-1} \circ u_0 : (U_0 \cap U_1) \times \mathbb{C} \to (U_0 \cap U_1) \times \mathbb{C}$ reads $([z_0 : z_1], \lambda) \mapsto \left([z_0 : z_1], \frac{z_1}{z_0}\lambda\right)$ so is a continuous family of $\mathbb{C}$-linear transformations.
- **Theorem.** The Hopf line bundle is not trivial.
- Equivalently we will establish that the underlying $\mathbb{C}^\times$-bundle is not trivial, namely $\mathbb{C}^2 \setminus \{0\} \to P$.

Example : the Hopf line bundle, II

- This is a classical degree argument. If the map $p : \mathbb{C}^2 \setminus \{0\} \to P \cong S^2$ admits a splitting f (i.e. $id_P = p \circ f$) then we would have a factorization

$$id : \pi_2(P) \xrightarrow{f_*} \pi_2(\mathbb{C}^2 \setminus \{0\}) \xrightarrow{p_*} \pi_2(P)$$

  contradicting $\pi_2(S^2) = \mathbb{Z}$ since $\pi_2(\mathbb{C}^2 \setminus \{0\}) = \pi_2(S^3) = 0$.
- We relate now to the usual Hopf bundle and consider the sphere $S^3 = \left\{(z_0, z_1) \in \mathbb{C}^2 \setminus \{0\} \mid |z_0|^2 + |z_1|^2 = 1\right\}$.
- The action of S^1 on $\mathbb{C}^2 \setminus \{0\}$ preserves S^3 so that we have a commutative diagram

$$\begin{tikzcd}
S^3 \arrow[r, hook] \arrow[d] & \mathbb{C}^2 \setminus \{0\} \arrow[r, "="] \arrow[d] & \mathbb{C}^2 \setminus \{0\} \arrow[d] \\
S^3/S^1 \arrow[r, hook] & (\mathbb{C}^2 \setminus \{0\})/S^1 \arrow[r] & (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times = P.
\end{tikzcd}$$

Hopf bundle, III

- We have a commutative diagram

$$\begin{tikzcd}
S^3 \arrow[r, hook] \arrow[d] & \mathbb{C}^2 \setminus \{0\} \arrow[r, "="] \arrow[d] & \mathbb{C}^2 \setminus \{0\} \arrow[d] \\
S^3/S^1 \arrow[r, hook] & (\mathbb{C}^2 \setminus \{0\})/S^1 \arrow[r] & (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times = P.
\end{tikzcd}$$

- It turns out that the bottom map is a homeomorphism. What is behind is the homeomorphism $\mathbb{C}^\times \cong \mathbb{R} \times S^1$ so that the left handside is the push-out by the projection $\mathbb{C}^\times \to S^1$
- We consider the long exact sequence of homotopy groups

$$\begin{tikzcd}
\ldots & \pi_2(S^3) \arrow[r, "p_*"] \arrow[d, equal] & \pi_2(S^2) \arrow[r] \arrow[d, equal] & \pi_1(S_1) \arrow[r] \arrow[d, equal] & \pi_1(S_3) \arrow[d, equal] & \ldots \\
& 0 & \mathbb{Z} & \mathbb{Z} & 0 &
\end{tikzcd}$$

Classification of lines bundles

If X is a nice enough topological space, that classification is remarkably simple. We assume then that X is paracompact and locally contractible (e.g. a compact smooth manifold).

- We denote by $\underline{O}_X$ the sheaf of complex continuous functions of X.
- The Čech cohomology group $\check{H}^1(X, \underline{O}_X^\times)$ classifies isomorphism line bundles of X (made of 1-cocycles) and agree with sheaf cohomology $H^1(X, \underline{O}_X^\times)$.
- We consider the exact sequence of sheaves on X

$$0 \to \underline{\mathbb{Z}}_X \to \underline{O}_X \xrightarrow{exp} \underline{O}_X^\times \to 0$$

  where $\underline{\mathbb{Z}}_X$ stands for the constant sheaf on X.
- We write the long cohomology exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}(X) = \mathbb{Z}^{\pi_0(X)} \longrightarrow C^0(X, \mathbb{C}) \xrightarrow{exp} C^0(X, \mathbb{C})^\times \longrightarrow \ldots\ldots$$

