Ruin probabilities in models with a Markov chain dependence structure

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Abstract

In this paper we derive explicit expressions for the probability of ruin in a renewal risk model with dependence among the increments $(Z_k)_{k>0}$ among claims. We study the case where the dependence structure among $(Z_k)_{k>0}$ is driven by a Markov chain with a transition kernel that can be described via ordinary differential equations with constant coefficients.

Keywords: Ruin probability, dependence, Markov chain, random walk, rational Laplace transform, ordinary differential equation with constant coefficients.

1 Introduction

Consider the classical collective risk model describing the evolution of the surplus of an insurance company

$$U(t) = u + ct - \sum_{k=1}^{N_t} X_k, \tag{1}$$

where u is the initial surplus, c represents the premium rate, X_k is the k-th claim amount and N_t represents the number of claims that occurred up to time t (see e.g. Asmussen and Albrecher (2010)). Immediately after the k-th claim, for $u \geq 0$, the surplus is

$$U(T_k) \quad = \quad u + cT_k - \sum_{i=1}^{k} X_i = u + c\sum_{i=1}^{k} \tau_i - \sum_{i=1}^{k} X_i = u - \sum_{i=1}^{k} Z_i.$$

The event of the surplus falling below zero for the first time is called ruin. Obviously, this can occur only at the time of a claim T_k. The probability of ruin to happen, in infinite time, is defined as

$$\psi(u) \quad = \quad \mathbb{P}(\sup_{k\geq 1} \sum_{i=1}^{k} Z_i > u).$$

While most classical models assume independence between inter-arrival times and claim amounts, we consider a special dependence between the positive random variable τ_k representing the $k-$th inter-arrival time and the subsequent claim size denoted by the positive random variable X_k. More specifically, we consider the *real-valued* random variable of the $k-$th increment

$$Z_k = X_k - c\tau_k, \quad k = 1, 2, \dots \tag{2}$$

where c represents the premium rate and $\mathbb{E}\{Z\} < 0$. Here X_k and τ_k can have any dependence structure. We are interested in providing explicit expressions for the probability of ruin.

The sum of increments

$$W_n = \sum_{k=1}^{n} Z_k$$

represents the loss of the company after n claims. When the increments Z_k are independent, W_n is a random walk. For an initial capital amount $u > 0$, the company is considered to be insolvent if $W_n > u$, and the probability $\psi(u)$ of this event is the ruin probability. W.l.o.g., in this paper $\mathbb{P}(Z = 0) = 0$.

By characterizing the dependence in renewal risk models via the series of identical distributed losses $(Z_k)_{k>0}$, one has two possible scenarios:

S1. $Z_1, Z_2, \dots$ are independent (random walk structure)

S2. $Z_1, Z_2, \dots$ are not independent (with multiple possible dependence structures among Z_ks).

Since in risk theory literature several cases preserving $(Z_k)_{k>0}$ independent (scenario S1) have already been addressed (see Albrecher and Teugels (2006), Ambagaspitiya (2009) or Badescu et al. (2009)), we will concentrate on a case pertaining to the second scenario. For a recent and more exhaustive survey of other dependence structures in risk theory see Asmussen and Albrecher (2010)[Chapter XIII].

The challenging part of choosing $(Z_k)_{k>0}$ to be dependent is the impossibility of relying on standard random walk theory. However, we show that one can still use the decomposition into positive and negative parts

$$Z = IZ^{+} + (1 - I)Z^{-}, \quad I \sim \text{Bernoulli}(p), \tag{3}$$

with $Z^{+} = \{Z \mid Z > 0\}$ and $Z^{-} = \{Z \mid Z < 0\}$, introduced in the random walk treatment, to find closed form solutions for the probability of ruin $\psi(u)$. Similarly to Albrecher et al. (2010), we derive exact forms via solving ordinary differential equations. The difference is that while in Albrecher et al. (2010) one exploits the independence of τ_k and X_k, here one focuses on the dependence structure of the losses Z_k, whereas the pair τ_k, X_k can have any dependence structure. Thus the necessary conditions to be imposed for calculating the ruin probability exactly are no longer on the densities of τ and X as in Albrecher et al. (2010), but on densities pertinent to Z or Z^{+} and Z^{-}.

In this paper we consider a dependence structure for the *dependent* losses $(Z_k)_{k>0}$ given by a Markov chain with a transition kernel $p_k(x,y)$ (starting at x, jumps from state x to state y at time k with probability p_k) that can be written as a product of two functions on each quadrant. The condition needed is that combinations of these functions further satisfy certain ODEs with constant coefficients (see Assumption 1). This condition is not intuitively evident. However, besides the mathematical amenability, the motivation of such a choice of dependent model is the following. The weight $p = \mathbb{P}(Z > 0)$ not being constant over time, but dependent on the previous step in a deterministic way, influences the tendency of the future behavior of the steps Z. Thus, by the choice of the mechanism to determine the next p, we can generate models where if one period was bad (we earn less than we spend i.e. $Z > 0$) it is more likely than in the independent case that the next period will also be bad.

To justify further our choice of such a Markov chain structure, we present as a limiting case, the case of *independent* $(Z_k)_{k>0}$, with densities of Z^+ and Z^- satisfying ODEs with constant coefficients. A further particular case of it is the case present in risk theory literature, of the density of Z satisfying an ODE with constant coefficients, as in Albrecher and Teugels (2006); Ambagaspitiya (2009); Badescu et al. (2009).

Similarly to Albrecher et al. (2010), where one transforms an integral equation for the probability of ruin into an ordinary differential equation, here we transform a *system* of integral equations into a *system* of ordinary differential equations that one can solve explicitly. This way we show that even under this dependence scenario, the probability of ruin has still a rational Laplace transform.

Thus, the classes of distributions for which one can identify the probability of ruin as having a rational Laplace transform are the following:

A. In the dependent case (S2): Markov chain structure on Z_k with a kernel satisfying certain ODE conditions, as in Propositions 1, 2 and 3.

B. In the independent case (S1): the classical result in random-walk theory of Z^+ with a rational Laplace transform, as in Proposition 4.

C. In the independent case (S1): the classical case in ruin theory literature of Z with rational moment generating function as in Proposition 5.

Note that for the distribution class C one can always find a dependence structure where the marginal distributions of τ and X have rational Laplace transform (see Lemma 1 in Appendix A). Equally natural, one can identify a dependence structure with marginals (of τ and X) not having a rational Laplace transform. For instance, one can have $\tau = Y_1 + Y_3$ and $X = Y_2 + Y_3$, where Y_1, Y_2 and Y_3 are independent, Y_1 and Y_2 have a rational Laplace transform, but Y_3 has not. Actually, Y_3 can have any positive distribution function, even heavy-tailed.

While the classes B and C are present in the applied probability literature, this paper brings forth class A and shows that as in the other two classes of distributions, the probability of ruin can be derived by means of ODEs with constant coefficients.

The paper is organized as follows. In Section 2 we analyze a risk model where the steps Z_k are dependent via a Markov chain. We consider a Markov chain structure with a transition kernel described by some ODEs with constant coefficients and show that the ruin probability has a rational Laplace transform. In Section 3, we present the case of independent steps Z_k as a limiting case of the Markov chain dependence from Section 2. We reformulate in ODE language some well-known results from random walk and risk literature. In Section 4, we look at a Markov chain dependent case with a transition kernel expressed through exponential densities, and for a numerical example we compare the values of probability of ruin under a Markov chain dependence structure with the one with independent increments. After the conclusions from Section 5, in the Appendix we present some additional lemmas and some deferred proofs.

Note regarding notation and terminology. Depending on relevance and purpose, we will switch back and forth between the following equivalent concepts:

- the density function f_X of the random variable X satisfies an ODE with constant coefficients
- the moment generating function (mgf) $M_X(s)$ of the random variable X is rational; or, equivalently, X has a rational Laplace transform.

