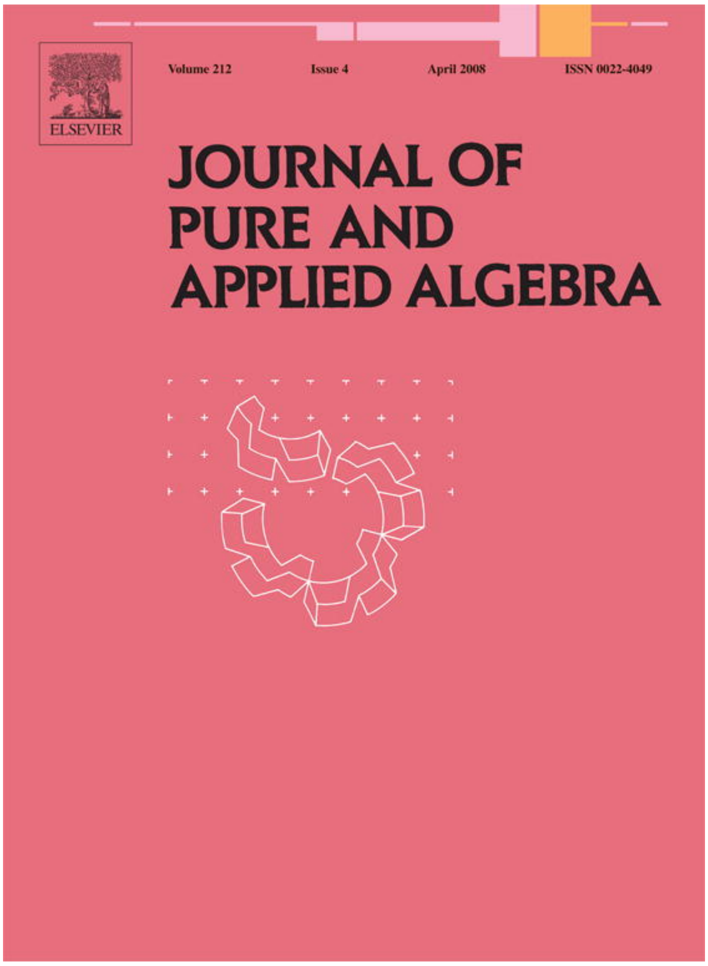
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Isotriviality and étale cohomology of Laurent polynomial rings

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Abstract

We provide a detailed study of torsors over Laurent polynomial rings under the action of an algebraic group. As applications we obtained variations of Raghunathan's results on torsors over affine space, isotriviality results for reductive group schemes and forms of algebras, and decomposition properties for Azumaya algebras.

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1. Introduction

That all topological vector bundles over $X = \mathbb{R}^n$ are trivial is an immediate consequence of the fact that the space X is contractible. The algebraic analogue of this problem, namely that if k is a field then all algebraic vector bundles over $X = \mathbb{A}_k^n = \mathrm{Spec}\,(k[t_1, \ldots, t_n])$ are trivial, is a much more delicate result due to Quillen and Suslin (the positive answer to the so called "Serre conjecture").

The essence of this problem is captured, in cohomological language, by the following two results:

$$H^1_{\mathrm{Zar}}(\mathbb{A}_k^n, \mathbf{GL}_{n,k}) = H^1_{\acute{e}t}(\mathbb{A}_k^n, \mathbf{GL}_{n,k}) = H^1_{fppf}(\mathbb{A}_k^n, \mathbf{GL}_{n,k}) \tag{1.1}$$

and

$$H^1_{\mathrm{Zar}}(\mathbb{A}_k^n, \mathbf{GL}_{n,k}) = 1. \tag{1.2}$$

The first of these two results, a version of Hilbert 90, asserts that the Zariski topology – which is in general too coarse to deal with principal bundles – is fine enough to measure algebraic vector bundles. The second result is equivalent to the theorem of Quillen and Suslin: All finitely generated projective modules over $k[t_1, \ldots, t_n]$ are free.

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It is a natural, difficult and interesting question to study (1.1) and (1.2) when $\mathbf{GL}_{n,k}$ is replaced by an arbitrary reductive k-group. Though in general the outcome to this problem is negative [33], the deep work of Raghunathan [38] gives a very good understanding of when a positive answer takes place.

The main objective of the present work is to look at the above two questions in the case when $k[t_1, \dots, t_n]$ is replaced by the ring $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials in n-variables. The importance of this case lies in its connections with infinite dimensional Lie theory as we now explain.

Assume k is algebraically closed and of characteristic 0, and set $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Consider the class of Lie algebras $\mathcal{L}$ over R_n with the property that

$$\mathcal{L} \otimes_{R_n} S \simeq \mathfrak{g} \otimes_k S \tag{1.3}$$

for some finite dimensional simple Lie algebra $\mathfrak{g}$, and some cover S of R on the étale topology. This is a very interesting class of algebras with intricate relations to the centreless cores of Extended Affine Lie Algebras (see [2] and [19] for details and further references). For $n = 1$ for example, the algebras in question turn out to be nothing but the affine Kac–Moody Lie algebras (derived modulo their centres) [35,36].

The algebras as in (1.3) are parametrized by $H^1_{ét}(R, \mathbf{Aut}(\mathfrak{g}))$, namely by torsors over R under $\mathbf{Aut}(\mathfrak{g})$. One of the major results of our work is that any such torsor is *isotrivial*, i.e., trivialized by a *finite* étale extension of R_n. This isotriviality theorem has important implications to the classification of Extended Affine Lie Algebras by cohomological methods [19, Theorem 5.13].

The present work contains complete proofs, and in some cases generalization, of all of the results announced in [18]. It is also an invaluable companion to [19]. The Appendix contains a summary, in suitable form and language, of most of the results on torsors that are needed for our work.

1.1. Notation and conventions

Throughout k denotes a field of characteristic 0. As usual, $\bar{k}$ will denote an algebraic closure of k. The tensor product $\otimes_k$ will be denoted by the unadorned symbol $\otimes$. The category of commutative associative unital k-algebras is denoted by $k\text{-}alg$.

For a given R in $k\text{-}alg$, by an *R-group* we will understand a group scheme over $\mathrm{Spec}(R)$. If $\mathbf{G}$ is an R-group, the pointed set of non-abelian Čech cohomology on the $fppf$ (resp. étale, resp. Zariski) site of $X = \mathrm{Spec}(R)$ with coefficients in $\mathbf{G}$, is denoted by $H^1_{fppf}(R, \mathbf{G})$ (resp. $H^1_{ét}(R, \mathbf{G})$, resp. $H^1_{\mathrm{Zar}}(R, \mathbf{G})$). These pointed sets measure the isomorphisms of sheaf torsors (with respect to a chosen one of our three topologies) over R under $\mathbf{G}$ (see Ch. IV Section 1 of [31] for basic definitions and references). Following customary usage depending on the context we also use the notation $H^1_{fppf}(X, \mathbf{G})$ instead of $H^1_{fppf}(R, \mathbf{G})$. Similarly for the étale and Zariski site.

If $\mathbf{G}$ is affine, flat and locally of finite presentation over R, then $\mathbf{G}$ is necessarily smooth over R.[1] By faithfully flat descent all of our sheaf torsors are representable (by affine R-schemes). They are thus *torsors* in the usual sense. Furthermore, the smoothness of $\mathbf{G}$ yields that all torsors are locally trivial for the étale topology. In particular, $H^1_{ét}(R, \mathbf{G}) = H^1_{fppf}(R, \mathbf{G})$. These assumption on $\mathbf{G}$ are present in most of the situations that arise in our work. At times, specially during proofs in order to cut down on notation, we write H^1 instead of $H^1_{ét}$.

Given an R-group $\mathbf{G}$ and a morphisms $R \to S$ in $k\text{-}alg$, we let $\mathbf{G}_S$ denote the S-group $\mathbf{G} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ obtained by base change. For convenience, we will under these circumstances denote most of the times $H^1_{ét}(S, \mathbf{G}_S)$ by $H^1_{ét}(S, \mathbf{G})$.

The expression *linear algebraic group (defined) over* k, is to be understood in the sense of Borel [6]. For a k-group $\mathbf{G}$, this is equivalent to requiring that $\mathbf{G}$ be affine of finite type ([44] $\mathrm{VI_B}.11.11$) because such a group is smooth by Cartier's theorem. The connected component of the identity of such group $\mathbf{G}$, will be denoted by $\mathbf{G}^0$.

$\mathbb{G}_{m,k}$ will denote the affine scheme $\mathrm{Spec}\, k[t^{\pm 1}]$. To avoid confusion, we will denote the corresponding (multiplicative group) by $\mathbf{G}_{m,k}$. As usual, n-dimensional affine k-space $\mathrm{Spec}\, k[t_1, \dots, t_n]$ is denoted by $\mathbb{A}^n_k$. If k is understood from the context, we simply write $\mathbb{G}_m$, $\mathbf{G}_m$, and $\mathbb{A}^n$ respectively.

---

[1] Since R is of characteristic zero, the geometric fibers of $\mathbf{G}$ are smooth by Cartier's theorem [13], II.6.1.1. The smoothness criterion on fibers [14], 17.8.1, shows that $\mathbf{G}$ is smooth over R.

A reductive R-group is to be understood in the sense of [44]. In particular, a reductive k-group is a reductive *connected* algebraic group defined over k in the sense of Borel.

For k fixed, we now list, for the reader's convenience, some notation which is extensively used throughout this work.

$$R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \qquad R_{n,d} = k[t_1^{\pm\frac{1}{d}}, \dots, t_n^{\pm\frac{1}{d}}] \quad \text{and} \quad R_{n,\infty} = \varinjlim_d R_{n,d}.$$

$$K_n = k(t_1^{\pm 1}, \dots, t_n^{\pm 1}), \qquad K_{n,d} = k(t_1^{\pm\frac{1}{d}}, \dots, t_n^{\pm\frac{1}{d}}) \quad \text{and} \quad K_{n,\infty} = \varinjlim_d K_{n,d}.$$

$$F_n = k((t_1))((t_2))\dots((t_n)) \quad \text{and} \quad F_{n,d} = k((t_1^{\pm\frac{1}{d}}))((t_2^{\pm\frac{1}{d}}))\dots((t_n^{\pm\frac{1}{d}}))$$

$\overline{F_n}$ is a fixed algebraic closure of F_n containing $\overline{k}$ and all $F_{n,d}$.

$\pi_1(R_n)$ denotes the algebraic fundamental group of $\mathrm{Spec}(R_n)$ at the geometric point $\mathrm{Spec}(\overline{F_n})$.

$(\zeta_n)_{n\geq 1}$ is a set of compatible primitive n-roots of unity, i.e., $\zeta_{\ell n}^{\ell} = \zeta_n$ (in the case when k is algebraically closed of characteristic 0).

2. Torsors over Laurent polynomial rings

Throughout this section k denotes a field of characteristic 0.

2.1. Locally trivial torsors

By a theorem of Quillen and of Suslin for $\mathbf{GL}_n$, and Raghunathan's work in general ([37] and [38]), one knows many examples of linear algebraic groups $\mathbf{G}/k$ for which $H^1_{\mathrm{Zar}}(\mathbb{A}^n_k, \mathbf{G}) = 1$. The vanishing of this H^1, however, does not hold in general (even in the case of the real numbers [33]). Our goal in this section is to derive analogous statements for $H^1_{\mathrm{Zar}}(R_n, \mathbf{G})$.

Lemma 2.1. *Let $\mathbf{G}$ be a linear algebraic group over k, and let Y/k be an affine variety for which $H^1_{\mathrm{Zar}}(Y_{k(t)}, \mathbf{G}) = 1$. Then the map*

$$H^1_{\mathrm{Zar}}(Y \times_k \mathbb{A}^1_k, \mathbf{G}) \to H^1_{\mathrm{Zar}}(Y \times_k \mathbb{G}_{m,k}, \mathbf{G})$$

is surjective.

Proof. Let $(U_i)_{i\in I}$ be a fundamental system of (affine) open neighborhoods of $0 \in \mathbb{A}^1$. Let $X = Y \times_k \mathbb{A}^1$, $X_0 = Y \times_k \mathbb{G}_m$, and $X_i = Y \times_k U_i$. The open set X_0 together with the X_i's form an open Zariski covering of X. We have $\varprojlim(X_i \cap X_0) = Y_{k(t)}$ where $k(t)$ is the function field of $\mathbb{A}^1$. Since our H^1 is defined à la Čech, it commutes with inverse limits (Appendix, Proposition A.4). More precisely,

$$\varinjlim_i H^1_{\mathrm{Zar}}(X_i \cap X_0, \mathbf{G}) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\varprojlim_i (X_i \cap X_0), \mathbf{G}) = H^1_{\mathrm{Zar}}(Y_{k(t)}, \mathbf{G}). \tag{2.1}$$

Let E_0 be a locally trivial $\mathbf{G}$-torsor over X_0. Since $H^1_{\mathrm{Zar}}(Y_{k(t)}, \mathbf{G}) = 1$, we obtain from (2.1) the existence of an $i \in I$ for which the restrictions of E_0 and $\mathbf{G}$ to $X_0 \cap X_i$ are isomorphic. Consider now the trivial torsor $E_1 = \mathbf{G}/X_i$. We can now glue E_1/X_i and E_0/X_0 along $X_0 \cap X_i$ to obtain a locally trivial $\mathbf{G}$-torsor E over X. □

Proposition 2.2. *Let $\mathbf{G}$ be a linear algebraic group over k. Let $\mathcal{E}$ be a collection of field extensions of k, with the property that whenever E is in $\mathcal{E}$, the field $E(t)$ is also in $\mathcal{E}$. If $H^1_{\mathrm{Zar}}(\mathbb{A}^n_E, \mathbf{G}) = 1$ for all E in $\mathcal{E}$, then $H^1_{\mathrm{Zar}}(E[t_1^{\pm 1}, \dots, t_r^{\pm 1}, t_{r+1}, \dots, t_n], \mathbf{G}) = 1$ for all $0 \leq r \leq n$. In particular, if k is in $\mathcal{E}$, then $H^1_{\mathrm{Zar}}(k[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \mathbf{G}) = 1$.*

Proof. First observe that if $H^1_{\mathrm{Zar}}(\mathbb{A}^n_E, \mathbf{G}) = 1$, then $H^1_{\mathrm{Zar}}(\mathbb{A}^m_E, \mathbf{G}) = 1$ for all $m < n$. To see this, simply consider the sequence $\mathbb{A}^m_E \to \mathbb{A}^n_E \to \mathbb{A}^m_E$, where the first map is the standard closed embedding, and the last map comes from specializing the last $n - m$ coordinates at 0.