Classification of lines bundles, II

- We write the long cohomology exact sequence

$$\begin{aligned}
0 \longrightarrow \underline{\mathbb{Z}}(X) = \mathbb{Z}^{\pi_0(X)} \longrightarrow C^0(X,\mathbb{C}) \xrightarrow{exp} C^0(X,\mathbb{C})^\times \longrightarrow \\
H^1(X,\underline{\mathbb{Z}}) \quad \longrightarrow H^1(X,\underline{O}_X) \xrightarrow{exp_*} H^1(X,\underline{O}_X^\times) \longrightarrow \\
H^2(X,\underline{\mathbb{Z}}) \quad \longrightarrow H^2(X,\underline{O}_X) \longrightarrow \dots
\end{aligned}$$

- Using partitions of unity on X, $\underline{O}_X$ is a flabby sheaf and then acyclic. We have $H^i(X,\underline{O}_X) = 0$ for each $i \geq 1$.
- We get an isomorphism $H^1(X,\underline{O}_X^\times) \xrightarrow{\sim} H^2(X,\underline{\mathbb{Z}})$.
- Since X is locally contractible, $H^2(X,\underline{\mathbb{Z}})$ agrees with the singular cohomology of X.

Classification of topological vector bundles over S^2

- **Theorem.** We have $H^1(S^2, \mathbb{C}^\times) \cong \mathbb{Z}$ and it is generated by $-[Hopf]$.
- We use the preceding fact and Poincaré duality $H^2(S^2, \underline{\mathbb{Z}}) = H^2(S^2, \mathbb{Z}) \cong \mathrm{Hom}(H_2(S^2, \mathbb{Z}), \mathbb{Z}) = \mathbb{Z}$.
- The point is that $\partial([Hopf]) = -1$. Of course you can apply the general formula from Hirzebruch's book !
- Here is an alternative trick based on the fact that this degree behave as (actually is) the degree arising from homotopy groups. Given a $\mathbb{C}^\times$-bundle $p : E \to P$, we have the exact sequence

$$\begin{tikzcd}
\dots & \pi_2(E) \arrow[r, "p_*"] & \pi_2(P) \arrow[r] \arrow[d, equal] & \pi_1(\mathbb{C}^\times) \arrow[d, equal] & \dots \\
 & & \mathbb{Z} \arrow[r, "d(E)"] & \mathbb{Z} &
\end{tikzcd}$$

.

which defines a degree such that $d(Hopf) = \pm 1$.

Classification of topological vector bundles over S^2

- $$\begin{tikzcd}
\cdots & \pi_2(E) \arrow[r, "p_*"] & \pi_2(P) \arrow[r] \arrow[d, equal] & \pi_1(\mathbb{C}^\times) \arrow[d, equal] & \cdots \\
 & & \mathbb{Z} \arrow[r, "d(E)"] & \mathbb{Z} & .
\end{tikzcd}$$

- This degree is additive so that the Hopf bundle must generate $\mathbb{Z}$.

- The next result is given for information, it will not be used further. It is based on homotopy theory.

- A vector bundle of rank $n \geq 1$ over S^2 is isomorphic to $\mathbb{C}^{n-1} \oplus Hopf^{\otimes r}$ for an unique $r \in \mathbb{Z}$.

- In particular, topological vector bundles are classified by their determinant (this holds for arbitrary dimension 2 manifolds).

(Analytic) vector bundles on a Riemann surface

- The definition is the same than for the topological case excepted that the transition maps $g_{i,j} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ are requested to be analytic.
- The Hopf line bundle is then analytic.
- We have then a dictionnary between vector bundles of degree n over the Riemann surface P and analytic $\mathrm{GL}_n(\mathbb{C})$-bundles.
- The Swan's correspondence has the Serre's analogue which is more complicated and involves that the presheaf of holomorphic functions on X is actually a sheaf $\mathscr{O}_X$ (in rings).
- The (analytic) section construction provides an equivalence of categories between the category of vector bundles over X and the category of locally free $\mathscr{O}_X$-sheaves of finite rank over .
- It attaches to V the sheaf of sections $U \to \mathrm{Hom}(\mathbb{C} \times U, V_{|U})$.