These are both equivalent to matrix exponentials (Bladt and Nielsen, 2010).

2 $(Z_k)_{k\geq 1}$ dependent

In this section we study the case where $(Z_k)_{k\geq 0}$ are dependent and assume to have a Markov chain structure. For a given $Z_0 = x$, we define the probability of ruin by

$$\psi(u,x) = \mathbb{P}\left(\sup_{k\geq 1}\sum_{i=1}^{k} Z_i > u \mid Z_0 = x\right). \tag{4}$$

We denote the transition density of Z by

$$p(x,y)dx := \mathbb{P}(Z_{i+1} \in dy | Z_i = x) = \mathbb{P}\left(Z_{i+1} \in dy | Z_i = x, Z_{i-1} = x, \dots\right).$$

For $u \geq 0$, by conditioning on Z_1 one has

$$\begin{aligned}\psi(u,x) &= \mathbb{P}\left(Z_1 > u \mid Z_0 = x\right) + \mathbb{P}\left(Z_1 \leq u, \sup_{k\geq 1}\sum_{i=2}^{k} Z_i > u - Z_1 \mid Z_0 = x\right) \\ &= \mathbb{P}(Z_1 > u \mid Z_0 = x) + \int_{-\infty}^{u} \mathbb{P}\left(\sup_{k\geq 1}\sum_{i=2}^{k} Z_i > u - Z_1 \mid Z_0 = x, Z_1 = y\right) p(x,y)\, dy \\ &= \mathbb{P}(Z_1 > u \mid Z_0 = x) + \int_{-\infty}^{u} \mathbb{P}\left(\sup_{k\geq 1}\sum_{i=2}^{k} Z_i > u - Z_1 \mid Z_1 = y\right) p(x,y)\, dy \\ &= \int_{u}^{\infty} p(x,y)\, dy + \int_{-\infty}^{u} \psi(u-y,y) p(x,y)\, dy. \end{aligned} \tag{5}$$

To obtain a system of ODEs for the ruin probability we will assume a rather special structure for the transition density, namely:

Assumption 1. *Let*

$$p(x,y) = \begin{cases} g^1(x)h^1(y), & (x,y) \in I_1 \\ g^2(x)h^2(y), & (x,y) \in I_2 \\ g^3(x)h^3(y), & (x,y) \in I_3 \\ g^4(x)h^4(y), & (x,y) \in I_4, \end{cases} \tag{6}$$

where I_k denotes the $k-th$ quadrant of the cartesian plane. Further let $g^1(x) > 0$, $g^4(x) > 0$ for $x > 0$, $g^1(x) = g^4(x) = 0$ for $x < 0$, $g^1(x)$, $g^4(x)$ are linearly independent and $g^1(x) + g^4(x) = 1$. Similarly $g^2(x) > 0$, $g^3(x) > 0$ for $x < 0$, $g^2(x) = g^3(x) = 0$ for $x > 0$, $g^2(x), g^3(x)$ are linearly independent and $g^2(x) + g^3(x) = 1$. Further assume that for

$$(k,m) \in \{(1,1),(1,2),(4,1),(4,2),(2,3),(2,4),(3,3),(3,4)\}$$

there exists polynomials $q^{k,m}(x) := \sum_{i=0}^{n} q_i^{k,m} x^i$ with

$$q^{k,m}\left(\frac{d}{du}\right) g^k(u)h^m(u) = 0 \quad u \neq 0$$

An interpretation of this kind of dependence is given in Section 4.

Remark 1. *One can easily extend the results of this paper to a transition density of the form*

$$p(x,y) = \begin{cases} \sum_{i=1}^{n_1} g_i^1(x)h_i^1(y), & (x,y) \in I_1 \\ \sum_{i=1}^{n_2} g_i^2(x)h_i^2(y), & (x,y) \in I_2 \\ \sum_{i=1}^{n_3} g_i^3(x)h_i^3(y), & (x,y) \in I_3 \\ \sum_{i=1}^{n_4} g_i^4(x)h_i^4(y), & (x,y) \in I_4 \end{cases} \tag{7}$$

with the obvious extension of the conditions to $g_i^1(x)$ and $h_i^1(y)$. Nevertheless note that this generalization is straight-forward (one needs to replace all matrices by corresponding block matrices). Hence, for ease of notation we consider only the case $n = 1$.

In order to evaluate the ruin probability we first need to check that for a given set of parameters the process tends to $-\infty$. Therefore one needs to evaluate the invariant distribution $f_Z(x)$ of $p(x,y)$ (see Lemma 4 in the Appendix), although this is not needed further for the derivation of the ruin probability. To find explicit expressions for $\psi(u,x)$ a key observation is:

Proposition 1. *If Assumption 1 is fulfilled, then the probability of ruin has the form*

$$\psi(u,x) = \begin{cases} g^1(x)\psi^1(u) + g^4(x)\psi^4(u), & x > 0 \\ g^2(x)\psi^2(u) + g^3(x)\psi^3(u), & x < 0, \end{cases} \tag{8}$$

where $\psi^i = \int_u^\infty h^i(y)dy + \int_0^u \psi(u-y,y)h^i(y)dy$, for $i = 1,3$ and $\psi^i = \int_{-\infty}^0 \psi(u-y,y)h^4(y)dy$, for $i = 2,4$. Further, for $i = 1,\dots,4$,

$$\lim_{u\to\infty} \psi^i(u) = 0.$$

Remark 2. *Note that for $x > 0$, $\psi^1(u)$ is the probability of ruin given that the first step (jump) is positive and $\psi^4(u)$ is the ruin probability given that the first step is negative. Similarly, for $x < 0$, $\psi^2(u)$ is the probability of ruin given that the first step is positive and $\psi^3(u)$ is the ruin probability given that the first step is negative.*

Next we provide a system of ODEs for the functions ψ^i.

Proposition 2. *If Assumption 1 is fulfilled, then $\psi^i(u)$, $i = 1, \dots, 4$ are a solution of the system of ODEs*

$$A\tilde{\psi} = 0,$$

where $\tilde{\psi} = (\psi^1, \psi^2, \psi^3, \psi^4)^T$,

$$A = \begin{pmatrix} \frac{(r^{1,1}-q^{1,1})q^{4,1}}{gcd(q^{1,1},q^{4,1})} & 0 & 0 & \frac{q^{1,1}r^{4,1}}{gcd(q^{1,1},q^{4,1})} \\ \frac{r^{1,2}q^{4,2}}{gcd(q^{1,2},q^{4,2})} & -\frac{q^{1,2}q^{4,2}}{gcd(q^{1,2},q^{4,2})} & 0 & \frac{q^{1,2}r^{4,2}}{gcd(q^{1,2},q^{4,2})} \\ 0 & \frac{r^{2,3}q^{3,3}}{gcd(q^{2,3},q^{3,3})} & \frac{(r^{3,3}-q^{3,3})q^{2,3}}{gcd(q^{2,3},q^{3,3})} & 0 \\ 0 & \frac{r^{2,4}q^{3,4}}{gcd(q^{2,4},q^{3,4})} & \frac{q^{2,4}r^{3,4}}{gcd(q^{2,4},q^{3,4})} & -\frac{q^{2,4}q^{3,4}}{gcd(q^{2,4},q^{3,4})} \end{pmatrix},$$

$gcd(a, b)$ denotes the greatest common divisor of the polynomials a and b and $r^{k,m}$ is defined by

$$M_{k,m}(-s) := M_{g^k h^m}(-s) := \int_{-\infty}^{\infty} e^{-sx} g^k(x) h^m(x) dx = \frac{r^{k,m}(s)}{q^{k,m}(s)}. \tag{9}$$

Further if $h^1(x) = h^2(x)$ then $\psi^1(u) = \psi^2(u)$ and if $h^3(x) = h^4(x)$ then $\psi^3(u) = \psi^4(u)$.