We now reason by contradiction. Choose first n and then r, both minimal, for which the result fails. Clearly $0 < r \leq n$. Thus $H^1_{\mathrm{Zar}}(E(t)[t_1^{\pm 1}, \dots, t_{r-1}^{\pm 1}, t_{r+1}, \dots, t_n], \mathbf{G}) = 1$ and $H^1_{\mathrm{Zar}}(E[t_1^{\pm 1}, \dots, t_{r-1}^{\pm 1}, t_r, \dots, t_n], \mathbf{G}) = 1$. Taking now $Y = \mathrm{Spec}(E[t_1^{\pm 1}, \dots, t_{r-1}^{\pm 1}, t_{r+1}, \dots, t_n])$, we can apply the previous lemma to Y/E to conclude that, contrary to our assumption, $H^1_{\mathrm{Zar}}(E[t_1^{\pm 1}, \dots, t_r^{\pm 1}, t_{r+1}, \dots, t_n], \mathbf{G}) = 1$. $\square$

Corollary 2.3. *Let* $\mathbf{U}$ *be the unipotent radical of* $\mathbf{G}$*. If all the* k*-simple components of the reductive group* $(\mathbf{G}/\mathbf{U})^0$ *are isotropic, then* $H^1_{\mathrm{Zar}}(k[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \mathbf{G}) = 1$.

Proof. It suffices to show that $H^1_{\mathrm{Zar}}(\mathbb{A}^n_k, \mathbf{G}) = 1$. For the reductive group $(\mathbf{G}/\mathbf{U})^0$, this is Theorem A of [38]. The general case follows by reasoning as in page 104 of [11]. $\square$

2.2. *Isotriviality*

Let $\epsilon_n : R_{n,\infty} \to k$ be the unique k-algebra homomorphism that maps all the $t_i^{1/d}$ to 1 (the direct limit of the usual “evaluation at 1” maps on the $R_{n,d}$). The kernel of ϵ_n will be denoted by $\mathbf{1}_n$. This is a rational point of the k-scheme $\mathrm{Spec}(R_{n,\infty})$. By means of ϵ_n, we obtain a natural map $H^1_{\acute{e}t}(R_{n,\infty}, \mathbf{G}) \to H^1(k, \mathbf{G})$ which we call the “specialization at 1”. The kernel of this specialization map will be denoted by $H^1_{\acute{e}t}(R_{n,\infty}, \mathbf{1}_n, \mathbf{G})$, or simply by $H^1_{\acute{e}t}(R_{n,\infty}, \mathbf{1}, \mathbf{G})$ if n is clear from the context. It consists of the classes of $\mathbf{G}$-torsors over $R_{n,\infty}$ that become trivial under the base change $\mathrm{Spec}(k) \to \mathrm{Spec}(R_{n,\infty})$ defined by ϵ_n.

Proposition 2.4. *Let* $\mathbf{G}$ *be a linear algebraic group over* k*. Then the canonical map*

$$H^1_{\mathrm{Zar}}(R_{n,\infty}, \mathbf{G}) \to H^1_{\acute{e}t}(R_{n,\infty}, \mathbf{1}_n, \mathbf{G}).$$

is bijective.

We first consider the case $n = 1$ separately, for which we prove a more precise statement.

Lemma 2.5. *Let* $\mathbf{G}$ *be a linear algebraic group over* k*.*

1. $H^1_{\acute{e}t}(R_{1,\infty}, \mathbf{1}_1, \mathbf{G}) = 1$.
2. $H^1_{\mathrm{Zar}}(R_{1,\infty}, \mathbf{G}) = 1$.
3. *The restriction map* $k \to R_{1,\infty}$ *induces a bijection* $H^1_{\acute{e}t}(k, \mathbf{G}) \simeq H^1_{\acute{e}t}(R_{1,\infty}, \mathbf{G})$.

To establish the lemma we need two preliminary results. The first is the following useful reduction argument.

Lemma 2.6. *Let* $\mathbf{G}$ *be a linear algebraic group over* k*, and let* $n \geq 0$ *be an integer.*

1. *The restriction map* $H^1(k, \mathbf{G}) \to H^1_{\acute{e}t}(\mathbb{A}^n_k, \mathbf{G})$ *is injective.*
2. *Let* $\mathbf{U}$ *be the unipotent radical of* $\mathbf{G}$*. If the restriction map* $H^1(k, \mathbf{G}/\mathbf{U}) \to H^1_{\acute{e}t}(\mathbb{A}^n_k, \mathbf{G}/\mathbf{U})$ *is bijective, then* $H^1(k, \mathbf{G}) \to H^1_{\acute{e}t}(\mathbb{A}^n_k, \mathbf{G})$ *is also bijective.*
3. *Let* $\mathbf{G}'$ *be a normal subgroup of* $\mathbf{G}$ *such that the quotient* $\mathbf{G}/\mathbf{G}'$ *is finite. If the restriction map* $H^1(k, \mathbf{G}') \to H^1_{\acute{e}t}(\mathbb{A}^n_k, \mathbf{G}')$ *is bijective, then* $H^1(k, \mathbf{G}) \to H^1_{\acute{e}t}(\mathbb{A}^n_k, \mathbf{G})$ *is also bijective.*

Proof. (1) The evaluation $H^1(\mathbb{A}^n, \mathbf{G}) \overset{ev_0}{\to} H^1(k, \mathbf{G})$ at 0 is a left inverse of the restriction map $H^1(k, \mathbf{G}) \to H^1(\mathbb{A}^n, \mathbf{G})$. The restriction map is thus injective.

(2) Consider the commutative diagram

$$\begin{array}{ccccc}
H^1(k, \mathbf{U}) & \xrightarrow{i_*} & H^1(k, \mathbf{G}) & \xrightarrow{p_*} & H^1(k, \mathbf{G}/\mathbf{U}) \\
\rho_{\mathbf{U}}\downarrow & & \rho_{\mathbf{G}}\downarrow & & \rho_{\mathbf{G}/\mathbf{U}}\downarrow \\
H^1(\mathbb{A}^n, \mathbf{U}) & \xrightarrow{i_*} & H^1(\mathbb{A}^n, \mathbf{G}) & \xrightarrow{p_*} & H^1(\mathbb{A}^n, \mathbf{G}/\mathbf{U}).
\end{array}$$

The rightmost vertical arrow is bijective by assumption. Next, we claim that $H^1(\mathbb{A}^n, {}_\gamma\mathbf{U}) = 1$ for all $[\gamma] \in H^1(\mathbb{A}^n, \mathbf{G})$. Since k is of characteristic 0, the group $\mathbf{U}$ has centre $\mathcal{Z} \simeq \mathbf{G}_a^\ell$ for some $\ell \geq 0$, where $\mathbf{G}_a$ is the additive k-group. By dévissage, we reduce to the case when ${}_\gamma\mathbf{U}$ is abelian. Since $\mathbb{A}^n$ is affine, we have $H^1(\mathbb{A}^n, {}_\gamma\mathbf{U}) = 0$ by Grothendieck's theorem. From this it follows that both horizontal maps p_* in the above diagram are injective, and we can now conclude as in the proof of (3) below.

(3) Let $\boldsymbol{\mu} = \mathbf{G}/\mathbf{G}'$, and consider the commutative diagram

$$\begin{tikzcd} H^1(k, \mathbf{G}') \arrow[r, "i_*"] \arrow[d, "\rho_{\mathbf{G}'}"'] & H^1(k, \mathbf{G}) \arrow[r, "p_*"] \arrow[d, "\rho_{\mathbf{G}}"'] & H^1(k, \boldsymbol{\mu}) \arrow[d, "\rho_{\boldsymbol{\mu}}"'] \\ H^1(\mathbb{A}^n, \mathbf{G}') \arrow[r, "i_*"] & H^1(\mathbb{A}^n, \mathbf{G}) \arrow[r, "p_*"] & H^1(\mathbb{A}^n, \boldsymbol{\mu}) \end{tikzcd}$$

The left vertical arrow is bijective by assumption. Let a be a geometric point of $\mathbb{A}^n$. Since $\mathbb{A}^n \times_k \overline{k}$ is simply connected, the canonical map $\pi_1(\mathbb{A}^n, a) \to \mathrm{Gal}(k)$ is an isomorphism ([43] théorème 6.1). On the other hand since k is of characteristic 0, the finite R-group $\boldsymbol{\mu}$ is étale, hence corresponds to a finite abstract group μ together with a continuous action of $\pi_1(\mathbb{A}^n, a)$. One has a bijection $H^1(\mathbb{A}^n, \boldsymbol{\mu}) \simeq H^1(\pi_1(\mathbb{A}^n, a), \mu)$ (see [43] XI.5). Since, as we have seen, $\pi_1(\mathbb{A}^n, a) \simeq \mathrm{Gal}(k)$, the right restriction map $\rho_{\boldsymbol{\mu}} : H^1(k, \boldsymbol{\mu}) \to H^1(\mathbb{A}^n, \boldsymbol{\mu})$ is bijective.

Let now $[\gamma] \in H^1(\mathbb{A}^n, \mathbf{G})$, and denote the class $ev_0[\gamma]$ of its evaluation at 0 by $[z]$. Assume first that $[z] = 1 \in H^1(k, \mathbf{G})$. Then $ev_0(p_*[\gamma]) = 1 \in H^1(k, \boldsymbol{\mu})$, hence $p_*[\gamma] = 1 \in H^1(\mathbb{A}^n, \boldsymbol{\mu})$ (because $\rho_{\boldsymbol{\mu}}$ is bijective). It follows that $[\gamma]$ belongs to the image of $H^1(\mathbb{A}^n, \mathbf{G}') \to H^1(\mathbb{A}^n, \mathbf{G})$, and we conclude that $[\gamma]$ comes from $H^1(k, \mathbf{G})$ under the restriction map $\rho_{\mathbf{G}}$ as desired. The general case follows by considering the twisted group ${}_z\mathbf{G}/k$, and following the twisting bijections as in (1) above. $\square$

We also need the following acyclicity observation due to M. Florence.

Proposition 2.7 *([15], Proposition 5.4). Let* $\mathbf{G}$ *be a linear algebraic group over* k*. The canonical map* $H^1(k, \mathbf{G}) \to \varinjlim_d H^1(k((t^{1/d})), \mathbf{G})$ *is bijective.*

Proof of Lemma 2.5. (1) Recall that $R_{1,d} = k[t^{\pm 1/d}]$. By direct limit considerations (see Appendix, Proposition A.4), we have

$$\varinjlim H^1(R_{1,d}, \mathbf{G}) \simeq H^1(R_{1,\infty}, \mathbf{G}). \tag{2.2}$$

It is immediate that under this correspondence we have

$$\varinjlim H^1(R_{1,d}, \mathbf{1}, \mathbf{G}) \simeq H^1(R_{1,\infty}, \mathbf{1}, \mathbf{G}), \tag{2.3}$$

where, as expected, $H^1(R_{1,d}, \mathbf{1}, \mathbf{G}) \subset H^1(R_{1,d}, \mathbf{G})$ accounts for the classes of $R_{1,d}$-torsors under $\mathbf{G}$ that become trivial under the specialization map $R_{1,d} \to k$ given by $t^{1/d} \mapsto 1$. Accordingly, it will suffice to show that the natural maps $H^1(R_{1,d}, \mathbf{1}, \mathbf{G}) \to H^1(R_{1,\infty}, \mathbf{G})$ are trivial. We may assume without loss of generality that $d = 1$. Let $[\gamma] \in H^1(R_{1,1}, \mathbf{1}, \mathbf{G})$. In view of Corollary A.8, the obstruction to extending γ to the full affine line is local at 0, and is given by the image $\gamma_{k((t))}$ of γ in

$$H^1(k((t)), \mathbf{G})/H^1(k[[t]], \mathbf{G}).$$

By Hensel's lemma ([7], lemme 3.2), the canonical map $H^1(k[[t]], \mathbf{G}) \to H^1(k, \mathbf{G})$ is bijective, so the previous obstruction actually belongs to

$$H^1(k((t)), \mathbf{G})/H^1(k, \mathbf{G}).$$

By Proposition 2.7 there exists $d > 0$ such that $[\gamma_{k((t^{1/d}))}] \in \mathrm{Im}\big(H^1(k, \mathbf{G}) \to H^1(k((t^{1/d})), \mathbf{G})\big)$. In other words, the obstruction to extending γ to the full affine line $\mathbb{A}_k^1$ is killed by the base change $R_{1,1} \to R_{1,d}$. So after replacing $R_{1,1}$ by $R_{1,d}$, we may assume then that γ comes from $H^1(\mathbb{A}_k^1, \mathbf{1}, \mathbf{G})$. By Raghunathan–Ramanathan's theorem [39] we have $H^1(\mathbb{A}_k^1, \mathbf{1}, \mathbf{G}) = 1$ whenever $\mathbf{G}$ is reductive, and we can now appeal to Lemma 2.6 to conclude that $H^1(\mathbb{A}_k^1, \mathbf{1}, \mathbf{G}) = 1$ in general. Thus γ is trivial in $H^1(R_{1,\infty}, \mathbf{G})$.