Grothendieck's theorem

- **Theorem.** If V is a homomorphic vector bundle of degree $n \geq 1$ over the Riemann sphere P, there exists a partition $n = n_1 + \cdots + n_c$ and an increasing sequence $a_1 < a_2 < \cdots < a_c$ such that

  $$V = (Hopf^{\otimes a_1})^{n_1} \oplus \cdots \oplus (Hopf^{\otimes a_c})^{n_c}.$$

  Furthermore the partition and the $a_i' s$ are unique.
- We denote by $\mathscr{O}(-1)$ is the invertible sheaf associated to the Hopf line bundle and put $\mathscr{O}(b) = \mathscr{O}(-1)^{\otimes(-b)}$. In terms of sheaves, it means that a locally free sheaf of rank n on P is the direct sum

  $$\mathscr{O}(b_1)^{n_1} \oplus \cdots \oplus \mathscr{O}(b_c)^{n_c}.$$

  with $b_1 > b_2 > \cdots > b_c$.

Grothendieck's theorem, II

- We shall reason in terms of sheaves so it requires some preparation. Firstly we deal with the map $v : \mathrm{Hopf} \to P \times \mathbb{C}$ which the composite $v : \mathrm{Hopf} \to P \times \mathbb{C}^2 \xrightarrow{p_0} P \times \mathbb{C}$. It provides a sequence of sheaves

$$0 \to \mathscr{O}(-1) \xrightarrow{v_*} \mathscr{O} \xrightarrow{ev_0} (i_0)_*(\mathbb{C}) \to 0$$

  where $(i_0)_*(\mathbb{C})$ stands for the skyscraper sheaf at 0.
- In concrete terms it means that u induces an exact sequence

$$0 \to \mathscr{O}(-1)(U) \to \mathscr{O}(U) \xrightarrow{ev_0} \mathbb{C} \to 0$$

  for each neighborhood U of 0 in P; the right handside map is the evaluation at zero.
- Indeed the condition for (z_0, l) to arise from some (z_0, z_1, l) with $(z_0, z_1) \in l$ is that $z_0 = 0$ if $l = [0 : 1]$.
- It follows that $H^0(P, \mathscr{O}(-1)) = 0$ and more generally that $H^0(P, \mathscr{O}(-m)) = 0$ for each $m \geq 1$.
- More delicate is to establish that $H^1(P, \mathscr{O}) = 0$.

$H^1(P, \mathscr{O}) = 0.$

- First we use Cartan's theorem B (alternatively some complex analysis) implying that $H^1(U_0, \mathscr{O}) = H^1(U_1, \mathscr{O}) = 0$. We are reduced to show that $\ker(H^1(P, \mathscr{O}) \to H^1(U_0, \mathscr{O}) \times H^1(U_1, \mathscr{O})) = 0$.
- In term of cocycles, that kernel is the obstruction to write holomorphic functions on $U_0 \cap U_1 = \mathbb{C}^\times$ as the difference $f_0 - f_1$ where $f_0 : U_0 \to \mathbb{C}$ and $f_0 : U_1 \to \mathbb{C}$.
- We remember (e.g. Ahlfors' book) that a holomorphic function $f : \mathbb{C}^\times \to \mathbb{C}$ admits a Laurent serie expansion

  $$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n = f_-(z^{-1}) + f_+(z)$$

  where $f_+ = \sum_{n \geq 0} a_n z^n$ (resp. f_-) is a power serie of radius ∞. This yields the desired vanishing.

Back to the statement

- **Statement.** A locally free sheaf of rank n on P is the direct sum
$$\mathscr{O}(b_1)^{n_1} \oplus \cdots \oplus \mathscr{O}(b_c)^{n_c}.$$
with $b_1 > \cdots > b_c$ and ; unicity holds for the n_i's and the b_i's.
- Unicity is the easy part. One way is to show that those numbers are intrinsic by computing the dimensions of $H^0(P, \mathscr{E} \otimes \mathscr{O}(m))$ for $m \in \mathbb{Z}$.
- One other is to appeal to the Atiyah-Krull-Schmidt's theorem.
- Note that $\mathscr{O}(1) \oplus \mathscr{O}(-1)$ is topologically trivial.
- A consequence is that the restriction of a vector bundle to $\mathbb{C}$ is trivial. On the other hand we know that holomorphic vector bundles on $\mathbb{C}$ are trivial (special case of Oka-Grauert's results for Stein spaces).