Proof. Substituting (8) into the last two lines of (18) leads to

$$\begin{aligned} & g^1(x) \left(\int_u^{\infty} h^1(y)dy + \int_0^u \left(\psi^1(u-y)g^1(y) + \psi^4(u-y)g^4(y) \right) h^1(y)dy \right) \\ & \quad + g^4(x) \int_{-\infty}^0 \left(\psi^2(u-y)g^2(y) + \psi^3(u-y)g^3(y) \right) h^4(y)dy \\ & = g^1(x)\psi^1(u) + g^4(x)\psi^4(u) \end{aligned}$$

Since $g^1(x)$ and $g^4(x)$ are linearly independent we get the equations

$$\begin{cases} \psi^1(u) - \int_0^u \left(\psi^1(u-y)g^1(y) + \psi^4(u-y)g^4(y) \right) h^1(y)dy & = \int_u^{\infty} h^1(y)dy \\ \psi^4(u) - \int_{-\infty}^0 \left(\psi^2(u-y)g^2(y) + \psi^3(u-y)g^3(y) \right) h^4(y)dy & = 0 \end{cases}$$

Analogously, using the linear independence of $g^2(x)$ and $g^3(x)$ from the last two lines of equation (19) one has the system

$$\begin{cases} \psi^2(u) - \int_0^u \left(\psi^1(u-y)g^1(y) + \psi^4(u-y)g^4(y) \right) h^2(y)dy & = \int_u^{\infty} h^2(y)dy \\ \psi^3(u) - \int_{-\infty}^0 \left(\psi^2(u-y)g^2(y) + \psi^3(u-y)g^3(y) \right) h^3(y)dy & = 0 \end{cases}$$

In order to write this system of equations in matrix form, we introduce the operators

$$K_+^{l,m}\psi := \int_0^u \psi(u-y)g^l(y)h^m(y)dy \quad \text{and} \quad K_-^{l,m}\psi := \int_{-\infty}^0 \psi(u-y)g^l(y)h^m(y)dy$$

(note that g^l, h^m vanish either on the positive or negative half-line). Further, denoting by 1 the identity operator, the above system of equations can be written as

$$\begin{pmatrix} K_+^{1,1}-1 & 0 & 0 & K_+^{4,1} \\ K_+^{1,2} & -1 & 0 & K_+^{4,2} \\ 0 & K_-^{2,3} & K_-^{3,3}-1 & 0 \\ 0 & K_-^{2,4} & K_-^{3,4} & -1 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} -\int_u^\infty h^1(y)dy \\ -\int_u^\infty h^2(y)dy \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

If $h^1(x) = h^2(x)$ then $K_+^{1,1} = K_+^{1,2}$ and $K_+^{4,1} = K_+^{4,2}$, hence it follows by subtracting the second line from the first line that $\psi^1(u) = \psi^2(u)$. Similarly if $h^3(x) = h^4(x)$, then we get by subtracting the fourth line from the third line that $\psi^3(u) = \psi^4(u)$, Hence the second part of the proposition follows.
For the first part we want to reduce this system of IDEs to a system of ODEs that one can solve. It is enough to act on the integral system with a matrix of the form

$$\begin{pmatrix} \frac{q^{1,1}q^{4,1}}{gcd(q^{1,1},q^{4,1})} & 0 & 0 & 0 \\ 0 & \frac{q^{1,2}q^{4,2}}{gcd(q^{1,2},q^{4,2})} & 0 & 0 \\ 0 & 0 & \frac{q^{2,3}q^{3,3}}{gcd(q^{2,3},q^{3,3})} & 0 \\ 0 & 0 & 0 & \frac{q^{2,4}q^{3,4}}{gcd(q^{2,4},q^{3,4})} \end{pmatrix}$$

formed by operators of the type $q^{k,m}\left(\frac{d}{du}\right)$ that annihilate each $K_\pm^{k,m}$. Applying this matrix to the right-hand side of the equation leads to a homogeneous system, since the left-hand side becomes zero after differentiation. More specifically, for $i = 1, 2$,

$$\begin{aligned} \int_u^\infty h^i(y)dy &= \int_u^\infty g^1(y)h^i(y)dy + \int_u^\infty g^4(y)h^i(y)dy \\ &= \int_0^\infty g^1(u+y)h^i(u+y)dy + \int_0^\infty g^4(u+y)h^i(u+y)dy. \end{aligned}$$

Each term in the sum is annihilated by the corresponding operator,

$$q^{1,i}(\frac{d}{du}) \int_0^\infty g^1(u+y)h^i(u+y)dy = 0,$$

respectively

$$q^{4,i}(\frac{d}{du}) \int_0^\infty g^4(u+y)h^i(u+y)dy = 0.$$

Now the result follows from Lemma 2 and 3 in the Appendix. □

From Proposition 2 we have that $\psi^i(u)$ is a solution of a system of linear ODEs, meaning

$$\psi^i(u) = \sum_{j=1}^{n} \sum_{k=0}^{m} c_{j,k}^i u^k e^{-\lambda_j u}. \tag{11}$$

The exponents λ_j are among the solutions of the equation $\det(A) = 0$. Recall that, in the independent case, solving this determinant equation is equivalent to solving the Lundberg equation, $M_Z(s) = 1$. Similarly, for the Markov chain described here one can derive the exponents λ_j by means of moment generating functions. We give the details in the following proposition (the proof is given in the Appendix).

Proposition 3. *If Assumption 1 is fulfilled, then $\psi^i(u)$ has the general form (11), with $-\lambda_j$ being either the negative roots of the polynomial*

$$\frac{q^{1,1}q^{1,2}q^{2,3}q^{2,4}q^{3,3}q^{3,4}q^{4,1}q^{4,2}}{gcd(q^{1,1},q^{4,1})gcd(q^{1,2},q^{4,2})gcd(q^{2,3},q^{3,3})gcd(q^{2,4},q^{3,4})}$$

or the roots of

$$\begin{aligned}1 = &-M_{1,1}(-s)M_{3,3}(-s) - M_{1,1}(-s) - M_{3,3}(-s)\\ &- M_{2,3}(-s)M_{3,4}(-s)M_{4,2}(-s)\left(M_{1,1}(-s)-1\right)\\ &+ M_{1,2}(-s)M_{2,3}(-s)M_{3,4}(-s)M_{4,1}(-s)\\ &+ M_{2,4}(-s)M_{4,2}\left(M_{1,1}(-s)M_{3,3}(-s) - M_{1,1}(-s) - M_{3,3}(-s)+1\right)\\ &- M_{1,2}(-s)M_{2,4}(-s)M_{4,1}(-s)\left(M_{3,3}(-s)-1\right),\end{aligned}$$

where $M_{k,m}$ are defined as in (9).

Note that for a fixed j, $c^i_{j,k}$ are linearly dependent. To get the $c^i_{j,k}$ we can plug $\psi^i(u)$ into Equation (10), which finally results in a linear system of equations for $c^i_{j,k}$. Further note that by setting $u=0$ in the first two lines of Equation (10) we get the conditions

$$\psi^1(0) = \int_0^\infty h^1(y)dy = 1 \quad \text{and} \quad \psi^2(0) = \int_0^\infty h^1(y)dy = 1.$$

An intuitive meaning of these expressions is that given that we have a positive jump at zero, if $u=0$ then ruin is certain.