(2) Clearly $H^1_{\mathrm{Zar}}(R_{1,\infty}, \mathbf{G}) \subset H^1(R_{1,\infty}, \mathbf{1}, \mathbf{G}) = 1$. Now (2) follows from (1).
(3) Let $Y \in H^1(R_{1,\infty}, \mathbf{G})$. Since $\varinjlim H^1(R_{1,d}, \mathbf{G}) \simeq H^1(R_{1,\infty}, \mathbf{G})$, we may assume that Y comes from an $R_{1,d}$-torsor under $\mathbf{G}$ by base change. By reasoning as in (1) above, the image of Y in some $H^1(R_{1,d'}, \mathbf{G})$ extends to a $\mathbf{G}$-torsor on the affine line. We are thus reduced to showing that the canonical map $H^1(k, \mathbf{G}) \to H^1(\mathbb{A}^n_k, \mathbf{G})$ is bijective. For $\mathbf{G}$ reductive, this is a theorem of Raghunathan and Ramanathan [39]. The general case follows from the last lemma. □

Proof of Proposition 2.4. We claim that the sequence of pointed sets

$$1 \to H^1_{\mathrm{Zar}}(R_{n,\infty}, \mathbf{G}) \to H^1(R_{n,\infty}, \mathbf{G}) \to H^1(K_{n,\infty}, \mathbf{G}) \tag{2.4}$$

is exact. For let $[E_\infty] \in H^1(R_{n,\infty}, \mathbf{G})$. Choose $d > 0$ and $[E] \in H^1(R_{n,d}, \mathbf{G})$ such that $E \mapsto E_\infty$ under the identification $\varinjlim H^1(R_{n,d}, \mathbf{G}) \simeq H^1(R_{n,\infty}, \mathbf{G})$. If E_∞ is rationally trivial, then $(E \times_{R_{n,d}} R_{n,\infty})(K_{n,\infty}) = E_\infty(K_{n,\infty}) \neq \emptyset$. It follows that $E(K_{n,\infty}) \neq \emptyset$, hence that $E(K_{n,d'}) \neq \emptyset$ for some $d' \geq d$ (this because E is of finite type over $R_{n,d}$). As a consequence of théorème 3.2 of [11], we can conclude that $E \times_{R_{n,d}} R_{n,d'}$ is locally trivial. Thus $E_\infty = E \times_{R_{n,d}} R_{n,d'} \times_{R_{n,d'}} R_{n,\infty}$ is also locally trivial, and our claim holds.

By taking (2.4) into account, Proposition 2.4 can be established by showing that the map $H^1(R_{n,\infty}, \mathbf{1}, \mathbf{G}) \to H^1(K_{n,\infty}, \mathbf{G})$ is trivial. To do this, we reason by induction on $n \geq 1$. The case $n = 1$ has already been established. Assume $n > 1$. Let $p : R_{n,\infty} \to K_{1,\infty} \otimes_k R_{n-1,\infty}$ be the natural map, and consider the commutative diagram

$$\begin{array}{ccc}
H^1(R_{n,\infty}, \mathbf{G}) & \xrightarrow{p_*} & H^1(K_{1,\infty} \otimes_k R_{n-1,\infty}, \mathbf{G}) \\
{\scriptstyle ev_{1_{n-1}}}\downarrow & & {\scriptstyle ev_{1_{n-1}}}\downarrow \\
H^1(R_{1,\infty}, \mathbf{G}) & \longrightarrow & H^1(K_{1,\infty}, \mathbf{G})
\end{array}$$

obtained by base change, where the vertical arrows are given by the evaluation maps $R_{n,d} \to R_{1,d}$ that set $t_2^{1/d} = \cdots = t_n^{1/d} = 1$. We have

$$ev_{1_{n-1}}\left(H^1(R_{n,\infty}, \mathbf{1}_n, \mathbf{G})\right) \subset H^1(R_{1,\infty}, \mathbf{1}_1, \mathbf{G}),$$

so the case $n = 1$ implies that $p_*\left(H^1(R_{n,\infty}, \mathbf{1}_n, \mathbf{G})\right) \subset H^1(K_{1,\infty} \otimes_k R_{n-1,\infty}, \mathbf{1}_{n-1}, \mathbf{G})$. We apply then the induction hypothesis to $K_{1,\infty} \otimes_k R_{n-1,\infty}$ with field of fractions $K_{n,\infty}$, to conclude that the map

$$H^1(K_{1,\infty} \otimes_k R_{n-1,\infty}, \mathbf{1}_{n-1}, \mathbf{G}) \to H^1(K_{n,\infty}, \mathbf{G})$$

is trivial. It now follows that $H^1(R_{n,\infty}, \mathbf{1}_n, \mathbf{G})$ consists of rationally trivial torsors. □

Remark 2.8. If we limit ourselves to the case of $\mathbf{G}$ finite and constant, the proof of Proposition 2.4 is much simpler, and can be established without appealing to the results of Colliot-Thélène and Ojanguren, and of Raghunathan and Ramanathan. The relevant point, namely that rationally trivial $\mathbf{G}$-torsors over R_n are trivial, is a general fact about finite connected étale coverings of normal noetherian schemes ([43] I.10.2). Note also that because of the canonical isomorphism between the absolute Galois group of k and that of $\varinjlim k((t^{1/d}))$ ([15], Section 2), Proposition 2.7 is clear in the finite constant case.

Theorem 2.9. *Let $\mathbf{G}$ be a linear algebraic group over k. If k is algebraically closed, then $H^1_{\acute{e}t}(R_{n,\infty}, \mathbf{G}) = 1$.*

Proof. If k is algebraically closed, then $H^1(R_{n,\infty}, \mathbf{G}) = H^1(R_{n,\infty}, \mathbf{1}_n, \mathbf{G})$, and therefore $H^1_{\mathrm{Zar}}(R_{n,\infty}, \mathbf{G}) = H^1(R_{n,\infty}, \mathbf{G})$ by Proposition 2.4. The group $(\mathbf{G}/\mathbf{U})^0$ of Corollary 2.3 is split, so $H^1_{\mathrm{Zar}}(R_n, \mathbf{G}) = 1$. It follows that $H^1_{\mathrm{Zar}}(R_{n,\infty}, \mathbf{G}) = \varinjlim H^1_{\mathrm{Zar}}(R_{n,d}, \mathbf{G}) = 1$, hence that $H^1(R_{n,\infty}, \mathbf{G}) = 1$ as desired. □

Assume k is algebraically closed, and fix a set $(\zeta_n)_{n\geq 1} \subset k^\times$ of compatible primitive n-roots of unity. We identify the elements of $(\mathbb{Z}/d\mathbb{Z})^n$ with automorphisms of the extension $R_{n,d}/R_n$ by ${}^{\bar{\mathbf{e}}}(t_i^{1/d}) = \zeta_d^{e_i} t_i^{1/d}$ for all $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ (where as usual, $^- : \mathbb{Z}^n \to (\mathbb{Z}/d\mathbb{Z})^n$ is the canonical map). One easily checks that $R_{n,d}/R_n$ is Galois, with Galois group $(\mathbb{Z}/d\mathbb{Z})^n$ as above.

We fix once and for all an algebraic closure $\overline{F_n}$ of F_n containing all of the $F_{n,d}$. This defines a geometric point a of $X = \operatorname{Spec}(R_n)$, and we consider the corresponding algebraic fundamental group $\pi_1(R_n, a)$ (see [43] V.7). Since X is connected, normal and noetherian, we have an explicit realization, henceforth denoted by X^{sc}, for the universal covering of X at a ([43] V.8.2). In particular, we have a canonical homomorphism $\operatorname{Gal}(F_n) \to \pi_1(R_n, a)$.

Corollary 2.10. *Assume k is algebraically closed. Then*

1. $\operatorname{Spec}(R_{n,\infty})$ *is simply connected (i.e. it does not have any non-trivial finite étale covers), and it is naturally isomorphic to the simply connected covering X^{sc} of* $\operatorname{Spec}(R_n)$ *at the geometric point a.*
2. $\pi_1(R_n, a)$ *is the Galois group of the pro-covering $R_{n,\infty}/R_n$, namely*
$$\pi_1(R_n, a) = \varinjlim_d \operatorname{Gal}(R_{n,d}/R_n) = \varinjlim_d (\mathbb{Z}/d\mathbb{Z})^n = (\widehat{\mathbb{Z}})^n = \widehat{\mathbb{Z}^n}.$$
3. *The canonical map* $\operatorname{Gal}(F_n) \to \pi_1(R_n, a)$ *is an isomorphism.*

Proof. (1) Given that the $R_{n,d}$'s are connected, we know that $\operatorname{Spec}(R_{n,\infty})/\operatorname{Spec}(R_n)$ is isomorphic to the quotient of X^{sc} by some closed subgroup H of $\pi_1(X, a)$ ([43] V.5.5). We have to show that H is trivial. Let H' be an open normal subgroup of H and set $G := H/H'$. Then $X^{sc}/H' \to X^{sc}/H \cong \operatorname{Spec}(R_{n,\infty})$ is a G-covering. It is non-trivial because X^{sc}/H' is connected. The isomorphism classes of principal covers of X with Galois group G, are parametrized by $H^1(R_{n,\infty}, \mathbf{G})$, where $\mathbf{G}$ is the constant group corresponding to the finite abstract group G ([43] XI.5). But by Theorem 2.9 this H^1 vanishes. So $G = 1$. All continuous finite quotients of the profinite group H are therefore trivial, hence $H = 1$. Thus $X^{sc} \simeq \operatorname{Spec}(R_{n,\infty})$ as desired.

(2) It is clear that the Galois group of the pro-covering $R_{n,\infty}/R_n$ is the group $(\widehat{\mathbb{Z}})^n$ described above.

(3) The canonical map $\operatorname{Spec}(F_n) \to \operatorname{Spec}(R_n)$ yields, by the functorial nature of π_1, a group homomorphism $\operatorname{Gal}(F_n) \to \pi_1(R_n, a)$. Under this map, the Galois extension $F_{n,d}/F$ corresponds to $R_{n,d}/R_n$. Since $\operatorname{Gal}(F_n) = \varprojlim_d \operatorname{Gal}(F_{n,d}/F_n)$ (as one can see by induction on n, the case $n = 1$ being covered in [17] II.7.1), (3) now follows from (2). $\square$

Remark 2.11. The isomorphism $\pi_1(R_n, a) \simeq (\widehat{\mathbb{Z}})^n$ of Corollary 2.10 is *not* canonical. It only becomes "natural" after a choice of compatible set of roots of unity in k is made.

Remark 2.12. Since R_n is connected, and $\pi_1(R_n, a)$ is abelian, the group $\pi_1(R_n, a)$ is independent of the choice of geometric point a (up to canonical isomorphism). Indeed, there exists a canonical isomorphism $\operatorname{Hom}_{ct}(\pi_1(R_n, a), \mathbb{Q}/\mathbb{Z}) \simeq H^1_{\acute{e}t}(R_n, \mathbb{Q}/\mathbb{Z})$ between the Pontryagin dual ([48] Section 6.4) of the profinite abelian group $\pi_1(R_n, a)$, and a group depending only on $\operatorname{Spec}(R_n)$. Since $\pi_1(R_n, a)$ can be recovered from its Pontryagin dual, $\pi_1(R_n, a)$ does not depend, up to a canonical isomorphism, on the choice of the geometric point a. Accordingly, we henceforth simply write $\pi_1(R_n)$.

Corollary 2.13 (*Comparison Theorem*). *Consider a geometric point $a' : \operatorname{Spec}(\overline{k}) \to \operatorname{Spec}(R_n)$ corresponding to a rational point of R_n. Then $\pi_1(R_n, a) \simeq \pi_1(R_n, a') \simeq (\widehat{\mathbb{Z}})^n \rtimes \operatorname{Gal}(k)$.*

Proof. The result follows from the previous corollary and remark by taking into account [43] IX théorème 6.1 and corollaire 6.4. $\square$

Corollary 2.14. *The canonical map* $\operatorname{Gal}(F_n) \to \pi_1(R_n, a)$ *is an isomorphism.*

Remark 2.15. Let $\mathbf{G}$ be a constant group (not necessarily finite). By [44, X.5.5], all sheaf torsors over R_n under $\mathbf{G}$ are representable, i.e., that they are torsors. Since R_n is normal, noetherian and connected, any R_n-torsor under $\mathbf{G}$ is isotrivial, i.e. split by a finite étale extension ([44] X.6). It follows from Corollary 2.10 that $\varinjlim_d H^1(R_{n,d}, \mathbf{G}) = 1$. Since $\mathbf{G}$ is locally of finite presentation Proposition A.4 of the Appendix yields $H^1(R_{n,\infty}, \mathbf{G}) = 1$.