  In terms of cocycles, we have a bijection
$$\mathrm{GL}_n(\mathscr{O}(U_0)) \backslash \mathrm{GL}_n(\mathscr{O}(U_0 \cap U_1)) \,/\, \mathrm{GL}_n(\mathscr{O}(U_1)) \xrightarrow{\sim} \check{H}^1(P, \underline{GL}_{n,P}).$$

Birkhoff's theorem

- In terms of cocycles, we have a bijection

  $$\mathrm{GL}_n(\mathscr{O}(U_0))\backslash\mathrm{GL}_n(\mathscr{O}(U_0\cap U_1))/\mathrm{GL}_n(\mathscr{O}(U_1)) \xrightarrow{\sim} \check{H}^1(P, \underline{GL}_{n,P})$$

  where $\underline{GL}_{n,P}$ is the sheaf of invertible matrices on P and the set $\check{H}^1(P, \underline{GL}_{n,P})$ classifies the equivalence classes of vector bundles of rank n over P.
- **Corollary.** For each $g \in \mathrm{GL}_n(\mathscr{O}(U_0 \cap U_1))$, there exists a decomposition

  $$g = g_0 \,\mathrm{diag}(z^{b_1}, \dots, z^{b_c})\, g_1$$

  with $g_0 \in \mathrm{GL}_n(\mathscr{O}(U_0))$ and $g_1 \in \mathrm{GL}_n(\mathscr{O}(U_1))$.
- This is the main result of Birkhoff's paper *A Theorem on Matrices of Analytic Functions*, Mathematische Annalen (1913).
- Birkoff's therorem together with the triviality of vector bundles over $\mathbb{C}$ implies Grothendieck's theorem. It is called now often Birkhoff-Grothendieck's theorem and plays an important role in differential equations.

The standard proof

It is what we find for example in Sabbah's book and is very close of Grothendieck's proof.

- It uses that for a finite locally free sheaf $\mathscr{E}$ (actually any coherent sheaf) over P, the space of global sections $H^0(P, \mathscr{E})$ is finite dimensional and also that $H^0(P, \mathscr{E}(-r)) = 0$ for $r >> 0$ and that $H^0(P, \mathscr{E}(r)) \neq 0$ for $r >> 0$ where $\mathscr{E}(r) = \mathscr{E} \otimes \mathscr{O}(r)$.
- The proof goes by induction on the rank. We are given a locally free sheaf $\mathscr{E}$ of rank n. Up to twist $\mathscr{E}$, we can assume that $H^0(P, \mathscr{E}) \neq 0$ and $H^0(P, \mathscr{E}(-i)) = 0$ for each $i > 0$.
- We pick a base $s_1, \dots, s_{n_1}$ of $V = H^0(P, \mathscr{E})$, and see them as $s_i : \mathscr{O}_P \to \mathscr{E}$. It defines a map $\mathscr{O}^{n_1} \to \mathscr{E}$. If one section $s \in V$ at a point $p \in P$, we can assume (by letting $\mathrm{PGL}_2(\mathbb{C})$ acting) that $p = 0$. Then s belongs to the image of $H^0(P, \mathscr{E}(-1)) \to H^0(P, \mathscr{E}(-1))$ contradicting the vanishing $H^0(P, \mathscr{E}(-1))$.
- We have then an exact sequence of locally free sheaves

$$0 \to \mathscr{O}^{n_1} \to \mathscr{E} \to \mathscr{E}' \to 0.$$

Proof of Grothendieck's theorem

- We have then an exact sequence of locally free sheaves

  $$0 \to \mathscr{O}^{n_1} \to \mathscr{E} \to \mathscr{E}' \to 0.$$

  The induction hypothesis yields that $\mathscr{E}' = \mathscr{O}(b_2)^{n_2} \oplus \cdots \oplus \mathscr{O}(b_c)^{n_c}$ and we want to show that this extension splits.
- *Claim* : $b_i < 0$ for $i = 2, .., c$.
- We consider the the exact sequence

  $$0 \to \mathbb{C}^{n_1} \xrightarrow{\sim} H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E}') \to H^1(X, \mathscr{O}) = 0$$

  so that $H^0(X, \mathscr{E}') = 0$. The Claim follows.