3 The connection with $(Z_k)_{k\geq 1}$ independent

If we set $h^1(x) = h^2(x)$ as the density of Z^+ satisfying an ODE with constant coefficients (e.g. phase-type distributions), $h^3(x) = h^4(x)$ as the density of Z^-, $g^1(x) = g^2(x) = \mathbb{P}(Z>0)$ and $g^3(x) = g^4(x) = \mathbb{P}(Z<0)$, then our model corresponds to the independent $(Z_k)_{k\geq 1}$ case. However, note that in this case Assumptions 1 are not fullfilled, since $g^1(x)$ and $g^4(x)$ are now not independent. A way around this problem is to set

$$g^1(x) = g^2(x) = \mathbb{P}(Z>0)e^{-\theta x} \quad \text{and} \quad g^3(x) = g^4(x) = 1 - g^1(x).$$

Thus, letting $\theta \to 0$ one recovers the independent case.

Moreover, taking $\theta \to 0$ in (10) and then applying the appropriate differential operator (to eliminate the integrals on the right-hand side), one obtains a new (simpler) system of ODEs

$$\begin{pmatrix} pr_+ - q_+ & pr_+ \\ (1-p)r_- & (1-p)r_- - q_- \end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},$$

where $p = \mathbb{P}(Z>0)$. Let $*$ stand for either $+$ or $-$. Then, here q_* is polynomial (as in characteristic equation) describing the ODE with constant coefficients that the densities of Z^*, whether r_* is defined through the rational moment generating functions of Z^*,

$$M_{Z^*}(-s) = \frac{r_*(s)}{q_*(s)}.$$

Standard ordinary differential equation theory says that $\psi^1(u)$, $\psi^4(u)$ and implicitly $\psi(u)$, are a linear combination of exponential functions, where the exponents are solutions of the polynomial equation

$$(pr^+ - q^+)((1-p)r^- - q^-) - p(1-p)r^+r^- = 0. \tag{12}$$

Equation (12) is equivalent to $M_Z(s) = 1$, further equivalent to $\psi(u)$ having a rational Laplace transform. This can be related to the result of Asmussen (1992)[Corollary 5.1.] for Z^+ phase-type distributed. In the following proposition, we (re)state a slight extension of this corollary (from phase-type to rational Laplace transform class) in our notation, to permit further connections with other existing results from the $(Z_k)_{k\geq 1}$ i.i.d. literature.

Proposition 4 (Corollary 5.1, Asmussen (1992)). *Assume that the density function of Z^+ satisfies an ODE with constant coefficients described by the polynomial $q_+(x) = q_0 + q_1 x + \cdots + q_n x^n$, with non-homogeneous boundary conditions, and denote by $M_{Z^-}(s)$ the moment generating function of Z^-. Define $r_+(s)$ through*

$$M_{Z^+}(-s) = \frac{r_+(s)}{q_+(s)},$$

and assume that for x_0 with $q_+(x_0) = 0$ we have $r_+(x_0) \neq 0$. Then

$$\psi(u) = \sum_{i=1}^{n} C_i u^{m_i} e^{-\lambda_i u}, \tag{13}$$

where C_i, and $\lambda_i \in \mathbb{C}$, are the solutions of the Lundberg equation

$$M_{Z^-}(s) = \frac{q_+(-s) - pr_+(-s)}{(1-p)q_+(-s)} \quad \text{or equivalently} \quad M_Z(s) = 1. \tag{14}$$

Further, $m_i \in \{0, \dots n\}$ is the multiplicity of the root λ_i minus 1.

Thus, for the slightly less general case, namely when Z itself has a rational moment generating function, one has the following result (see e.g the Notes and Remarks on page 270 in Asmussen and Albrecher (2010)).

Proposition 5. *Assume that the moment generating function of Z is rational and*

$$M_Z(-s) = \frac{r(s)}{q(s)}$$

and also assume that for x_0 with $q(x_0) = 0$ we have $r(x_0) \neq 0$. Then the probability of ruin is a solution of the ODE

$$\left(q\left(\frac{d}{du}\right) - r\left(\frac{d}{du}\right)\right)\psi(u) = 0.$$

Namely it has form (13)*, where* λ_i *are the solutions of the characteristic equation*

$$q(s) - r(s) = 0, \tag{15}$$

with multiplicity m_i *plus* 1*. This is equivalent to saying that* ψ *has a rational Laplace transform,*

$$\hat{\psi}(s) = \frac{((q-r)_+(s) - dq_+(s))}{s(q-r)_+(s)} \tag{16}$$

where $(q-r)_+(s)$ *is the polynomial that has as only roots the positive roots of the characteristic equation* (15) *and* $d = (q-r)_+(0)/q_+(0)$.

Remark 3. *After partial fractions decomposition and inversion of the Laplace transform, one obtains the explicit form of the probability of ruin this time down to constants, not only up to constants as in the theorem.*

As examples of models where Z has rational moment generating functions, one can revisit all the models considered by Ambagaspitiya (2009) and derive exact expressions for the ruin probability. For example, we consider in the following the Kibble and Moran's bivariate Gamma joint density for τ and X, defined through

$$M_Z(s) = \mathbb{E}(e^{sZ}) = \mathbb{E}(e^{-c\tau+X}) = M_{\tau,X}(s) = \frac{1}{\left((1+\frac{cs}{\beta_1})(1-\frac{s}{\beta_2}) + \rho\frac{cs^2}{\beta_1\beta_2}\right)^m},$$

where $\rho \geq 0$ is the correlation coefficient between τ and X.

Example 1 (Kibble and Moran's bivariate Gamma joint density)**.** *According to Proposition 5 one needs to first solve the Lundberg equation*

$$M_Z(s) = 1 \quad \textit{or equivalently} \quad \left((1+\frac{cs}{\beta_1})(1-\frac{s}{\beta_2}) + \rho\frac{cs^2}{\beta_1\beta_2}\right)^m = 1.$$

Assuming that $c\beta_2 > \beta_1$*, we get the roots (compare Ambagaspitiya (2009))*

$$\lambda^i_{\pm} = \frac{c\beta_2 - \beta_1 \pm \sqrt{(c\beta_2-\beta_1)^2 + 4c(1-\rho)\beta_1\beta_2(1-e^{4\pi k/m})}}{2c(1-\rho)},$$

where $Re(\lambda^i_+) > 0$ *and* $Re(\lambda^i_-) \leq 0$*. Hence* ψ *satisfies the ODE*

$$\left(\left(\left(1-\frac{c}{\beta_1}\frac{d}{du}\right)\left(1+\frac{1}{\beta_2}\frac{d}{du}\right) + \rho\frac{c^2}{\beta_1\beta_2}\left(\frac{d}{du}\right)^2\right)^m - 1\right)\psi(u) = 0,$$

with solution

$$\psi(u) = \sum_{i=1}^m c_i e^{-\lambda^i_+ u}.$$

Further, note that for $i = 1, \ldots, m-1$*,*

$$c_i = \left.\frac{s+\lambda_i}{s}\frac{(q-r)_+(s) - dq_+(s)}{(q-r)_+(s)}\right|_{s=-\lambda_i}.$$

4 $(Z_k)_{k\geq 1}$ dependent with exponential densities

In this section we return to the dependent case endowed with a Markov chain structure and provide an example for the results derived. We are again in the set-up of $g^i(x)$, $i = 1, \ldots 4$ being non-constant functions of the initial step $Z_0 = x$. Let

$$\begin{cases} g^1(x) = 1 - g^4(x) = 1 - e^{-\theta x} & (I) \\ g^2(x) = 1 - g^3(x) = 1 - e^{\eta x} & (1-I) \\ h^1(y) = \lambda_1 e^{-\lambda_1 y}, \quad h^2(y) = \lambda_2 e^{-\lambda_2 y} & (Z^+) \\ h^3(y) = \lambda_3 e^{\lambda_3 y}, \quad h^4(y) = \lambda_4 e^{\lambda_4 y} & (Z^-). \end{cases}$$

More precisely we will further assume that $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. The interpretation of such a model is the following: The size of the jumps are exponentially distributed, having different parameters for positive and negative jumps. Thus, the functions h^1, h^2 and h^3, h^4 respectively play the same role as the densities of Z^+ and Z^- of (3) in the independent random-walk case of Asmussen (1992). The parameters θ or η control the probability of going up or down in the next step of the random walk. Functions g^1, g^4 (reps. g^2, g^3) correspond to $1-I$, I for $x > 0$ (respectively $x < 0$) as in the decomposition (3). If θ is large, then it is more likely that a positive jump is followed by another positive jump. On the contrary, if η is close to zero then it is more likely that a negative jump is followed by another negative jump (see Figure 1 compare the graph to paths when sampled from the invariant distribution in Figure 2). Since the lucky events are when the random walk goes to $-\infty$, choosing a large θ one would emphasize the bad tendencies. On the other hand, choosing a η close to zero would emphasize the good tendencies. Also, note that choosing a small θ would lead to some counter-cyclical, tamed behavior (see Figure 4 the invariant distribution is the same as the one of the model in Figure 1).