Corollary 2.16. *Assume that k is algebraically closed. Then*

1. *If $\boldsymbol{\mu}$ is a finite twisted constant R_n-group, then $\boldsymbol{\mu} \times_{R_n} R_{n,\infty}$ is constant. Furthermore $H^1(R_{n,\infty}, \boldsymbol{\mu}) = 1$.*
2. *If $\mathbf{G}$ is a reductive R_n-group, then $\mathbf{G} \times_{R_n} R_{n,\infty}$ is split. Furthermore $H^1(R_{n,\infty}, \mathbf{G}) = 1$.*

3. *For every R_n-group $\mathbf{F}$ which is an extension of a twisted finite constant group by a reductive group, there exists a canonical correspondence $H^1_{ct}(\pi_1(R_n), \mathbf{F}(R_{n,\infty})) \simeq H^1(R_n, \mathbf{F})$. (The étale cohomology of the R_n-group $\mathbf{F}$ is thus given by the usual Galois cocycles.)*

Proof. (1) Let $\boldsymbol{\mu}$ be a finite twisted constant R_n-group. Then $\boldsymbol{\mu}$ is isotrivial. This means that there exists a finite étale covering S/R_n such that $\boldsymbol{\mu} \times_{R_n} S$ is a constant S-group. It then follows from Corollary 2.10 that $\boldsymbol{\mu} \times_{R_n} R_{n,\infty}$ is constant. In particular, $\boldsymbol{\mu} \times_{R_n} R_{n,\infty}$ is obtained by the base change $R_{n,\infty}/k$ from a linear algebraic group over k (namely the k-group corresponding to the finite abstract group underlying $\boldsymbol{\mu} \times_{R_n} R_{n,\infty}$). By the Theorem, $H^1(R_{n,\infty}, \boldsymbol{\mu}) = 1$.
(2) For showing that $\mathbf{G} \times_{R_n} R_{n,\infty}$ is split we proceed by dévissage.
First case : $\mathbf{G}$ is semisimple: By the structure theorem of Demazure, there exists a unique Chevalley $\mathbb{Z}$-group $\mathbf{G}_0$ such that $\mathbf{G}$ is an R_n-form of $\mathbf{G}_0 \times_{\mathbb{Z}} R_n$. We have an exact sequence of affine $\mathbb{Z}$-groups

$$1 \to \mathbf{G}_0^{ad} \to \mathbf{Aut}(\mathbf{G}_0) \to \mathbf{Out}(\mathbf{G}_0) \to 1,$$

with $\mathbf{Out}(\mathbf{G}_0)$ finite constant ([44] XXIV théorème 1.3 and Cor. 1.6). This yields the exact sequence of pointed sets

$$H^1(R_{n,\infty}, \mathbf{G}_0^{ad}) \to H^1\left(R_{n,\infty}, \mathbf{Aut}(\mathbf{G}_0)\right) \to H^1\left(R_{n,\infty}, \mathbf{Out}(\mathbf{G}_0)\right). \tag{2.5}$$

We have $H^1(R_{n,\infty}, \mathbf{Out}(\mathbf{G}_0)) = 1$ by (1), and $H^1(R_{n,\infty}, \mathbf{G}_0^{ad}) = 1$ by Theorem 2.9. Thus $H^1(R_{n,\infty}, \mathrm{Aut}(\mathbf{G}_0)) = 1$. Since this last pointed set classifies the isomorphism classes of $R_{n,\infty}$-forms of $\mathbf{G}_0 \times_{\mathbb{Z}} R_{n,\infty}$ ([44] XXIV Cor. 1.18), the $R_{n,\infty}$-groups $\mathbf{G} \times_{R_n} R_{n,\infty}$ and $\mathbf{G}_0 \times_{\mathbb{Z}} R_{n,\infty}$ are isomorphic. In particular, $\mathbf{G} \times_{R_n} R_{n,\infty}$ is split.
Second case : $\mathbf{G}$ is an R_n-torus: Since R_n is connected, there exists an integer r such that $\mathbf{G}$ is locally isomorphic to $\mathbf{G}_m^r$ for the $fpqc$ topology [44] IX.1.3. By [44] IX.2.1(b), it follows that $\mathbf{G}$ is of finite type over R_n. Since R_n is noetherian and normal, $\mathbf{G}$ is isotrivial ([44] IX.5.16). Thus $\mathbf{G} \times_{R_n} R_{n,\infty}$ is split (i.e., isomorphic to $\mathbf{G}_m^r$).
General case: By [44] XXII, 4.3 and 6.2.4, we have an exact sequence of R_n-group schemes

$$1 \to \boldsymbol{\mu} \to \mathbf{G}' \times_{R_n} \mathrm{rad}(\mathbf{G}) \to \mathbf{G} \to 1$$

where $\mathbf{G}'$ is the derived group of $\mathbf{G}$, $\mathrm{rad}(\mathbf{G})$ its radical torus and $\boldsymbol{\mu}$ is a finite R_n-group of multiplicative type which is central in $\mathbf{G}' \times_{R_n} \mathrm{rad}(\mathbf{G})$. The first (resp. second) case shows that $\mathbf{G}' \times_{R_n} R_{n,\infty}$ (resp. $\mathrm{rad}(\mathbf{G}) \times_{R_n} R_{n,\infty}$) is split. Since a central quotient of a split reductive group is split ([44] XXII.4.3.1), the group $\mathbf{G} \times_{R_n} R_{n,\infty}$ is split.

We have proven that $\mathbf{G} \times_{R_n} R_{n,\infty} \cong \mathbf{G}_0 \times_{\mathbb{Z}} R_{n,\infty}$ where $\mathbf{G}_0/\mathbb{Z}$ is the Chevalley form of $\mathbf{G}$. We have then $H^1(R_{n,\infty}, \mathbf{G}) \cong H^1(R_{n,\infty}, \mathbf{G}_0)$, thus $H^1(R_{n,\infty}, \mathbf{G})$ vanishes by Theorem 2.9.

Variations on the proof of splitness: We have the same sequence (2.5) where $\mathbf{G}_0$ is the unique reductive Chevalley $\mathbb{Z}$-group $\mathbf{G}_0$ such that $\mathbf{G}$ is an R_n-form of $\mathbf{G}_0 \times_{\mathbb{Z}} R_n$. The first and last term of this sequence vanish (the leftmost H^1 by Theorem 2.9, the rightmost H^1 by Remark 2.15), and we can conclude by [44] XXIV Cor. 1.18 as above.

Yet another argument is as follows. The case of tori above shows that after a finite étale base change $\mathrm{rad}(\mathbf{G})$ is split. Thus after replacing R_n by $R_{n,m}$ there exists an isomorphism $f : \mathrm{rad}(\mathbf{G}) \simeq \mathrm{rad}(\mathbf{G}_0)$. The set $X = \{\phi \in \mathrm{Isom}(\mathbf{G}, \mathbf{G}_0) : \phi|_{\mathrm{rad}(\mathbf{G})} = f\}$ is a torsor under a group $\mathbf{H}$ which fits into an exact sequence $1 \to \mathbf{G}_0^{ad} \to \mathbf{H} \to \mathbf{F} \to 1$ with $\mathbf{F}$ finite and constant ([44] XXIV Cor 2.16). Then X is isotrivial by the same reasoning use in the semisimple case above.
(3) We have canonical identifications

$$\ker\left(H^1(R_n, \mathbf{F}) \longrightarrow H^1(R_{n,d}, \mathbf{F})\right) = H^1(R_{n,d}/R_n, \mathbf{F}) = H^1\left(\mathrm{Gal}(R_{n,d}/R_n), \mathbf{F}(R_{n,d})\right),$$

where this last is the "usual" non-abelian cohomology of the finite group $\mathrm{Gal}(R_{n,d}/R_n)$ acting on $\mathbf{F}(R_{n,d})$. With the aid of Proposition A.4 of the Appendix we obtain

$$H^1_{ct}(\pi_1(R_n), \mathbf{F}(R_{n,\infty})) \cong \ker\left(H^1(R_n, \mathbf{F}) \longrightarrow H^1(R_{n,\infty}, \mathbf{F})\right),$$

where the profinite group $\pi_1(R_n)$ acts continuously of the discrete group $\mathbf{F}(R_{n,\infty})$. Now (3) follows from (1) and (2). □

Corollary 2.17. *Let $\mathbf{G}$ be a reductive R_n-group. There exists an integer $d \geq 0$, and a finite Galois extension K/k, such that $\mathbf{G} \times_{R_n} (K \otimes_k R_{n,d})$ is split. In particular, $\mathbf{G}$ is isotrivial.*

Proof. This easily follows from Corollary 2.16(2), when taking into account that $\mathbf{G}/R_n$ is finitely presented. One can also bypass Corollary 2.16(2), and give a direct proof of the present corollary along the lines of Theorem 2.9. □

2.3. Applications to forms of algebras

Throughout A will denote a *finite dimensional* k-algebra (not necessarily unital or associative; for example a Lie algebra). All rings are assumed to be commutative and unital.

Recall that if R is a ring, then by an *R-form of A* we understand an algebra $\mathcal{L}$ over R for which there exists a faithfully flat and finitely presented extension S/R in $k\text{-}alg$ such that

$$\mathcal{L} \otimes_R S \simeq_S A \otimes S$$

(isomorphism of S-algebras). Thus by definition, forms are trivialized on some $fppf$ cover S/R. In the case of Laurent polynomials, we have very precise control over the trivializing base change.

Theorem 2.18 (*Isotriviality*). *Let A be a finite dimensional k-algebra. Every R_n-form $\mathcal{L}$ of A is isotrivial (i.e. trivialized by a finite étale cover of R_n). More precisely, there exist a finite Galois extension K/k and a positive integer d, such that*

$$\mathcal{L} \otimes_{R_n} (R_{n,d} \otimes K) \simeq_{R_{n,d}\otimes K} A \otimes (R_{n,d} \otimes K).$$

Proof. Let $\bar{k}$ be the algebraic closure of k. We will use $\bar{\ }$ to denote the objects obtained by the base change $\bar{k}/k$; for example $\bar{\mathcal{L}} = \mathcal{L} \otimes \bar{k}$, and $\bar{A} = A \otimes \bar{k}$. Note that $\varinjlim_d \bar{R}_{n,d} \simeq \bar{R}_{n,\infty}$.

Our first objective is to show that $\bar{\mathcal{L}}$, with its natural $\bar{R}_n$-algebra structure, is a form of $\bar{A}$. Indeed, if $\phi : \mathcal{L} \otimes_{R_n} S \to A \otimes S$ is an S-algebra isomorphism, then the composite map $\bar{\phi}$ given by the sequence

$$\begin{aligned}\bar{\mathcal{L}} \otimes_{\bar{R}_n} \bar{S} &\simeq (\mathcal{L} \otimes \bar{k}) \otimes_{R_n \otimes \bar{k}} (S \otimes \bar{k}) \\ &\simeq (\mathcal{L} \otimes_{R_n} S) \otimes \bar{k} \simeq (A \otimes S) \otimes \bar{k} \simeq (A \otimes \bar{k}) \otimes_{\bar{k}} (S \otimes \bar{k}) \simeq \bar{A} \otimes_{\bar{k}} \bar{S}\end{aligned}$$

is an $\bar{S}$-algebra isomorphism.

It follows at once that $\bar{\mathcal{L}}_\infty = \bar{\mathcal{L}} \otimes_{\bar{R}_n} \bar{R}_{n,\infty}$ is an $\bar{R}_{n,\infty}$-form of $\bar{A}$. Now $\bar{R}_{n,\infty}$-isomorphism classes of such forms are parametrized by

$$H^1\left(\bar{R}_{n,\infty}, \mathbf{Aut}(\bar{A})\right) \simeq H^1\left(\varinjlim_d \bar{R}_{n,d}, \mathbf{Aut}(\bar{A})\right) \simeq \varinjlim_d H^1\left(\bar{R}_{n,d}, \mathbf{Aut}(\bar{A})\right)$$

(see Appendix, Proposition A.4). Since $\mathbf{Aut}(\bar{A})$ is a linear algebraic group over an algebraically closed field of characteristic zero, Theorem 2.9 shows that $H^1\left(\bar{R}_{n,\infty}, \mathbf{Aut}(\bar{A})\right)$ vanishes. As a consequence, there exist an $\bar{R}_{n,d}$-algebra isomorphism

$$\psi : \bar{\mathcal{L}} \otimes_{\bar{R}_n} \bar{R}_{n,d} \simeq \bar{A} \otimes_{\bar{k}} \bar{R}_{n,d}$$

for some $d > 0$.

Since A is finite dimensional, there exists a finite Galois subextension K/k of $\bar{k}$ such that $A \otimes 1 \subset \psi(\mathcal{L} \otimes_k K) \otimes_{R_n \otimes K} (R_{n,d} \otimes K)$. Given that ψ is $\bar{R}_{n,d}$-linear, ψ induces, by restriction, an $(R_n \otimes K)$-algebra isomorphism $(\mathcal{L} \otimes_k K) \otimes_{R_n \otimes K} (R_{n,d} \otimes K) \simeq A \otimes_k (R_{n,d} \otimes K)$. In view of the canonical isomorphisms $\mathcal{L} \otimes_{R_n} (R_{n,d} \otimes K) \simeq (\mathcal{L} \otimes_{R_n} R_{n,d}) \otimes K \simeq (\mathcal{L} \otimes K) \otimes_{R_n \otimes K} (R_{n,d} \otimes K)$, the result follows. $\square$

3. Étale cohomology of $\mathbf{R}_n$

Let k be a field of characteristic 0, and $\bar{k}$ a (fixed) algebraic closure of k. For convenience we will denote $\boldsymbol{\mu}_{l k}$ simply by $\boldsymbol{\mu}_l$. Then $\boldsymbol{\mu}_\infty = \varinjlim_l \boldsymbol{\mu}_l$ is the Galois module of the roots of unity. We recall first the definition of Tate twist following [32, Section 7.3.6]. We have $\widehat{\mathbb{Z}}^\times = \mathrm{Aut}(\boldsymbol{\mu}_\infty)$ and we denote by $\chi_{cycl} : \mathrm{Gal}(\bar{k}/k) \to \widehat{\mathbb{Z}}^\times$ the cyclotomic character, namely the action of the absolute Galois group of k on the roots of unity of $\bar{k}$. Given a Galois module A and $i \in \mathbb{Z}$, we denote by $A(i)$ the Galois module which is equal to A as an abelian group, and in which $\mathrm{Gal}(\bar{k}/k)$ acts by

$$\sigma(a) = \chi_{cycl}(\sigma)^i \cdot \sigma(a).$$

$A(i)$ is called the i-th *Tate twist* of A. In particular, $A = A(0)$ and $A(i + j) = A(i) \otimes A(j)$ for all $i, j \in \mathbb{Z}$. For $i \geq 1$, we have $\mathbb{Z}/l\mathbb{Z}(i) = \boldsymbol{\mu}_l^{\otimes i} = \boldsymbol{\mu}_l \otimes \cdots \otimes \boldsymbol{\mu}_l$ (i times) and $\mathbb{Z}/l\mathbb{Z}(-i) \cong \mathrm{Hom}(\boldsymbol{\mu}_l^{\otimes i}, \mathbb{Z}/l\mathbb{Z})$. Recall also that $\mathbb{Q}/\mathbb{Z}(i) = \varinjlim \mathbb{Z}/l\mathbb{Z}(i)$ for all $i \in \mathbb{Z}$. Finally, set $\mathbb{Z}(i) = \varprojlim \mathbb{Z}/l\mathbb{Z}(i)$, which is a projective limit of Galois modules.