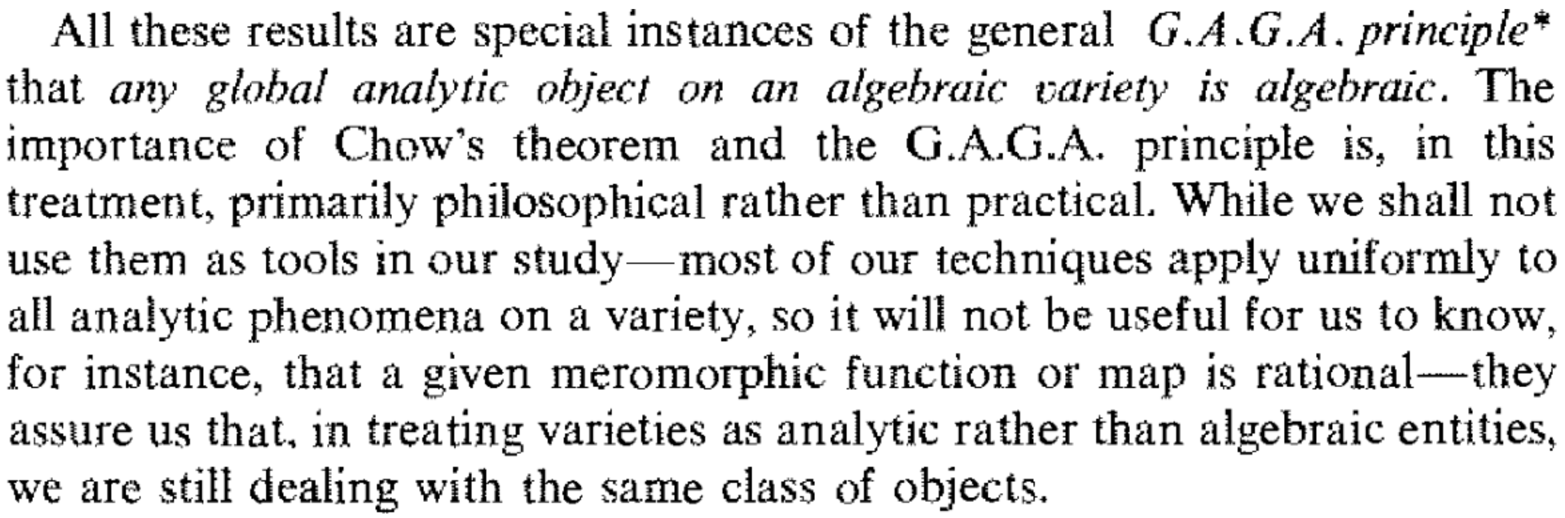
Proof, II

- We have an exact sequence
$$0 \to \mathscr{O}^{n_1} \to \mathscr{E} \to \bigoplus_{i=2}^{c} \mathscr{O}(b_i)^{n_i} \to 0$$
with $b_i \leq -1$ for $i = 2, ..., c$.
- The obtruction for splitness is the same than for the dual sequence
$$0 \to \bigoplus_{i=2}^{c} \mathscr{O}(-b_i)^{n_i} \to \mathscr{E}^{\vee} \to \mathscr{O}^{n_1} \to 0,$$
which is a class of $\left(\bigoplus_{i=2}^{c} H^1(P, \mathscr{O}(-b_i))^{n_i}\right)^{n_1}$.
- From induction from the case $\mathscr{O}$, we have $H^1(P, \mathscr{O}(d)) = 0$ for any $d \geq 0$.
- The obstruction is then zero and we are done.

A more algebraic proof

Grothendieck's proof uses algebraic geometry and it is of interest to see exactly where it matters.

- This goes a bit against the following opinion by Griffiths and Harris :



> All these results are special instances of the general *G.A.G.A. principle** that *any global analytic object on an algebraic variety is algebraic*. The importance of Chow's theorem and the G.A.G.A. principle is, in this treatment, primarily philosophical rather than practical. While we shall not use them as tools in our study—most of our techniques apply uniformly to all analytic phenomena on a variety, so it will not be useful for us to know, for instance, that a given meromorphic function or map is rational—they assure us that, in treating varieties as analytic rather than algebraic entities, we are still dealing with the same class of objects.

- We see now P as the analytic space associated to the projective line $\mathbb{P}^1_{\mathbb{C}}$ of algebraic geometry which is obtained by glueing two copies of the affine line $U_0 = \mathrm{Spec}(\mathbb{C}[t])$ and $U_1 = \mathrm{Spec}(\mathbb{C}[t^{-1}])$.