Next we give a numerical example. We use the parameters $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 2$ and $\eta = 0.4$ (see Figure 3 for a plot of some paths of this process). Using Lemma 4 from the Appendix we derive the density of the invariant distribution,

$$f_Z(x) = \begin{cases} \frac{1}{2} e^{-x} & x \geq 0 \\ \frac{1}{2}\frac{4}{5} e^{\frac{4}{5}x} & x < 0. \end{cases}$$

With Plots 1 and 4 we emphasize the difference in the behavior of the paths when θ and η have different values, even though they have the same invariant distribution. Since $\psi^1(u) = \psi^2(u)$ and $\psi^3(u) = \psi^4(u)$, to calculate $\psi^i(u)$ one needs to solve the system of ODEs

$$A \begin{pmatrix} \psi^1 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case, the matrix A is

$$A = \begin{pmatrix} \frac{(r^{1,1} - q^{1,1}) q^{4,1}}{gcd(q^{1,1}, q^{4,1})} & \frac{q^{1,1} r^{4,1}}{gcd(q^{1,1}, q^{4,1})} \\ \frac{r^{2,3} q^{3,3}}{gcd(q^{2,3}, q^{3,3})} & \frac{(r^{3,3} - q^{3,3}) q^{2,3}}{gcd(q^{2,3}, q^{3,3})} \end{pmatrix},$$

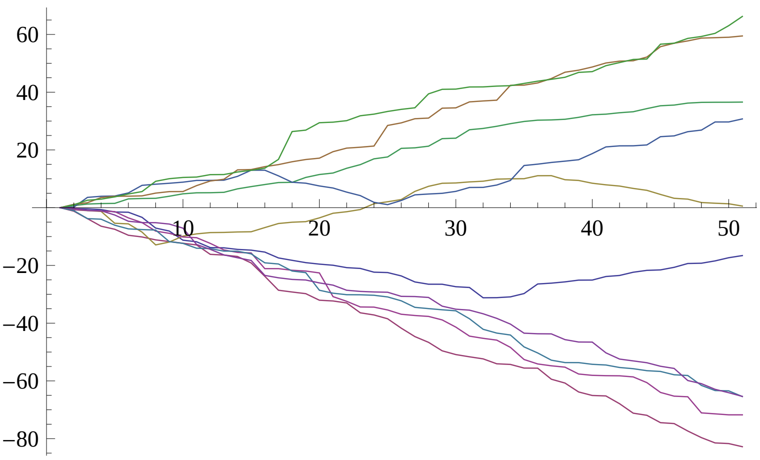

Figure 1: Paths of the risk process when $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 100$ and $\eta = 0.008$

with the determinant

$$\det(A) = s\left(s^3 + \frac{14s^2}{5} - \frac{87s}{25} - \frac{6}{25}\right).$$

Given the infinity condition, one needs to consider only the negative roots of the polynomial $\det(A) = 0$. Thus, the solution of the system of ODEs has the form

$$\psi^i(u) = c_2^i e^{-3.7185u} + c_2^i e^{-0.0655856u},$$

which once plugged back into Equation (10) leads to

$$\psi^1(u) = \psi^2(u) = 0.0488089e^{-3.7185u} + 0.951191e^{-0.0655856u}$$
$$\psi^3(u) = \psi^4(u) = 0.000839295e^{-3.7185u} + 0.75528e^{-0.0655856u}$$

and thus

$$\psi(u,x) = \begin{cases} (1 - e^{-2x})\psi^1(u) + e^{-2x}\psi^3(u) & x > 0 \\ (1 - e^{\frac{4}{10}x})\psi^1(u) + e^{-\frac{4}{10}x}\psi^3(u) & x < 0. \end{cases}$$

Further, if we assume that Z_0 is distributed according to the invariant distribution, then

$$\psi(u) = 0.0248241e^{-3.7185u} + 0.853236e^{-0.0655856u}.$$

We compare this to the ruin probability of a process with iid increments, which follows the same invariant distribution,

$$\psi_Z(u) = \frac{9}{10}e^{-\frac{u}{10}}.$$

Numerical values of these ruin probabilities are plotted in Figures 5 and 6. One can clearly see that, in this case, dependence causes an increase of the risk of ruin.

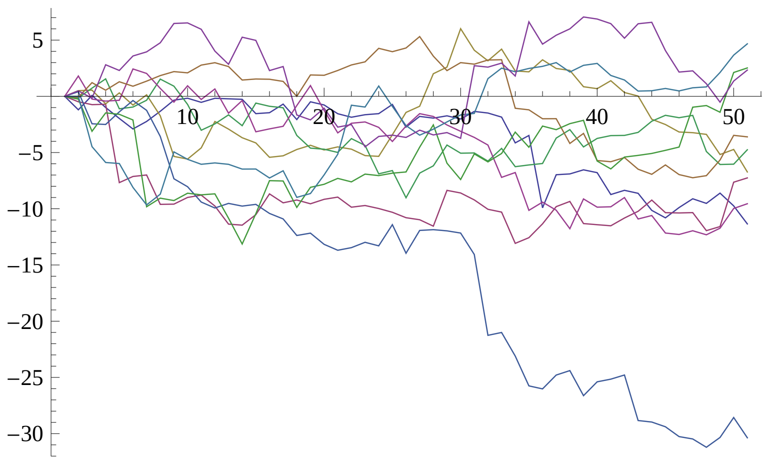

Figure 2: Paths of the risk process when the claims are iid distributed with the invariant distribution of a process with $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 100$ and $\eta = 0.008$

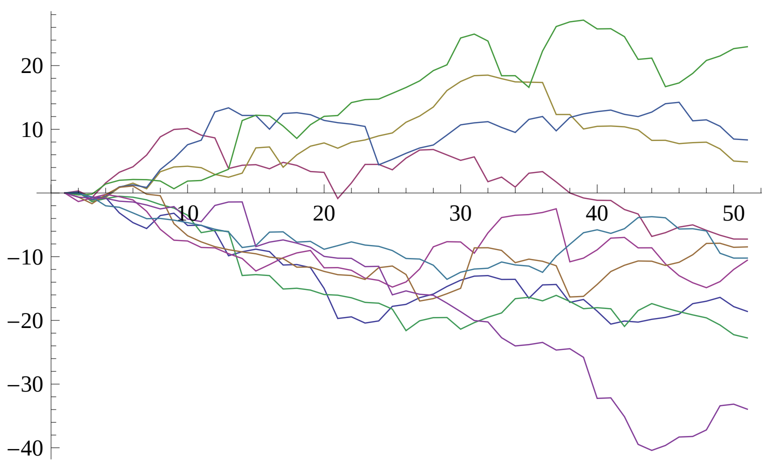

Figure 3: Paths of the risk process when $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 2$ and $\eta = 0.4$