3.1. Standard coefficients

We begin by relating the étale cohomology of R_n with coefficients $\mathbb{Z}/l\mathbb{Z}(r)$ or $\mathbf{G}_m$, to the Galois cohomology of k and F_n.

Proposition 3.1. 1. *The base change map $H^q_{\acute{e}t}(R_n, \mathbb{Z}/l\mathbb{Z}(r)) \to H^q(F_n, \mathbb{Z}/l\mathbb{Z}(r))$ is an isomorphism for $q \geq 0$, and we have the decomposition*

$$H^q_{\acute{e}t}(R_n, \mathbb{Z}/l\mathbb{Z}(r)) = \bigoplus_{i=0}^{n} H^{q-i}(k, \mathbb{Z}/l\mathbb{Z}(r-i))^{\binom{n}{i}}.$$

2. *The base change map $H^q_{\acute{e}t}(R_n, \mathbf{G}_m) \to H^q(F_n, \mathbf{G}_m)$ is an isomorphism for $q \geq 2$, and we have the decomposition*

$$H^q_{\acute{e}t}(R_n, \mathbf{G}_m) = H^q(k, \mathbf{G}_m) \oplus \bigoplus_{i=1}^{n} H^{q-i}(k, \mathbb{Q}/\mathbb{Z}(1-i))^{\binom{n}{i}}.$$

Lemma 3.2. *Let X/k be a smooth variety. Then*

1. *$H^q_{\acute{e}t}(X, \mathbf{G}_m)$ is a torsion group for all $q \geq 2$;*
2. *We have an exact sequence*

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \to \varinjlim_{l} H^2_{\acute{e}t}(X, \boldsymbol{\mu}_n) \to \mathrm{Br}(X) \to 0.$$

3. *For $q \geq 3$, we have an isomorphism*

$$\varinjlim_{l} H^q_{\acute{e}t}(X, \boldsymbol{\mu}_l) \simeq H^q_{\acute{e}t}(X, \mathbf{G}_m).$$

4. *The canonical map $H^*_{\acute{e}t}(X, \mathbf{G}_m) \to H^*_{\acute{e}t}(X \times_k \mathbb{A}^1_k, \mathbf{G}_m)$ is an isomorphism.*

Proof. (1) This is Proposition 1.4 of [23].
(2) and (3). Given $l \geq 1$, the Kummer exact sequence $1 \to \boldsymbol{\mu}_l \to \mathbf{G}_m \to \mathbf{G}_m \to 1$ gives rise to the long exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, \boldsymbol{\mu}_l) & \longrightarrow & H^0(X, \mathbf{G}_m) & \xrightarrow{\times l} & H^0(X, \mathbf{G}_m) & \longrightarrow & \\ & & H^1(X, \boldsymbol{\mu}_l) & \longrightarrow & H^1(X, \mathbf{G}_m) & \xrightarrow{\times l} & H^1(X, \mathbf{G}_m) & \longrightarrow & \cdots \end{array}$$

For each $q \geq 1$, we have the short exact sequence

$$0 \longrightarrow H^{q-1}(X, \mathbf{G}_m) \otimes \mathbb{Z}/l\mathbb{Z} \longrightarrow H^q(X, \boldsymbol{\mu}_l) \longrightarrow {}_lH^q(X, \mathbf{G}_m) \longrightarrow 0.$$

By passing to the limit with respect to l, we get

$$0 \longrightarrow H^{q-1}(X, \mathbf{G}_m) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \varinjlim_{l} H^q(X, \boldsymbol{\mu}_l) \longrightarrow H^q(X, \mathbf{G}_m)_{tors} \longrightarrow 0$$

Assertions (2) and (3) readily follow of (1).
(4) For $q = 0$, we have $H^0(X, \mathbf{G}_m) = \mathbf{G}_m(X) = \mathbf{G}_m(X \times \mathbb{A}^1) = H^0(X \times \mathbb{A}^1, \mathbf{G}_m)$. For $q = 1$, it is well known that $H^1(X, \mathbf{G}_m) = \mathrm{Pic}(X) = \mathrm{Pic}(X \times \mathbb{A}^1) = H^1(X \times \mathbb{A}^1, \mathbf{G}_m)$ (see [20], Proposition A.4.4). Since acyclicity of the morphism $X \times \mathbb{A}^1 \to X$ holds for the Picard functor and étale cohomology with finite coefficients (Artin, see [31], IV.4.20), the exact sequence of Part (2) yields acyclicity for the Brauer group, i.e. $\mathrm{Br}(X) \cong \mathrm{Br}(X \times \mathbb{A}^1)$. For higher indices $q \geq 3$, it follows from (3) that $H^q(X, \mathbf{G}_m) \simeq H^q(X \times \mathbb{A}^1, \mathbf{G}_m)$. □

Proof of Proposition 3.1. Assume that $n \geq 1$.
(1) By taking $X = \mathrm{Spec}(R_{n-1}[t_n])$, $U = \mathrm{Spec}(R_n) \subset \mathrm{Spec}(R_{n-1}[t_n])$ and $Z = X \setminus U = \mathrm{Spec}(R_{n-1}) \times \{0\}$ as entries in the long exact sequence of localization in étale cohomology (see [9], Theorem 3.4.1), we obtain the exact sequence

$$\cdots \longrightarrow H^q(R_{n-1}[t_n], \mathbb{Z}/l\mathbb{Z}(r)) \longrightarrow H^q(R_n, \mathbb{Z}/l\mathbb{Z}(r)) \xrightarrow{\partial} H^{q-1}(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r-1)) \longrightarrow H^{q+1}(R_{n-1}[t_n], \mathbb{Z}/l\mathbb{Z}(r)) \longrightarrow \cdots$$

By homotopy invariance ([9] 3.1, and also [45] Exp. XV Cor. 2.2), we have $H^q(R_{n-1}[t_n], \mathbb{Z}/l\mathbb{Z}(r)) \simeq H^q(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r))$. Since the morphism $\mathrm{Spec}(R_n) \to \mathrm{Spec}(R_{n-1})$ is split, it follows that the map

$$H^q(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r)) \simeq H^q(R_{n-1}[t_n], \mathbb{Z}/l\mathbb{Z}(r)) \to H^q(R_n, \mathbb{Z}/l\mathbb{Z}(r))$$

is also split. Hence we have a short split exact sequence

$$0 \to H^q(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r)) \longrightarrow H^q(R_n, \mathbb{Z}/l\mathbb{Z}(r)) \xrightarrow{\partial} H^{q-1}(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r-1)) \to 0.$$

On the other hand, we have a localization sequence of Galois cohomology ([20], Section 6.8). The compatibility between the two residues (which a priori agree only up to sign) is non-trivial to check, and has been worked out by E. Frossard ([16], corollaire A.2). The signs are indeed the same, and we thus have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^q(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r)) \arrow[r] \arrow[d] & H^q(R_n, \mathbb{Z}/l\mathbb{Z}(r)) \arrow[r, "\partial"] \arrow[d] & H^{q-1}(R_{n-1}, \mathbb{Z}/l\mathbb{Z}(r-1)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^q(F_{n-1}, \mathbb{Z}/l\mathbb{Z}(r)) \arrow[r] & H^q(F_n, \mathbb{Z}/l\mathbb{Z}(r)) \arrow[r, "\partial"] & H^{q-1}(F_{n-1}, \mathbb{Z}/l\mathbb{Z}(r-1)) \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma, and an induction argument on n, we get $H^q(R_n, \mathbb{Z}/l\mathbb{Z}(r)) \simeq H^q(F_n, \mathbb{Z}/l\mathbb{Z}(r))$ for all $q \geq 0$. Moreover, the splitting above yields the decomposition

$$H^q(F_n, \mathbb{Z}/l\mathbb{Z}(r)) \overset{s_{t_n} \oplus \partial_{t_n}}{\xrightarrow{\sim}} H^q(F_{n-1}, \mathbb{Z}/l\mathbb{Z}(r)) \oplus H^{q-1}(F_{n-1}, \mathbb{Z}/l\mathbb{Z}(r-1))$$

([20], Section 6.8). By an easy induction on n, we get the desired decomposition, namely

$$H^q(F_n, \mathbb{Z}/l\mathbb{Z}(r)) = \bigoplus_{i=0}^{n} H^{q-i}(k, \mathbb{Z}/l\mathbb{Z}(r-i))^{\binom{n}{i}}.$$

(2) Assume $q \geq 2$, and consider the commutative square

$$\begin{tikzcd}
\varinjlim_l H^q(R_n, \boldsymbol{\mu}_l) \arrow[r] \arrow[d] & H^q(R_n, \mathbf{G}_m) \arrow[d] \\
\varinjlim_l H^q(F_n, \boldsymbol{\mu}_l) \arrow[r] & H^q(F_n, \mathbf{G}_m).
\end{tikzcd}$$

The case of finite coefficients implies that the left vertical map is an isomorphism, and we can appeal to Lemma 3.2(1) to conclude that the horizontal maps are also isomorphisms. Hence the right vertical map is an isomorphism. It remains to pass to the limit in the decomposition

$$H^q(R_n, \boldsymbol{\mu}_l) \cong H^q(F_n, \boldsymbol{\mu}_l) \cong \bigoplus_{i=0}^{n} H^{q-i}\,(k, \mathbb{Z}/l\mathbb{Z}(1-i))^{\binom{n}{i}}.$$

This is allowed by a technical result of Grothendieck ([45], exp. VII, théorème 5.7 and Cor. 5.8). Since by definition $\mathbb{Q}/\mathbb{Z}(1-i) = \varinjlim \mathbb{Z}/l\mathbb{Z}(1-i)$, we obtain

$$H^q(R_n, \mathbf{G}_m) \cong \varinjlim_l H^q(R_n, \boldsymbol{\mu}_l) \cong H^q(k, \mathbb{Q}/\mathbb{Z}(1)) \oplus \bigoplus_{i=1}^{n} H^{q-i}\,(k, \mathbb{Q}/\mathbb{Z}(1-i))^{\binom{n}{i}}$$

as desired. $\square$

Corollary 3.3. *R_n^{sc} is an acyclic object for étale cohomology with constant finite coefficients, i.e., $H^q_{\acute{e}t}(R_n^{sc}, \mathbf{F}) = 0$ for any finite constant abelian R_n-group $\mathbf{F}$, and all $q \geq 1$.*

Proof. We may assume that k is algebraically closed, and that $\mathbf{F} = \boldsymbol{\mu}_l$. Recall from Section 2.2 that $R_n^{sc} = R_{n,\infty} = \varinjlim_d R_{n,d}$. By the technical result on étale cohomology mentioned above, we have

$$H^q(R_n^{sc}, \boldsymbol{\mu}_l) = \varinjlim_d H^q(R_{n,d}, \boldsymbol{\mu}_l).$$

Recall that $F_{n,d} = k((t_1^{\frac{1}{d}}))((t_2^{\frac{1}{d}}))\ldots((t_n^{\frac{1}{d}}))$ and $F_{n,\infty} = \varinjlim_d F_{n,d}$. By Puiseux's theorem, $F_{n,\infty}$ is algebraically closed. With the help of Proposition 3.1(1) we now obtain

$$H^q(R_n^{sc}, \boldsymbol{\mu}_l) = \varinjlim_d H^q(R_{n,d}, \boldsymbol{\mu}_l) = \varinjlim_d H^q(F_{n,d}, \boldsymbol{\mu}_l) = H^q(F_{n,\infty}, \boldsymbol{\mu}_l) = 0$$

for $q \geq 1$. □

3.2. Groups of multiplicative type

Proposition 3.4. 1. *The base change $\mathbf{H} \to \mathbf{H}_{F_n}$ induces an equivalence between the category of multiplicative R_n-groups of finite type, and multiplicative F_n-groups of finite type. Furthermore, the groups and equivalence between these two categories, correspond to finitely generated abelian groups together with a compatible continuous action of the profinite groups $\mathrm{Gal}(F_n) \to \pi_1(R_n)$.*