A more algebraic proof, II

- Algebraic vector bundles are defined in the same way that the analytic ones provided we restrict to Zariski open covers and we require that the transition maps are algebraic.
- A special case of Serre's GAGA principle is that there is an equivalence of categories between the category of algebraic vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ and the category of holomorphic vector bundles over P (complex analysis occurs).
- An advantage of the algebraic setting is to use that the finitely generated $\mathbb{C}[t]$-projective modules are free. Using Serre-Swan's dictionnary, it follows that the restriction of an algebraic vector bundle on $\mathbb{P}^1_{\mathbb{C}}$ over U_0 (resp. U_1) is trivial.
- Using the 1-cocycles approach, we are reduced to show an algebraic version of Birkhoff's decomposition. We mean that any $g \in \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ can be written as

$$g = g_0 \, \mathrm{diag}(t^{b_1}, \ldots, t^{b_n}) \, g_1$$

with $g_0 \in \mathrm{GL}_n(\mathbb{C}[t])$ and $g_1 \in \mathrm{GL}_n(\mathbb{C}[t^{-1}])$.

Dedekind-Weber's decomposition

- Can we write $g \in \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ as
$$g = g_0 \,\mathrm{diag}(t^{b_1}, \dots, t^{b_n})\, g_1$$
with $g_0 \in \mathrm{GL}_n(\mathbb{C}[t])$ and $g_1 \in \mathrm{GL}_n(\mathbb{C}[t^{-1}])$?
- This statement is a result by Dedekind and Weber (Crelle Journal, 1882) !
- One proof of Grothendieck's result is then GAGA together with this classical result (noticed by Hazewinkel and Martin, 1982).
- The way it is written is not with matrices but in terms of linear transformations.
- If $h : \left(\mathbb{C}[t, t^{-1}]\right)^n \to \left(\mathbb{C}[t, t^{-1}]\right)^n$ is a $\mathbb{C}[t, t^{-1}]$-linear isomorphism, then there exists a $\mathbb{C}[t^{-1}]$-base (e_i) of $(\mathbb{C}[t^{-1}])^n$ (resp. a $\mathbb{C}[t]$-base (f_i) of $\mathbb{C}[t]^n$) such that $h(e_i) = t^{b_i} f_i$.
- This statement is similar with that for maps $\mathbb{C}[t]^n \to \mathbb{C}[t]^n$ which permits the classification of projective $\mathbb{C}[t]$-modules.

Dedekind-Weber's decomposition II

- It was no hidden in the literature. It is in Hasse's book *Zahlentheorie* where it is called the *Wittscher Hilfssatz*.
- The decomposition can be proven in an elementary way by using $\mathbb{C}[t]$-row/$\mathbb{C}[t^{\pm 1}]$-column transformations in the manner of
$$M_n(\mathbb{C}[t]) = \mathrm{SL}_n(\mathbb{C}[t])\, D\, \mathrm{SL}_n(\mathbb{C}[t])$$
where $D \subset M_n(\mathbb{C}[t])$ consists in the diagonal matrices.
- I shall explain a more conceptual way to catch the decomposition. We consider the ring of formal series $R = \mathbb{C}[[t^{-1}]]$ and its fraction field $K = \mathbb{C}((t^{-1}))$ at ∞.
- Notation : We put
$$\Delta = \left\{ \mathrm{diag}(t^{b_1}, \ldots, t^{b_n}) \ \mid \ (b_1, ..., b_n) \in \mathbb{Z}^n \right\} \subset \mathrm{GL}_n(\mathbb{C}[t^{\pm 1}]).$$
- **Serre-Soulé's decomposition.** We have
$$\mathrm{GL}_n(K) = \mathrm{GL}_n(\mathbb{C}[t])\, \Delta\, \mathrm{GL}_n(R).$$
- It implies Dedekind-Weber's decomposition.