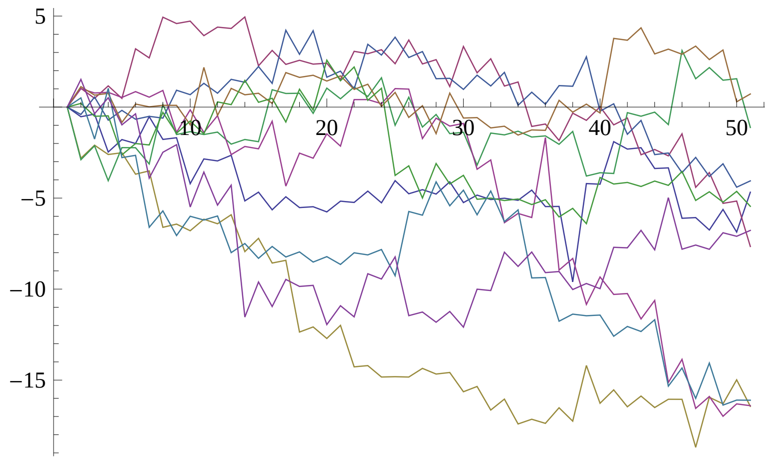

Figure 4: Paths of the risk process when $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 100^{-1}$ and $\eta = 80$

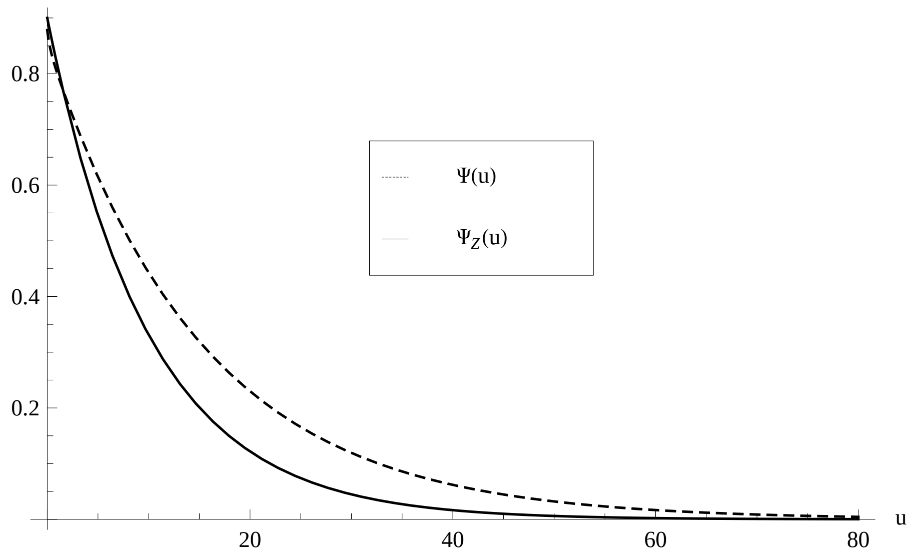

Figure 5: Comparison of the ruin probability (absolute values) of the Markov chain with $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 2$ and $\eta = 0.4$, and the corresponding process of independent Z with the invariant distribution.

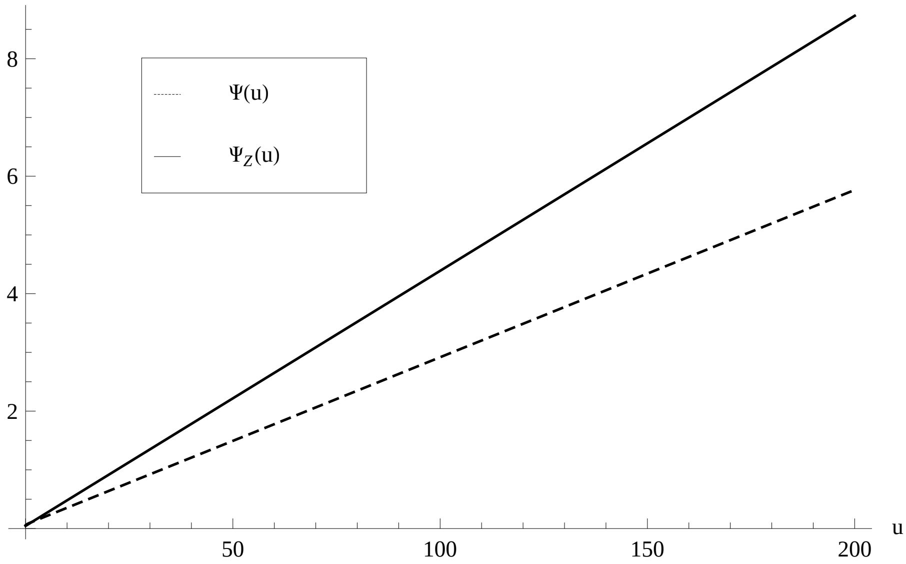

Figure 6: Comparison of the ruin probability ($-\log_{10}$ of the values) of the Markov chain with $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 4/5$, $\theta = 2$ and $\eta = 0.4$, and the corresponding process of independent Z with the invariant distribution.

5 Conclusions

One of the fundamental problems in random walk theory is the computation of the ladder height distributions. More specifically, for a random walk $W_n = \sum_{k=1}^{n} Z_k$, with increments having the distribution F_Z and ladder epochs defined as $\tau_- = \inf\{n \geq 1 | W_n \geq 0\}$ respectively $\tau_+ = \inf\{n \geq 1 | W_n > 0\}$, the ladder height distributions are $G_-(u) = \mathbb{P}(Z_{\tau_-} \leq u)$ and $G_+(u) = \mathbb{P}(Z_{\tau_+} \leq u)$. These quantities are relevant in sequential, queuing and risk theory. Results have been derived for increments having a structure $Z = pZ_1 + (1-p)Z_2$ (Feller, 1971) or a difference structure $Z = \tau - X$ (Asmussen, 1992).

Focusing on a risk theory model, one could analyze the probability of ruin in this random-walk/ladder-height setting. In this paper we base our analysis on the real-valued random variable $Z_k = -c\tau_k + X_k$ describing the difference between the inter-arrival time τ_k and its consecutive claim size X_k. In the classical model, τ and X are assumed to be independent. When Z_k are independent, we are in the random walk setting. A natural condition for Z in order to be able to derive explicit forms of the ruin probability is that its mgf is rational. Or, the more general version of it, that only the mgf of $Z^+ = \{Z \mid Z > 0\}$ is rational (Asmussen, 1992).

We introduced a Markov chain dependence which still permits the exact calculation of the probability of ruin. A motivation for such a model, besides the mathematical amenability, consists in the fact that one can implement a counter-cyclic behavior of the sample paths.

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A

Lemma 1.

1. *If the joint moment generating function $M_{\tau,X}(t,s)$ of the vector (τ, X) is rational, then the moment generating functions of Z^+ and Z^-, M_{Z^+} and respectively M_{Z^-} are also rational.*

2. *If M_{Z^+} and M_{Z^-} are rational, then there exists a vector (τ, X) such that $M_{\tau,X}(t,s)$ is rational.*

Proof. Ad 1. Since $M_{\tau,X}(t,s)$ is rational, the density F_Z of $Z = X - c\tau = IZ^+ + (1-I)Z^-$ has a rational Fourier transform, implying

$$f_Z(x) = \sum_{i=1}^{n} c_i x^{m_i} e^{-\lambda_i x} 1_{\{\mathrm{sign}(\mathrm{Re}(\lambda_i))[0,\infty]\}}(x).$$

Now 1 follows from

$$pf_{Z^+}(x) = \sum_{i=1,\mathrm{Re}(\lambda_i)>0}^{n} c_i x^{m_i} e^{-\lambda_i x} 1_{\{(0,\infty)\}}(x)$$

and

$$(1-p)f_{Z^-}(x) = \sum_{i=1,\mathrm{Re}(\lambda_i)<0}^{n} c_i x^{m_i} e^{-\lambda_i x} 1_{\{(-\infty,0)\}}(x).$$