2. *Let $\boldsymbol{\mu}$ be a finite multiplicative R_n-group. The canonical maps $H^q_{ct}(\pi_1(R_n), \boldsymbol{\mu}(R_n^{sc})) \to H^q_{\acute{e}t}(R_n, \boldsymbol{\mu}) \to H^q(F_n, \boldsymbol{\mu}_{F_n})$ are isomorphisms for all $q \geq 0$. Furthermore, if k is algebraically closed, these H^q vanish for $q \geq n+1$.*
3. *For any multiplicative R_n-group $\mathbf{H}$ of finite type, and for all $q \geq 1$, the canonical maps $H^q_{\acute{e}t}(R_n, \mathbf{H}) \to H^q(F_n, \mathbf{H}_F)$ obtained by base change are isomorphisms. Furthermore, if k is algebraically closed, these H^q vanish for $q \geq n+1$.*

Proof. (1) Since the canonical map $\mathrm{Gal}(F_n) \to \pi_1(R_n)$ is an isomorphism (Corollary 2.14), (1) follows from [44] Exp.X Cor 1.2, Rem. 1.3 and Prop. 1.4.
(2) Since $\mathrm{Gal}(F_n) \simeq \pi_1(R_n)$ and $\boldsymbol{\mu}(R^{sc}) = \boldsymbol{\mu}(\overline{F}_n)$, we have an isomorphism $H^*(\pi_1(R_n), \boldsymbol{\mu}(R_n^{sc})) \simeq H^*(\mathrm{Gal}(F_n), \boldsymbol{\mu}(\overline{F}_n)) = H^*(F_n, \boldsymbol{\mu})$. So we have to prove that the map $H^*(\pi_1(R_n), \boldsymbol{\mu}(R_n^{sc})) \to H^*(R_n, \boldsymbol{\mu})$ is an isomorphism. By the spectral sequence in [31] III 2.21(b), it suffices to show that $H^q(R_n^{sc}, \boldsymbol{\mu}) = 0$ for $i \geq 1$. This boils down to the vanishing of $H^q_{\acute{e}t}(R_n^{sc}, \boldsymbol{\mu}_l)$ for $q \geq 1$, which is true by the acyclicity of R_n^{sc} (Corollary 3.3). If $k = \overline{k}$, we have $\mathrm{Gal}(F_n) = (\widehat{\mathbb{Z}})^n$, so $cd(F_n) = n$ ([42], I.3, prop. 15). It follows that $H^q(F_n, \boldsymbol{\mu}) = 0$ for $q \geq n+1$.
(3) For $\mathbf{G}_m$ and $q \geq 1$, we have $H^q(R_n, \mathbf{G}_m) \simeq H^q(F_n, \mathbf{G}_m)$ by Proposition 3.1(1). By Shapiro's formula, we have $H^q(R_n, \mathbf{E}) \simeq H^q(F_n, \mathbf{E})$ for any induced R_n-torus $\mathbf{E}$ and any $q \geq 1$. Next we consider the case of an R_n-torus $\mathbf{T}$. By (1) we can appeal to the following trick due to Ono (see [41], lemme 1.7 and [34], Theorem 1.5.1): There exists an integer m such that $\mathbf{T}^m$ fits in an exact sequence $1 \to \boldsymbol{\mu} \to \mathbf{E}_1 \to \mathbf{T}^m \times_{R_n} \mathbf{E}_2 \to 1$, where $\mathbf{E}_i$ is an induced R_n-torus, and $\boldsymbol{\mu}$ a finite multiplicative R_n-group. By (2), we have $H^q(R_n, \boldsymbol{\mu}) \simeq H^q(F_n, \boldsymbol{\mu})$ for all q. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 = H^1(R_n, \mathbf{E}_1) & \to & H^1(R_n, \mathbf{T}^m \times_{R_n} \mathbf{E}_2) & \to & H^2(R_n, \boldsymbol{\mu}) & \to & H^2(R_n, \mathbf{E}_1) & \to & H^2(R_n, \mathbf{T}^m \times_{R_n} \mathbf{E}_2) \cdots \\
 & & \downarrow & & \downarrow \wr & & \downarrow \wr & & \downarrow \\
0 = H^1(F_n, \mathbf{E}_1) & \to & H^1(F_n, \mathbf{T}^m \times_{F_n} \mathbf{E}_2) & \to & H^2(F_n, \boldsymbol{\mu}) & \to & H^2(F_n, \mathbf{E}_1) & \to & H^2(F_n, \mathbf{T}^m \times_{F_n} \mathbf{E}_2) \cdots .
\end{array}$$

The canonical maps $H^q(R_n, \mathbf{T}^m \times_{R_n} \mathbf{E}_2) \to H^q(F_n, \mathbf{T}^m \times_{R_n} \mathbf{E}_2)$ are isomorphisms (for $q = 1$ this is a diagram chase, while for $q \geq 2$ we can appeal to the five lemma). From this we conclude that the canonical maps $H^q(R_n, \mathbf{T}) \to H^q(F_n, \mathbf{T})$ are isomorphisms for all $q \geq 1$.

For the general case, we fit our $\mathbf{H}$ into an exact sequence $1 \to \mathbf{T} \to \mathbf{H} \to \boldsymbol{\mu} \to 1$, where $\mathbf{T}$ is a torus and $\boldsymbol{\mu}$ is finite. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
H^0(R_n, \boldsymbol{\mu}) & \longrightarrow & H^1(R_n, \mathbf{T}) & \longrightarrow & H^1(R_n, \mathbf{H}) & \longrightarrow & H^1(R_n, \boldsymbol{\mu}) & \longrightarrow & \cdots \\
\downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \\
H^0(F_n, \boldsymbol{\mu}) & \longrightarrow & H^1(F_n, \mathbf{T}) & \longrightarrow & H^1(F_n, \mathbf{T}) & \longrightarrow & H^1(F_n, \boldsymbol{\mu}) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^2(R_n, \mathbf{T}) & \longrightarrow & H^2(R_n, \mathbf{H}) & \longrightarrow & H^2(R_n, \boldsymbol{\mu}) \to \cdots \\
& & \downarrow\wr & & \downarrow & & \downarrow\wr \\
\cdots & \longrightarrow & H^2(F_n, \mathbf{T}) & \longrightarrow & H^2(F_n, \mathbf{H}) & \longrightarrow & H^2(F_n, \boldsymbol{\mu}) \to \cdots
\end{array}$$

The five lemma implies that the maps $H^q(R_n, \mathbf{H}) \to H^q(F_n, \mathbf{H})$ are isomorphisms for $q \geq 1$.

If $k = \overline{k}$, we have $cd(F_n) = n$, which implies that $H^q(F_n, \mathbf{G}_m) = 0$ for $q \geq n + 1$ ([42], III.2.3, prop. 3). Then $H^q(F_n, \mathbf{E}) = 0$ for $q \geq n + 1$ and any induced R_n-torus $\mathbf{E}$. Given an R_n-torus $\mathbf{T}$, the resolution $1 \to \boldsymbol{\mu} \to \mathbf{E}_1 \to \mathbf{T}^m \times_{R_n} \mathbf{E}_2 \to 1$ above yields the exact sequence

$$H^q(F_n, \mathbf{E}_1) \to H^q(F_n, \mathbf{T})^m \oplus H^q(F_n, \mathbf{E}_2) \to H^{q+1}(F_n, \boldsymbol{\mu}).$$

For $q \geq n + 1$, we have $H^q(F_n, \mathbf{E}_1) = 0$ and $H^{q+1}(F_n, \boldsymbol{\mu}) = 0$, hence $H^q(F_n, \mathbf{T}) = 0$. For the general case, we again fit $\mathbf{H}$ into an exact sequence $1 \to \mathbf{T} \to \mathbf{H} \to \boldsymbol{\mu} \to 1$ as above, and consider the resulting exact sequence

$$H^q(F_n, \mathbf{T}) \to H^q(F_n, \mathbf{H}) \to H^q(F_n, \boldsymbol{\mu}).$$

For $q \geq n + 1$, we have $H^q(F_n, \mathbf{T}) = H^q(F_n, \boldsymbol{\mu}) = 0$, hence $H^q(F_n, \mathbf{T}) = 0$. $\square$

4. Azumaya algebras

4.1. Generalities

We follow Grothendieck [22,23]. Let R be a commutative ring and let n be a positive integer. An Azumaya algebra over R of degree n is a form for the $fppf$-topology of the matrix algebra M_n. By descent theory, the isomorphism classes of such Azumaya algebras are classified by $H^1_{fppf}(R, \mathbf{PGL}_n)$. This means that, given such an algebra A, there exists a $fppf$-covering S/R, and a cocycle $z \in \mathbf{PGL}_n(S \otimes_R S)$, such that $A \cong {}_z(M_n)$ is the twist of M_n by z, i.e.,

$$A \cong \left\{x \in M_n(S) \mid z.p_2^*(x) = p_1^*(x) \in M_n(S \otimes_R S)\right\}.$$

The exact sequence $1 \to \mathbf{G}_m \to \mathbf{GL}_n \xrightarrow{p} \mathbf{PGL}_n \to 1$ induces the sequence of pointed sets

$$\mathrm{Pic}(R) \to H^1_{fppf}(R, \mathbf{GL}_n) \to H^1_{fppf}(R, \mathbf{PGL}_n) \xrightarrow{\delta} H^2_{fppf}(R, \mathbf{G}_m) = \mathrm{Br}(R). \tag{4.1}$$

We denote again by $[A] \in \mathrm{Br}(R)$ the class of $\delta([A])$ in the cohomological Brauer group. To such an algebra, we can associate the group scheme $\mathbf{GL}_1(A)$ of invertible elements of A, i.e. defined by

$$\mathbf{GL}_1(A)(S) = (A \otimes_R S)^\times \quad \text{for all } R\text{-algebras } S/R.$$

By descent theory $\mathbf{GL}_1(A)$ is affine. We have a canonical embedding $\mathbf{G}_m \to \mathbf{GL}_1(A)$ whose cokernel $\mathbf{PGL}_1(A)$ is representable (again by descent theory). Indeed, if $A = {}_z(M_n)$, $\mathbf{PGL}_1(A)$ is nothing but ${}_z\mathbf{PGL}_n$, i.e. the twist of $\mathbf{PGL}_n$ by z. Similarly $\mathbf{GL}_1(A) = {}_z\mathbf{GL}_n$ [23, Section 7].

Given a positive integer n' and an Azumaya algebra A' of degree n', the algebra $A \otimes_R A'$ is a form of the matrix algebra $M_{nn'} \cong M_n \otimes M_{n'}$, hence an Azumaya algebra of degree nn'. The isomorphism $M_{nn'} \cong M_n \otimes M_{n'}$ gives rise to the following homomorphisms

$$\rho_{n,n'} : \mathbf{GL}_n \times \mathbf{GL}_{n'} \to \mathbf{GL}_{nn'}, \quad \overline{\rho}_{n,n'} : \mathbf{PGL}_n \times \mathbf{PGL}_{n'} \to \mathbf{PGL}_{nn'}.$$

If $A = {}_zM_n$ and $A' = {}_{z'}M_{n'}$, we have $A \otimes_R A' =_{\rho_{n,n'}(z,z')}(M_{nn'})$. Moreover, we have $[A \otimes_R A'] = [A]+[A'] \in \mathrm{Br}(R)$ [21, V.4.6, V.4.3.(1)].

We now look at the special case $M_r(A) = M_r \otimes_R A$ where r is a positive integer. It is convenient to set $\mathbf{GL}_r(A) = \mathbf{GL}_1(M_r(A))$ and $\mathbf{PGL}_r(A) = \mathbf{PGL}_1(M_r(A))$.

The homomorphism $\rho_{r,n} : \mathbf{GL}_r \times \mathbf{GL}_n \to \mathbf{GL}_{rn}$ induces a homomorphism $\lambda_{n,rn} = \rho_{r,n}(? \otimes id) : \mathbf{GL}_n \to \mathbf{GL}_{dn}$ and $\lambda_{n,rn} = \mathbf{PGL}_n \to \mathbf{PGL}_{rn}$. Then the induced map $\lambda_{r,rn,*} : H^1_{fppf}(R, \mathbf{PGL}_n) \to H^1_{fppf}(R, \mathbf{PGL}_{rn})$ sends $[A]$ to $[M_r(A)]$.

4.2. *Locally free modules*

Recall that the set $H^1_{fppf}(R, \mathbf{GL}_n)$ classifies isomorphism classes of projective R-modules of rank n. Given such a module P, we can associate the Azumaya R-algebra $\mathrm{End}_R(P)$. In the exact sequence (4.1) we have $p_*([P]) = [\mathrm{End}_R(P)] \in H^1_{fppf}(R, \mathbf{PGL}_n)$ [21, V.4.5]. Our goal is to provide a similar description for $H^1_{fppf}(R, \mathbf{GL}_r(A))$. Since $\mathbf{GL}_r(A)$ is the automorphism group of the right A-module A^r, descent theory shows that $H^1_{fppf}(R, \mathbf{GL}_r(A))$ classifies the right A-modules P such that

(1) P is locally isomorphic to A^r for the $fppf$ topology, namely there exists S/R faithfully flat of finite presentation such that $P \otimes_R S \cong (A \otimes_R S)^r$ as $A \otimes_R S$ right modules.

By taking a covering S/R which trivializes also A, we see that (1) is equivalent to

(2) there exists S/R faithfully flat of finite presentation such that $(A \otimes_R S, P \otimes_R S) \cong (M_n(S), M_n(S)^r)$.

Remark 4.1. Let P be a right A-module satisfying the equivalent conditions (1) and (2) above. By faithfully flat descent, as an R-module, hence also as an A-module, P is finitely presented. Again, by faithfully flat descent, it follows that P is a projective A-module (see [26, III lemma 5.1.8]).