Serre-Soulé's decomposition

- **Serre-Soulé's decomposition.** $R = \mathbb{C}[[t^{-1}]]$ and $K = \mathbb{C}((^{-1}))$; we have
$$\mathrm{GL}_n(K) = \mathrm{GL}_n(\mathbb{C}[t])\,\Delta\,\mathrm{GL}_n(R).$$
- It implies Dedekind-Weber's decomposition. Indeed a matrix $g \in \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ writes
$$g = g_0\,\mathrm{diag}(t^{b_1}, \ldots, t^{b_n})\,g_1$$
with $g_0 \in \mathrm{GL}_n(\mathbb{C}[t])$ and $g_1 \in \mathrm{GL}_n(R) \cap \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ $= \mathrm{GL}_n(\mathbb{C}[t^{-1}])$.
- Serre-Soulé's approach involves the Goldman-Iwahori space (which is a special case of an euclidean building). The point is that $\mathrm{GL}_n(K)/\mathrm{GL}_n(R)$ is the space Λ of R–lattices in K^n seen as subspace of the Goldman-Iwahori space of metrics on K^n. The above decomposition rephrases to say that the set Σ of lattices $\quad Rt^{b_1}e_1 \oplus \cdots \oplus Rt^{b_n}e_n \qquad (b_1, ..., b_n) \in \mathbb{Z}^n$
satisfies $\mathrm{GL}_n(\mathbb{C}[t])\,.\,\Sigma = \Lambda$.

Serre-Soulé's decomposition ; II

- $\mathrm{GL}_n(\mathbb{C}[t])\,.\,\Lambda \;=\; \mathrm{GL}_n(K)/\mathrm{GL}_n(R)$
- This is a connection with arithmetic groups, think about the action of $\mathrm{SL}_n(\mathbb{Z})$ on the symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$.

Deformations of vector bundles on P

- Geometers are interested in family of objects. For example if X is an analytic manifold, what can we say about the vector bundles on $P \times X$?
- This is the matter of moduli spaces of vector bundles, ask Mihai Pavel for example. Today we focus only on what happens locally namely at the neighborhood of a point x.
- The Grothendieck theorem provides a type function

  $$type : \mathrm{Vect}^n(P \times X) \to \mathbb{Z}^n$$

  which associates the shape $(b_1(x), \dots, b_n(x))$ (with $b_1(x) \geq b_2(x) \geq \cdots \geq b_n(x)$) of the fiber at x of a vector bundle V on $P \times X$.
- **Claim.** The sum $b_1(x) + \cdots + b_n(x)$ is a locally constant function.
- This is the degree of the determinant bundle attached to V.

Deformations of vector bundles on P, II

- **Claim.** The sum $b_1(x) + \cdots + b_n(x)$ is a locally constant function.
- Since it is the degree of the determinant bundle attached to V so boils down to the case of line bundles. In that case, we have
$$H^1(P \times X, \mathscr{O}_{P\times X}) \cong H^2(P \times X, \mathbb{Z}) =$$
$$H^2(P, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) = \mathbb{Z} \oplus H^2(X, \mathbb{Z})$$
in view of the Künneth formula for singular cohomology. If X is contractible, we have $H^2(X, \mathbb{Z}) = 0$ and the above degree function is constant.
- One example of deformation is for $X = \mathbb{C}$ with parameter t is based on the fact $\mathbb{C} = H^1(P, \mathscr{O}(-2)) = \mathrm{Ext}^1(\mathscr{O}, \mathscr{O}(-2))$.
- We have then a map $H^0(X, \mathscr{O}_X) \to H^1(P \times X, \mathscr{O}(-2) \boxtimes \mathscr{O}_X)$ so that the identity function $f : X \to \mathbb{C}$ defines an extension
$$0 \to \mathscr{O}_{P\times X}(-2) \to \mathscr{E}_f \to \mathscr{O}_{P\times X} \to 0.$$

Deformations, III

- We deal with the extension

  $$0 \to \mathscr{O}_{P\times X}(-2) \to \mathscr{E}_f \to \mathscr{O}_{P\times X} \to 0.$$

  This extension is not split and its specialization at $t_0 \neq 0$ is not as well.

- The exact sequence

  $$\begin{tikzcd}
0 \arrow[r] & H^0(P,\mathscr{O}(-2)) \arrow[r] & H^0(P,\mathscr{E}_{t_0}) \arrow[r] & H^0(P,\mathscr{O}) \arrow[r] \arrow[d, equal] & H^1(P,\mathscr{O}(-2)) \arrow[d, equal] \\
 & & & \mathbb{C} \arrow[r, "id"] & \mathbb{C}
\end{tikzcd}$$

  shows that $H^0(P,\mathscr{E}_{t_0}) = 0$. Hence $\mathscr{E}_{t_0} \ncong \mathscr{O} \oplus \mathscr{O}(-2)$.