Ad 2. We have

$$M_Z(s) = pM_{Z^+}(s) + (1-p)M_{Z^-}(s) = \frac{p}{1-s+s} M_{Z^+}(s) + \frac{1-p}{1-s+s} M_{Z^-}(s).$$

We use

$$M_{\tau,X}(t,s) = p\frac{1}{1-t-s} M_{Z^+}(s) + (1-p)\frac{1}{1-t-s} M_{Z^-}(-t).$$

To show that this is the moment generating function of a random vector, let I be Bernoulli(p) and Y be Exp(1). Then the random vector

$$(\tau, X) := \mathbb{P}\left(Y, Y + Z^+\right) + (1-P)\left(Y - Z^-, Y\right)$$

has $M_{\tau,X}(t,s)$ as moment generating function. □

Lemma 2. *Let $g(y)$ and $\psi(y)$ be two functions that are sufficiently often differentiable and bounded for $y > 0$. Assume that there exists a polynomial $q(x) = q_0 + q_1 x + \cdots + q_n x^n$ with*

$$q\left(\frac{d}{dy}\right) g(y) = 0, \quad y > 0.$$

Further define the polynomial

$$r(x) = \sum_{l=0}^{n-1} \sum_{i=l+1}^{n} q_i g^{(i-l-1)}(0) x^l.$$

Then the Laplace transform of g

$$\int_0^\infty e^{-sy} g(y) dy = \frac{r(s)}{q(s)},$$

and

$$q\left(\frac{d}{du}\right) \int_0^u \psi(u-y) g(y) dy = r\left(\frac{d}{du}\right) \psi(u).$$

Proof. Note that for $s > 0$, using integration be parts

$$\int_0^\infty e^{-sy} g(y) dy = \sum_{k=1}^{l} \frac{1}{s^k} g^{(k-1)}(0) + \int_0^\infty \frac{1}{s^l} e^{-sx} g^{(l)}(x) dx.$$

Multiplying with $q(s)$ leads to

$$\begin{aligned} q(s) \int_0^\infty e^{-sy} g(y) dy &= \sum_{l=0}^{n} \left(q_l \sum_{k=1}^{l} s^{l-k} g^{(k-1)}(0) + \int_0^\infty e^{-sx} q_l g^{(l)}(x) dx \right) \\ &= \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} q_l g^{(l-k-1)}(0) s^k + \int_0^\infty e^{-sx} q\left(\frac{d}{dx}\right) g(x) dx \\ &= r(s). \end{aligned}$$

For the second part of the lemma note that by induction

$$\begin{aligned} \left(\frac{d}{du}\right)^n \int_0^u \psi(u-y) g(y) dy &= \left(\frac{d}{du}\right)^n \int_0^u \psi(y) g(u-y) dy \\ &= \sum_{k=0}^{n-1} \psi^{(k)}(u) g^{(n-k-1)}(0) + \int_0^u \psi(y) g^{(n)}(u-y) dy. \end{aligned}$$

It follows

$$\begin{aligned} q\left(\frac{d}{du}\right) \int_0^u \psi(u-y) g(y) dy &= \sum_{l=0}^{n} q_l \sum_{k=0}^{l-1} \psi^{(k)}(u) g^{(n-k-1)}(0) \\ &\quad + \int_0^u \psi(y) q\left(\frac{d}{du}\right) g(u-y) dy = r\left(\frac{d}{du}\right) \psi(u). \end{aligned}$$

□

Lemma 3. *Let $g(y)$ and $\psi(y)$ be two functions that are sufficiently often differentiable, where $g(y)$ is bounded for $y < 0$ and $\psi(y)$ is bounded for $y > 0$. Assume that there exists a polynomial $q(x) = q_0 + q_1 x + \cdots + q_n x^n$ with*

$$q\left(\frac{d}{dy}\right) g(y) = 0, \quad y < 0.$$

Further define the polynomial

$$r(x) = -\sum_{l=0}^{n-1} \sum_{i=l+1}^{n} q_i g^{(i-l-1)}(0) x^l.$$

Then

$$\int_{-\infty}^{0} e^{-sy} g(y) dy = \frac{r(s)}{q(s)}$$

and

$$q\left(\frac{d}{du}\right) \int_{-\infty}^{0} \psi(u-y) g(y) dy = r\left(\frac{d}{du}\right) \psi(u).$$

Proof. At first note that by partial integration

$$\int_{-\infty}^{0} e^{-sy} g(y) dy = -\sum_{k=1}^{l} \frac{1}{s^k} g^{(k-1)}(0) + \int_{-\infty}^{0} \frac{1}{s^l} e^{-sx} g^{(l)}(x) dx.$$

Multiplying with $q(s)$ leads to

$$\begin{aligned} q(s) \int_{-\infty}^{0} e^{-sy} g(y) dy &= -\sum_{l=0}^{n} \left(q_l \sum_{k=1}^{l} s^{l-k} g^{(k-1)}(0) - \int_{-\infty}^{0} e^{-sx} q_l g^{(l)}(x) dx \right) \\ &= -\sum_{k=0}^{n-1} \sum_{l=k+1}^{n} q_l g^{(l-k-1)}(0) s^k + \int_{0}^{\infty} e^{-sx} q\left(\frac{d}{dx}\right) g(x) dx \\ &= r(s). \end{aligned}$$

For the second part of the lemma note that by induction

$$\begin{aligned} \left(\frac{d}{du}\right)^n \int_{-\infty}^{0} \psi(u-y) g(y) dy &= \left(\frac{d}{du}\right)^n \int_{u}^{\infty} \psi(y) g(u-y) dy \\ &= -\sum_{k=0}^{n-1} \psi^{(k)}(u) g^{(n-k-1)}(0) + \int_{u}^{\infty} \psi(y) g^{(n)}(u-y) dy. \end{aligned}$$

It follows

$$\begin{aligned} q\left(\frac{d}{du}\right) \int_{u}^{\infty} \psi(u-y) g(y) dy = &-\sum_{l=0}^{n} q_l \sum_{k=0}^{l-1} \psi^{(k)}(u) g^{(n-k-1)}(0) \\ &+ \int_{u}^{\infty} \psi(y) q\left(\frac{d}{du}\right) g(u-y) dy = r\left(\frac{d}{du}\right) \psi(u). \end{aligned}$$

□

Lemma 4. *Let Z be a Markov chain with transition density satisfying the Assumption 1. Denote with*

$$K^{l,m} = \begin{cases} \int_0^\infty g^l(x) h^m(x) dx, & \text{if } l \in \{1,4\},\ m \in \{1,2\} \\ \int_{-\infty}^0 g^l(x) h^m(x) dx, & \text{if } l \in \{2,3\},\ m \in \{3,4\} \end{cases}.$$

Then the invariant distribution has the density

$$f_Z(y) = \frac{K^{1,2}K^{2,3}h^1(y) + K^{2,3}K^{4,1}h^2(y)}{K^{1,2}K^{2,3} + 2K^{2,3}K^{4,1} + K^{3,4}K^{4,1}}, \quad y > 0,$$
$$f_Z(y) = \frac{K^{3,4}K^{4,1}h^3(y) + K^{2,3}K^{4,1}h^4(y)}{K^{1,2}K^{2,3} + 2K^{2,3}K^{4,1} + K^{3,4}K^{4,1}}, \quad y < 0,$$

Proof. The invariant distribution is defined through

$$f_Z(y) = \int_0^\infty p(x,y) f_Z(x) dx.$$

We can split this equation into two equations, for $y > 0$ and $y < 0$,

$$f_Z(y) = h^1(y) \int_0^\infty g^1(x) f_Z(x) dx + h^2(y) \int_{-\infty}^0 g^2(x) f_Z(x) dx, \quad y > 0,$$
$$f_Z(y) = h^4(y) \int_0^\infty g^4(x) f_Z(x) dx + h^3(y) \int_{-\infty}^0 g^3(x) f_Z(x) dx, \quad y < 0.$$