The main result is the following

Theorem 4.2 (*Hilbert–Grothendieck 90*).

$$H^1_{\mathrm{Zar}}(R, \mathbf{GL}_r(A)) \cong H^1_{\acute{e}t}(R, \mathbf{GL}_r(A)) \cong H^1_{fppf}(R, \mathbf{GL}_r(A)).$$

Proof. Assume first that R is local. By a result of DeMeyer ([12], see also [26, lemma 5.2.2]), all projective A-module of finite type having the same rank as R-modules are isomorphic. This, together with the previous remark, shows that $H^1_{fppf}(R, \mathbf{GL}_r(A)) = 1$. Thus, the theorem holds in the local case.

In general, we are given a $\mathbf{GL}_r(A)$-torsor E/R and we have to show that it is locally split for the Zariski topology. Let $\mathfrak{m}$ be a prime ideal of R. Since $R_\mathfrak{m} = \varinjlim_{f \notin \mathfrak{m}} R_f$, we have

$$\varinjlim_{f} H^1_{fppf}(R_f, \mathbf{GL}_r(A)) \cong H^1_{fppf}(R_\mathfrak{m}, \mathbf{GL}_r(A)) = 1$$

by Proposition A.4 of the Appendix. Hence there exists $f \notin \mathfrak{m}$ such that $E \times_R R_f \cong \mathbf{GL}_r(A) \times_R R_f$. Thus E is locally trivial for the Zariski topology on $\mathrm{Spec}(R)$. $\square$

Hence $H^1_{fppf}(R, \mathbf{GL}_r(A))$ classifies right A-modules P satisfying the following condition:

(3) there exists a Zariski covering $(S_i)_{i=1,\ldots,l}$ of R such that $P \otimes_R S_i \cong (A \otimes_R S_i)^r$ as $A \otimes_R S_i$-right modules for $i = 1, \ldots, l$.

The A-modules satisfying the equivalent properties (1), (2) and (3) are called locally free right A-modules[2] of relative rank r.

Lemma 4.3. *Assume that* $A = {}_zM_n$.

[2] These modules are often called projective (right) A-modules of rank r.

1. *The diagram*

$$\begin{array}{ccccc} H^1(R, \mathbf{GL}_r(A)) & \xrightarrow{p_*} & H^1(R, \mathbf{PGL}_r(A)) & \longrightarrow & \mathrm{Br}(R) \\ & & \theta_{\rho(z)} \downarrow \wr & & \downarrow ?-\delta([A]) \\ & & H^1(R, \mathbf{PGL}_{nr}) & \longrightarrow & \mathrm{Br}(R). \end{array}$$

commutes and is exact, where θ stands for the torsion map ([13], III.4.3.4). Furthermore

$$\theta_{\rho(z)}\,(p_*([P])) = [\mathrm{End}_A(P)]$$

for any locally free right A-module P of relative rank r.

2. *Let P and P' be locally free right A-modules of relative rank r. Then $\mathrm{End}_A(P) \cong \mathrm{End}_A(P')$ if and only if $P \cong \mathcal{L} \otimes_R P'$ for some line bundle $\mathcal{L}/R$.*
3. *Assume $\mathrm{Pic}(R) = 0$. Then $\mathrm{End}_A(P) \cong \mathrm{End}_A(P')$ if and only if $P \cong P'$.*

Proof. (1) For the commutativity and the exactness of the diagram, see [21, IV.4.3.4]. We proceed to the proof of the formula. Condition (2) above provides a covering S/R such that $A \otimes_R S \cong M_n(S)$ and $P \otimes_A (A \otimes_R S) \cong M_n(S)^d$. Let $z \in \mathbf{PGL}_n(S \otimes_R S)$ (resp. $f \in \mathbf{GL}_r(A \otimes_R (S \otimes_R S))$) be a cocycle such that $A \cong {}_zM_n$ (resp. $P = {}_f(A^r)$). We have

$$\mathrm{End}_A(P) \cong {}_f\mathrm{End}_A(A^r) = {}_f({}_{\rho(z)}M_{rn}) = {}_{\theta_{\rho(z)(f)}}M_{rn},$$

hence the desired formula $\theta_{\rho(z)}(p_*([P])) = [\mathrm{End}_A(P)]$.
(2) This is [25, prop. 2.2].
(3) Follows from (2). □

By looking at the boundary map, we recover the following fact:

Lemma 4.4. *Let B be an Azumaya algebra of degree rn. Then A, B have the same class in the Brauer group if and only if there exists a locally free right A-module P of relative rank r such that $B \cong \mathrm{End}_A(P)$.*

In particular, if $r = 1$, an Azumaya algebras A' of degree n has the same Brauer class than A if and only if $A' \cong \mathrm{End}_A(\mathcal{L})$ where $\mathcal{L}$ locally free right A-module of relative rank 1.

Proof. We may assume that $A =_z M_n$ for some $fppf$ cocycle $z \in \mathbf{PGL}_n(S \otimes_R S)$. By Lemma 4.3.1, we have the following diagram of pointed sets

$$\begin{array}{ccccc} & & H^1_{fppf}(R, \mathbf{PGL}_{rn}) & \longrightarrow & H^2_{fppf}(R, \mathbf{G}_m) \\ & & \theta_z \uparrow \wr & & \uparrow \wr\, ?+[A] \\ H^1_{fppf}(R, \mathbf{GL}_r(A)) & \xrightarrow{{}_zp} & H^1_{fppf}(R, \mathbf{PGL}_r(A)) & \longrightarrow & H^2_{fppf}(R, \mathbf{G}_m) \end{array}$$

If $[A] = [B] \in H^2_{fppf}(R, \mathbf{G}_m)$, a diagram chase shows that $\theta_z([B]) = {}_zp([P])$ for some locally free right A-module P of relative rank r. Lemma 4.3.1 shows that $[B] = [\mathrm{End}_A(P)] \in H^1_{fppf}(R, \mathbf{PGL}_n)$. The converse is obvious. □

Example 4.5 (*Symbol Algebras*). Let n be an integer and assume that our base field k is equipped with a choice of a primitive n-root of unity ζ_n. Assume that R is a k-algebra. Given units $u, v \in R^\times$ and an integer m we define the *symbol algebra* $(u, v)_n^m$ to be the (associative unital) R-algebra generated by the two symbols X, Y subject to the relations

$$X^n = u, \qquad Y^n = v^m, \qquad XY = \zeta_n Y X.$$

It is well known that $(u, v)_n^m$ is an R-Azumaya algebra of degree n.

4.3. Central simple algebras over F_n

Henceforth we assume that k is algebraically closed. The field F_n is of significant interest for the theory of central simple algebras, and has been investigated in detail by Amitsur [1], Tignol [46] and Tignol–Wadworth [47] among others. Saltman has proved that central division F_n-algebras are always tensor products of cyclic algebras [40]. In this paper, we shall use an effective version of this fact pointed out by Brussel [8] using Tignol's "armatures". Since $\mathrm{Br}(F_n) = H^2(F_n, \mathbf{G}_m)$, Proposition 3.1(2) reads

$$\mathrm{Br}(R_n) \cong \mathrm{Br}(F_n) \cong \mathrm{Br}(k) \oplus H^1(k, \mathbb{Q}/\mathbb{Z})^n \oplus H^0(k, \mathbb{Q}/\mathbb{Z}(-1))^{\binom{n}{2}}.$$

Since $\mathrm{Br}(k) = 0$ and $H^1(k, \mathbb{Q}/\mathbb{Z}) = 0$, we have an isomorphism

$$\mathrm{Br}(R_n) \cong \mathrm{Br}(F_n) \cong (\mathbb{Q}/\mathbb{Z})^{\binom{n}{2}} = \bigoplus_{1 \le i < j \le n} \mathbb{Q}/\mathbb{Z}. \tag{4.2}$$

Note that we recover a result of Magid for the Brauer group of the torus $\mathrm{Spec}(R_n)$ [28]. By mean of our choice of roots of unity we identify $\mathbb{Q}/\mathbb{Z} \cong (\mathbb{Q}/\mathbb{Z})(-1)$. Recall that an element $\frac{p}{q} \in \mathbb{Q}/\mathbb{Z}$ is in *reduced form* if p, q are coprime and $1 \le p \le q$. It is easy to identify F_n-algebras representing given classes in the Brauer group.

Lemma 4.6. *Let* $\left(\frac{p_{i,j}}{q_{i,j}}\right)_{1 \le i < j \le n}$ *be a sequence of elements of* $\mathbb{Q}/\mathbb{Z}$ *in reduced form. Then the Brauer class of the* R_n*-Azumaya algebra*

$$\bigotimes_{1 \le i < j \le n} (t_i, t_j)_{q_{i,j}}^{p_{i,j}}$$

is $\left(\frac{p_{i,j}}{q_{i,j}}\right)_{1 \le i < j \le n}$.

Proof. By additivity, the formula is nothing but [8, (1.2)]. $\square$

The tensor products of the last lemma are not in general minimal in terms of dimension for representing classes in the Brauer group, and we next turn our attention to this problem. Consider the group $\mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z})$ of alternate degree n matrices with values in $\mathbb{Q}/\mathbb{Z}$ (i.e. antisymmetric and zero diagonal). We have $\mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z}) = \mathrm{Alt}_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ and we denote by $(e_{i,j})$ the canonical basis of $M_n(\mathbb{Z})$. The composition of the isomorphism (4.2) with the isomorphism

$$\begin{array}{ccc} \bigoplus_{1 \le i < j \le n} \mathbb{Q}/\mathbb{Z} & \xrightarrow{\sim} & \mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z}) \\ (a_{i,j}) & \mapsto & \sum_{1 \le i < j \le n} a_{i,j}\,(e_{i,j} - e_{j,i}) \end{array}$$

defines the isomorphism

$$\mathrm{Inv} \colon \mathrm{Br}(R_n) \cong \mathrm{Br}(F_n) \cong \mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z}). \tag{4.3}$$

Theorem 4.7 (*Brussel, Theorem 2 of [8]*). *Let $\mathcal{A}$ be an element of* $\mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z})$. *Define $\rho(\mathcal{A})$ to be the $\mathcal{A}$'s row space, namely the finite subgroup of $(\mathbb{Q}/\mathbb{Z})^n$ generated by the rows of $\mathcal{A}$, and let $|\rho(\mathcal{A})|$ denote its order.*

1. *If A is central simple F_n-algebra of Brauer invariant* $\mathrm{Inv}(A) = \mathcal{A}$, *then* $\mathrm{Ind}(A) = \sqrt{|\, \rho(\mathcal{A})\, |}$.
2. *There exists $g \in \mathbf{GL}_n(\mathbb{Z})$, a non-negative integer $l \le [\frac{n}{2}]$ and $(r_i/s_i)_{i=1,\dots,l} \in \mathbb{Q}/\mathbb{Z}$ such that the central F_n-algebra*

$$\bigotimes_{i=1}^{l} \left(g^*(t_{2i-1}), g^*(t_{2i})\right)_{s_i}^{r_i}$$

is a division algebra with Brauer invariant $\mathcal{A}$.

The notation $g^*(t_i)$ means the base change for the natural action of $\mathbf{GL}_n(\mathbb{Z})$ on R_n. We call the algebra above a Brussel representant of $\mathcal{A}$; it is not unique. Note that l and the set $\{r_1/s_1, \dots, r_l/s_l\}$ are well determined, in particular $l \prod_i s_i = \sqrt{|\, \rho(\mathcal{A})\, |}$.

4.4. The case of Laurent polynomials

We can pass now to Azumaya algebras over R_n by using the isomorphism $\mathrm{Br}(R_n) \cong \mathrm{Br}(F_n) \cong \mathrm{Alt}^n(\mathbb{Q}/\mathbb{Z})$ (Proposition 3.1 and (4.3)) and the fact that Brussel representants arise from R_n-Azumaya algebras. Moreover the isomorphism

$$\mathrm{Br}(R_n) \cong \mathrm{Alt}^n(\mathbb{Q}/\mathbb{Z}).$$

is equivariant with respect to the natural action of $\mathbf{GL}_n(\mathbb{Z})$ on R_n.

Proposition 4.8. *Let $\mathcal{A}$ be an element of $\mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z})$. Let $g \in \mathbf{GL}_n(\mathbb{Z})$, l, (r_i/s_i) be as in Theorem 4.7, and define the R_n-Azumaya algebra*

$$A_0/R_n := \bigotimes_{i=1}^{l} (t_{2i-1}, t_{2i})_{s_i}^{r_i}.$$

1. *The Azumaya algebra $g^*(A_0) = \bigotimes_{i=1}^{l} (g^*(t_{2i-1}), g^*(t_{2i}))_{s_i}^{r_i}$ is of minimal degree among the R_n-Azumaya algebras of Brauer invariant $\mathcal{A}$.*
2. *Let A be a R_n-Azumaya algebra with Brauer invariant $\mathcal{A}$. Then*
   (a) $\deg(A_0)$ *divides* $\deg(A)$*;*
   (b) *If $r = \deg(A)/\deg(A_0)$, there exists a locally free right A_0-module P_0 of relative rank r, unique up to isomorphism, such that $A \cong g^*\big(\mathrm{End}_{A_0}(P_0)\big)$.*

Proof. From the $\mathbf{GL}_n(\mathbb{Z})$-equivariance of $\mathrm{Br}(R_n) \cong \mathrm{Alt}^n(\mathbb{Q}/\mathbb{Z})$, we may assume, without loss of generality, that $g = 1$.
(1) Theorem 4.7(2) states that the algebra $A_0 \otimes_{R_n} F_n$ is division. By Wedderburn's theorem, $A_0 \otimes_{R_n} F_n$ is of minimal degree among the F_n-algebras which are Brauer equivalent to $A_0 \otimes_{R_n} F_n$. A fortiori, A_0 is of minimal degree among the R_n-Azumaya algebras which Brauer equivalent to A_0.
(2) Given an R_n-Azumaya algebra A which is Brauer equivalent to A_0, Wedderburn's theorem states that $A \otimes_{R_n} F_n$ is a matrix algebra, say of size r, with coefficients in the division algebra $A_0 \otimes_{R_n} F_n$. Thus $\deg(A_0)$ divides $\deg(A)$. Lemma 4.4 shows that $A \cong \mathrm{End}_{A_0}(P_0)$ for some locally free right A_0-module P_0 of relative rank r. Since $\mathrm{Pic}(R_n) = 0$, the unicity of P_0 follows from Lemma 4.3(3). □

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Appendix. Generalities on group schemes and torsors

A.1. Reductive and semisimple group schemes

Let X be a scheme. Following [44], we define a reductive (resp. semisimple, resp. semisimple simply connected, resp. semisimple adjoint) X-group, to be a smooth affine group scheme $\mathbf{G}$ over X for which all geometric fibers of $\mathbf{G}$ are connected reductive (resp. semisimple, resp. semisimple simply connected, resp. semisimple adjoint) linear algebraic groups in the sense of [5].