- Actually we have $\mathscr{E}_t \cong \mathscr{O}(-1) \oplus \mathscr{O}(-1)$. We have $\mathscr{E}_t \cong \mathscr{O}(d) \oplus \mathscr{O}(-d-2)$ for an integer d. Since $h^0(\mathscr{E}_t) = 0$, we have $d \leq -1$ and $-d-2 \leq -1$, hence $d = -1$.

Deformations, IV

- In terms of cocycles, the above family corresponds to

$$g(t) = \begin{pmatrix} z^{-2} & t\,z^{-1} \\ 0 & 1 \end{pmatrix}.$$

- Indeed by $\mathbb{C}[t]$-row/$\mathbb{C}[t^{-1}]$-column operations we have for $t \neq 0$

$$g(t) = \begin{pmatrix} z^{-2} & t\,z^{-1} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & t\,z^{-1} \\ -t^{-1}\,z^{-1} & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 0 & -t^{-1}z^{-1} \\ -tz^{-1} & 0 \end{pmatrix} \sim \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

- Summarizing we have $\mathit{type}(0) = (-2, 0)$ and $\mathit{type}(t_0) = (-1, -1)$ for $t_0 \neq 0$. We say that $(-2, 0)$ is a degeneration of $(-1, -1)$.

Ramanathan's theorem

- **Ramanathan's theorem (1983)**. Let V be a vector bundle on $P \times X$ of rank n. Let $x \in X$ be a point. Then x admits an open neighborhood $U \subset X$ such that for each $y \in U$, we have the inequalities

  $$b_i(y) - b_{i+1}(y) \leq b_i(x) - b_{i+1}(x) \quad (i = 1, ..., n-1).$$

  Furthermore there are no other general inequalities.

- **Corollary.** The continuity domain of the type function is the preimage of $\left\{(b_1, ..., b_n) \mid b_1 \leq b_n + 1\right\}$.
  In particular the type function is continuous at the preimage of $(0, \ldots, 0)$.

- We say also that the vector bundles whose type is diagonal are rigid.

Further topics

- One is Antshuetz interpretation of the monoidal category $\mathit{Vect}(\mathbb{P}^1_{\mathbb{C}})$ by the functor
$$\mathit{Rep}(\mathbb{G}_m) \to \mathit{Vect}(\mathbb{P}^1_{\mathbb{C}}).$$
The category $\mathit{Rep}(\mathbb{G}_m)$ is the category of finite dimensional representations of $\mathbb{G}_m$. To a representation $\lambda : \mathbb{G}_m \to \mathrm{GL}_n$, we associate the vector bundle $\lambda_*^{-1}(\mathrm{Hopf})$.
- The category $\mathit{Rep}(\mathbb{G}_m)$ is equivalent to the category of graded f.d. $\mathbb{C}$-vector spaces.
- It is remarkable that the above functor induces a bijection on isomorphism classes of objects. This is not an equivalence of categories !
- Indeed $\mathit{Rep}(\mathbb{G}_m)$ is semisimple when $\mathit{Vect}(\mathbb{P}^1_{\mathbb{C}})$ admits non trivial extensions.
- There is a categorical proof of Grothendieck's theorem by O'Sullivan in his preprint *Principal bundles under reductive groups*.

Rationally connected varieties

- This theory is due to Kollár and Mori. Let X be a complex smooth projective variety of dimension d (over $\mathbb{C}$). We are interested in morphisms
$$f : \mathbb{P}^1 \to X$$
which are *very free*, that is, the vector bundle $f^* T_X$ is ample, equivalently $f^* T_X \cong \mathscr{O}(b_1) \oplus \cdots \oplus \mathscr{O}(b_d)$ with $b_1 \geq b_2 \geq \cdots \geq b_d \geq 1$.
- That condition permits deformation and the existence of one such rational curve implies that there exists plenty of such curves. One can prescribe for example that a finite given set belongs to the image. For example Fano varieties are rationally connected (Kollár-Miyaoka-Mori, Campana).
- Let us quote the result of Graber-Harris-Starr (2001). Let $h : X \to \mathbb{P}^1$ be a morphism of projective smooth varieties whose geometric fiber is rationally connected (i.e. on the algebraic closure of $\mathbb{C}(t)$). Then h admits a section.

Further topics, II

- Remark : there is a generalization of rationally connected varieties in compact Kähler geometry.

- In their work on p-adic Hodge theory, Fargues and Fontaine abstracted the notion of Riemann sphere in order to have Grothendieck's decomposition theorem. This applies to the Fargues-Fontaine's curve.

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