It follows that the invariant distribution can be written as

$$f_Z(y) = c^1 h^1(y) + c^2 h^2(y), \quad y > 0,$$
$$f_Z(y) = c^3 h^3(y) + c^4 h^4(y), \quad y < 0,$$

and we are left with determining the constants c^l, which are the solution of the system of linear equations

$$\begin{cases} c^1 &= K^{1,1}c^1 + K^{1,2}c^2 \\ c^2 &= K^{2,3}c^4 + K^{2,4}c^4 \\ c^3 &= K^{3,3}c^4 + K^{3,4}c^4 \\ c^4 &= K^{4,1}c^1 + K^{4,2}c^2. \end{cases}$$

In our matrix notation

$$\begin{pmatrix} K^{1,1}-1 & K^{1,2} & 0 & 0 \\ 0 & -1 & K^{2,3} & K^{2,4} \\ 0 & 0 & K^{3,3}-1 & K^{3,4} \\ K^{4,1} & K^{4,2} & 0 & -1 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \\ c^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

Using the fact that $\int h^i(x)dx = 1$ and $1 - g^i(x) = g^{5-i}(x)$ one can rewrite the matrix in (17) as

$$\begin{pmatrix} -K^{4,1} & 1-K^{4,2} & 0 & 0 \\ 0 & -1 & K^{2,3} & K^{2,4} \\ 0 & 0 & -K^{2,3} & 1-K^{2,4} \\ K^{4,1} & K^{4,2} & 0 & -1 \end{pmatrix}$$

Moreover, by adding all other rows to the last one, it becomes

$$\begin{pmatrix} -K^{4,1} & 1-K^{4,2} & 0 & 0 \\ 0 & -1 & K^{2,3} & K^{2,4} \\ 0 & 0 & -K^{2,3} & 1-K^{2,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \\ c^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It follows that

$$\begin{aligned}
c^1 &= tK^{1,2}\left(\frac{K^{3,4}+K^{2,4}}{K^{4,1}}\right) = t\frac{K^{1,2}}{K^{4,1}} \\
c^2 &= t\left(K^{3,4}+K^{2,4}\right) = t \\
c^3 &= t\frac{K^{3,4}}{K^{2,3}} \\
c^4 &= t.
\end{aligned}$$

Since $f_Z(x)$ is a density function, $t^{-1} = \frac{K^{1,2}}{K^{4,1}} + \frac{K^{3,4}}{K^{2,3}} + 2$. □

Remark 4. *In the general case we have that* $\int h^i(x)dx = 1$ *and* $1 - \sum_j g_j^i(x) = \sum_j g_j^{5-i}(x)$ *and hence for fixed* j *and* m*,*

$$\sum_i K_{i,j}^{l,m} + K_{i,j}^{5-l,m} = 1$$

($K_{i,j}^{l,m}$ *is defined as* $K^{l,m}$ *by replacing* h^m *by* h_j^m *and* g^l *by* g_i^l*). If we interpret the matrix* (17) *as a block matrix where* $K^{l,m}$ *corresponds to the matrix with elements* $K_{i,j}^{l,m}$ *and* 1 *to the identity matrix, then the sum of all rows is* 0*. Which means that we get non-trivial candidates for* f_Z*.*

Proof of Proposition 1. Equation (5) for $\psi(u,x)$ will have two expressions, depending on the sign of x. For $x \in I_1 \cup I_4$, $(x > 0)$

$$\begin{aligned}
\psi(u,x) &= \int_u^\infty g^1(x)h^1(y)dy + \int_{-\infty}^0 \psi(u-y,y)g^4(x)h^4(y)dy \\
&\quad + \int_0^u \psi(u-y,y)g^1(x)h^1(y)dy \\
&= g^1(x)\left(\int_u^\infty h^1(y)dy + \int_0^u \psi(u-y,y)h^1(y)dy\right) \\
&\quad + g^4(x)\int_{-\infty}^0 \psi(u-y,y)h^4(y)dy \\
&= g^1(x)\psi^1(u) + g^4(x)\psi^4(u). \qquad (18)
\end{aligned}$$

Analogously for $x \in I_2 \cup I_3$, $(x < 0)$

$$\begin{aligned}
\psi(u,x) &= \int_u^\infty g^2(x)h^2(y)dy + \int_{-\infty}^0 \psi(u-y,y)g^3(x)h^3(y)dy \\
&\quad + \int_0^u \psi(u-y,y)g^2(x)h^2(y)dy \\
&= g^2(x)\left(\int_u^\infty h^2(y)dy + \int_0^u \psi(u-y,y)h^2(y)dy\right) \\
&\quad + g^3(x)\int_{-\infty}^0 \psi(u-y,y)h^3(y)dy \\
&= g^2(x)\psi^2(u) + g^3(x)\psi^3(u). \qquad (19)
\end{aligned}$$

Thus the first part of the theorem follows. The second part of the theorem follows from $\lim_{u\to\infty}\psi(u,x) = 0$ for every x. □

Proof of Theorem 3. From the fact that ψ^i solve a system of ODEs with boundary conditions $\lim_{u\to\infty}\psi^i(u)=0$, we have

$$\psi^i(u)=\sum_{j=1}^{n}\sum_{k=0}^{m}c^i_{j,k}u^k e^{-\lambda_j u},$$

where $-\lambda_j$ are the negative roots of the polynomial $\det(A)=0$, equivalent to

$$\begin{aligned}0=&\frac{(r^{1,1}-q^{1,1})(r^{3,3}-q^{3,3})q^{1,2}q^{2,3}q^{2,4}q^{3,4}q^{4,1}q^{4,2}}{gcd(q^{1,1},q^{4,1})gcd(q^{1,2},q^{4,2})gcd(q^{2,3},q^{3,3})gcd(q^{2,4},q^{3,4})}\\ &+\frac{\left(r^{2,3}r^{3,4}q^{2,4}q^{3,3}-r^{2,4}(r^{3,3}-q^{3,3})q^{2,3}q^{3,4}\right)\left(r^{4,2}(r^{1,1}-q^{1,1})q^{1,2}q^{4,1}-r^{1,2}r^{4,1}q^{1,1}q^{4,2}\right)}{gcd(q^{1,1},q^{4,1})gcd(q^{1,2},q^{4,2})gcd(q^{2,3},q^{3,3})gcd(q^{2,4},q^{3,4})}.\end{aligned}$$

Dividing by

$$\frac{q^{1,1}q^{1,2}q^{2,3}q^{2,4}q^{3,3}q^{3,4}q^{4,1}q^{4,2}}{gcd(q^{1,1},q^{4,1})gcd(q^{1,2},q^{4,2})gcd(q^{2,3},q^{3,3})gcd(q^{2,4},q^{3,4})}$$

leads to the equation

$$\begin{aligned}0=&\frac{(r^{1,1}-q^{1,1})(r^{3,3}-q^{3,3})}{q^{1,1}q^{3,3}}+\frac{r^{2,3}r^{3,4}r^{4,2}(r^{1,1}-q^{1,1})}{q^{1,1}q^{2,3}q^{3,4}q^{4,2}}-\frac{r^{1,2}r^{2,3}r^{3,4}r^{4,1}}{q^{1,2}q^{2,3}q^{3,4}q^{4,1}}\\ &-\frac{(r^{1,1}-q^{1,1})(r^{3,3}-q^{3,3})r^{2,4}r^{4,2}}{q^{1,1}q^{2,4}q^{3,3}q^{4,2}}+\frac{(r^{3,3}-q^{3,3})r^{1,2}r^{2,4}r^{4,1}}{q^{1,2}q^{2,4}q^{3,3}q^{4,1}}\end{aligned}$$

The claim follows from the definition of $M_{k,m}$. □

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