Let $\mathbf{G}$ be a reductive X-group. Assume X is connected. The main result of [44] states that $\mathbf{G}$ is locally isomorphic for the étale topology, to a unique Chevalley reductive $\mathbb{Z}$-group[3] $\mathbf{G}_0$. If $\mathbf{G}$ is isomorphic to $\mathbf{G}_0 \times_{\mathbb{Z}} X$, we say that $\mathbf{G}$ is split. The Dynkin diagram of $\mathbf{G}_0$ then defines the type of $\mathbf{G}$. The construction of simply connected coverings and adjoint groups can be done fiberwise ([44], Section XXII.4.3, XXIV.5) : if $\mathbf{G}$ is a semisimple X-group, then there exists a unique (up to unique isomomorphism) isogeny $\lambda : \widetilde{\mathbf{G}} \to \mathbf{G}$ of semisimple S-groups with $\widetilde{\mathbf{G}}/S$ simply connected. Similarly there exists unique isogeny $\mathbf{G} \to \mathbf{G}^{ad}$ of semisimple S-groups with $\mathbf{G}^{ad}$ is adjoint. The X-group scheme $\widetilde{\mathbf{G}}$ (resp. $\mathbf{G}_{ad}$) is the simply connected covering (resp. the adjoint group) of $\mathbf{G}$.

[3] Strictly speaking, a $\mathrm{Spec}(\mathbb{Z})$-group.

A.2. Torsors as quotient spaces

Definition A.1. Let X be scheme and $\mathbf{G}$ a group scheme over X. Let E be a scheme over X, and $E \times_X \mathbf{G} \to E$ an action. The scheme E/X is an homogeneous space under $\mathbf{G}$ (resp. is a $\mathbf{G}$-torsor) if the following conditions hold:

(a) The morphism $E \to X$ is a covering morphism for the $fppf$-topology.

(b) The morphism $E \times_X \mathbf{G} \to E \times_X E$, $(x, g) \mapsto (x, xg)$ is a covering morphism for the $fppf$-topology (resp. is an isomorphism).

A $\mathbf{G}$-torsor X is locally trivial (resp. étale locally trivial) if it admits a trivialization by an open Zariski (resp. étale) covering. If $\mathbf{G}$ is smooth, any $\mathbf{G}$-torsor is étale locally trivial (cf. [44], Exp. XXIV). If $\mathbf{G}/X$ is affine, flat and locally of finite presentation, $\mathbf{G}$-torsors over X are classified by the pointed set of cohomology $H^1_{fppf}(X, \mathbf{G})$ defined by means of cocycles à la Čech.

The next result provides plenty of examples of $\mathbf{G}$-torsors.

Theorem A.2 (*Raynaud–Seshadri, See [10], Proposition 6.11*). *Let X be a noetherian scheme and $\mathbf{G}$ a reductive X-group. Let E be an affine scheme of finite type over X, and $E \times_X \mathbf{G} \to E$ an action. Assume that*

(1) *the action is free, i.e. the map $E \times_X \mathbf{G} \to E \times_X E$ is a closed immersion,*

(2) *the action is linearizable, i.e. there exists a vector bundle $\mathcal{E}/X$ containing E/X as a closed subscheme, such that the action extends to a linear action on $\mathcal{E}/X$.*

Then the étale quotient sheaf $E/\mathbf{G}$ is representable by an affine scheme of finite type. Moreover, the canonical morphism $E \to E/\mathbf{G}$ is a $\mathbf{G}$-torsor which is étale locally trivial.

In the case of fields, the second hypothesis can be dropped.

Theorem A.3 (*Luna, [27]*). *Let $\mathbf{G}$ be a linearly reductive algebraic group over k. Let X be an affine variety over k, and $X \times_k \mathbf{G} \to X$ a free action. Then the étale quotient sheaf $X/\mathbf{G}$ is representable by an affine variety of finite type. Moreover, the canonical morphism $X \to X/\mathbf{G}$ is a $\mathbf{G}$-torsor which is étale locally trivial.*

A.3. Passage to the limit

The following is a special case of Grothendieck–Margaux's result (see [45] VII 5.7 and [29]) about passage to the limit in non-abelian Čech cohomology.

Proposition A.4. *Let R be a ring, and let $R_\infty = \varinjlim R_\lambda$ be a direct limit in R-alg based on some directed set Λ. Let $\mathbf{G}$ be a group scheme over* $\mathrm{Spec}(R)$ *which is locally of finite presentation, and set $\mathbf{G}_\lambda = \mathbf{G} \times_R R_\lambda$ and $\mathbf{G}_\infty \times_R R_\infty$. The canonical map*

$$\varinjlim H^1_{fppf}(R_\lambda, \mathbf{G}_\lambda) \to H^1_{fppf}(R_\infty, \mathbf{G}_\infty)$$

is bijective. Similarly for $H^1_{\acute{e}t}$ and H^1_{Zar}.

A.4. Patching torsors in codimension 1 and 2

Given a group scheme $\mathbf{G}$ over X, and a dense open subset U of X, it is natural to ask whether a $\mathbf{G}$-torsor over U extends to X or not. This is somehow a local question.

Lemma A.5. *Let X be scheme and $\mathbf{G}$ an group scheme over X which is affine, flat, and locally of finite presentation. Let $X = U \cup V$ be a cover by Zariski open subsets. Then the following commutative diagram of pointed sets*

$$\begin{array}{ccc} H^1_{fppf}(X, \mathbf{G}) & \longrightarrow & H^1_{fppf}(U, \mathbf{G}) \\ \downarrow & & \downarrow \\ H^1_{fppf}(V, \mathbf{G}) & \longrightarrow & H^1_{fppf}(U \cap V, \mathbf{G}) \end{array}$$

is cartesian.

Proof. We use that $H^1_{fppf}(X, \mathbf{G})$ classifies $\mathbf{G}$-torsors over X. We are given classes $\alpha = [E] \in H^1_{fppf}(U, \mathbf{G})$ and $\beta = [F] \in H^1_{fppf}(U, \mathbf{G})$ such that $\alpha_{|U \cap V} = \beta_{|U \cap V} \in H^1_{fppf}(U \cap V, \mathbf{G})$. Then there exists an isomorphism of $\mathbf{G}$-torsors $\phi : E/U \cap V \cong F/U \cap V$. By glueing E and F along $U \cap V$ by ϕ, we get a $\mathbf{G}$-torsor P on X which extends E and F. So its class extends α and β as desired. $\square$

In codimension one, the obstruction is of infinitesimal nature. The following generalizes a result by Harder [24], lemma 4.1.3 (where $\mathbf{G}_K$ is assumed reductive).

Proposition A.6. *Let A be a discrete valuation ring and K its fraction field. Denote by $\widehat{A}$ and $\widehat{K}$ the completions of A and K respectively. Let $\mathbf{G}$ be a group scheme which is affine, flat and locally of finite presentation over A. Then the diagram of pointed sets*

$$\begin{array}{ccc} H^1_{fppf}(A, \mathbf{G}) & \longrightarrow & H^1_{fppf}(\widehat{A}, \mathbf{G}) \\ \downarrow & & \downarrow \\ H^1_{fppf}(K, \mathbf{G}) & \longrightarrow & H^1_{fppf}(\widehat{K}, \mathbf{G}). \end{array}$$

is cartesian.

Proof. We are given classes $\alpha = [\widehat{E}] \in H^1_{fppf}(\widehat{A}, \mathbf{G})$ and $\beta = [F] \in H^1_{fppf}(K, \mathbf{G})$ mapping to the same class in $H^1_{fppf}(\widehat{K}, \mathbf{G})$. So $\widehat{E}/A$ and F/K lead (after taking the appropriate base change) to the same class in $H^1_{fppf}(\widehat{K}, \mathbf{G})$. Thus, there exists an isomorphism of $\mathbf{G}$-torsors $\phi : \widehat{E}/\widehat{K} \cong F/\widehat{K}$. Since $\widehat{E}$ (resp. F) is affine over $\widehat{A}$ (resp. K), a result of descent theory ([4], Section 6, Proposition D4) shows that the data $(F, \widehat{E}, \phi)$ is equivalent to the data of an affine scheme E over A such that $E \otimes_A \widehat{A} \cong \widehat{E}$ and $E \otimes_A K \cong F$. Furthermore, E is canonically equipped with a right action of $\mathbf{G}$. We claim that E/A is a $\mathbf{G}$-torsor which extends $\widehat{E}$ and F. Since $\mathrm{Spec}(\widehat{A}) \sqcup \mathrm{Spec}(K)$ is an $fpqc$ covering of $\mathrm{Spec}(A)$, the morphism $E \times_A \mathbf{G} \to E \times_A E$ is an isomorphism. By descent theory, E is faithfully flat ([14] Cor. 2.2.13) and locally of finite type ([14] Prop. 2.7.1) over A. Thus E/A admits a trivialization for the $fppf$ topology, hence is a $\mathbf{G}$-torsor. By construction E extends both $\widehat{E}$ and F, hence its class extends both α and β as desired. $\square$

Remark A.7. For higher dimensional analogues, see [3] and [30].

Proposition A.4 enables us to state the following global result.

Corollary A.8. *Let X be a regular noetherian integral scheme of function field K. For $x \in X^{(1)}$, denote by $\widehat{O}_x$ the completion of the local ring $O_{X,x}$, and $\widehat{K}_x$ its fraction field. Let $\mathbf{G}$ be an affine flat group scheme locally of finite presentation over X. Let U be an open subscheme of X and $\gamma \in H^1_{fppf}(U, \mathbf{G})$. For any $x \in X^{(1)} \setminus U$, assume that*

$$\gamma_{\widehat{K}_x} \in \mathrm{Im}\left(H^1_{fppf}(\widehat{O}_x, \mathbf{G}) \to H^1_{fppf}(\widehat{K}_x, \mathbf{G})\right).$$

Then there exists an open subscheme $\widetilde{U} \subset X$ containing U such that $\mathrm{codim}_X(X \setminus \widetilde{U}) \geq 2$ and $\gamma \in \mathrm{Im}\left(H^1_{fppf}(\widetilde{U}, \mathbf{G}) \to H^1_{fppf}(U, \mathbf{G})\right)$.

Proof. Let $x_1, \ldots, x_n \in X$ be the points of codimension one of X outside U. By induction on n, we can assume that X has only one point x of codimension one outside U. Proposition A.6 shows that there exists $\alpha \in H^1_{fppf}(O_{X,x}, \mathbf{G})$ such that α and γ coincide over $U \times_X \mathrm{Spec}(O_{X,x}) = \mathrm{Spec}(K)$. Proposition A.4 shows that there exists an affine open neighbour V_x of x in X such that α comes from $\widetilde{\alpha} \in H^1_{fppf}(V_x, \mathbf{G})$. So $\widetilde{\alpha}$ and γ coincide at the generic point, therefore they coincide over some open dense affine subset W_0 of X (again by Proposition A.4). We claim that there exists an open neighbourhood W of x such that $U \cap W \subset W_0$. Consider the decomposition in irreducible components of $X \setminus W_0 = \overline{\{x\}} \cup Z_1 \cup \cdots \cup Z_n$. Since x is of codimension one, x does not belong to Z_i for $i = 1, \ldots n$. Hence $W := X \setminus Z_1 \cup \cdots \cup Z_n$ is an open subset of X which contains x and satisfies $U \cap W \subset W_0$. Up to replacing W by $W \cap V_x$, we may assume that $W \subset V_x$. Lemma A.5 implies that we can patch the classes γ and $\widetilde{\alpha}_W$ along $U \cap W$; in other words, $\gamma \in \mathrm{Im}\left(H^1_{fppf}(U \cup W, \mathbf{G}) \to H^1_{fppf}(U, \mathbf{G})\right)$. Since $\widetilde{U} := U \cup W$ contains all points of codimension one of X, the proof is complete. $\square$

Remark A.9. This last statement is quite useful since one can control the sets $H^1(\widehat{K}_x, \mathbf{G})$ by Bruhat–Tits theory [7].

The following result states that torsors over punctured surfaces extend uniquely.

Theorem A.10 (*[10], théorème 6.13*). *Let X be a regular integral scheme of dimension 2. Let U be an open subset of X such that $\operatorname{codim}_X(X \setminus U) = 2$. Let $\mathbf{G}$ be a reductive X-group. Then the canonical map $H^1_{fppf}(U, \mathbf{G}) \to H^1_{fppf}(X, \mathbf{G})$ is bijective.*

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