INVARIANT MEASURES FOR PIECEWISE CONVEX TRANSFORMATIONS OF AN INTERVAL

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Abstract. In this article we investigate the existence and ergodic properties of absolutely continuous invariant measures for a class of piecewise monotone and convex self-maps of the unit interval. Our assumption entails a type of average convexity which strictly generalizes the case of individual branches being convex, as investigated by Lasota and Yorke (1982). Along with existence, we identify tractable conditions for the invariant measure to be unique and such that the system has exponential decay of correlations on bounded variation functions and Bernoulli natural extension. In the case when there is more than one invariant density we identify a dominant component over which the above properties also hold. Of particular note in our investigation is the lack of smoothness or uniform expansiveness assumptions on the map or its powers.

1. Introduction and Statement of Results

We study nonsingular transformations T from the unit interval $I = [0,1]$ into I that are piecewise monotone and continuous. Specifically, there is a finite partition of I given by $0 = a_0 < a_1 < a_2 < \cdots < a_N = 1$ and such that on each (a_{i-1}, a_i) we have $T_i = T|_{(a_{i-1},a_i)}$ being continuous and (strictly) increasing. Since each T_i is 1-1, they have well defined inverses $\phi_i = T_i^{-1}$ which may be extended continuously to increasing (and hence a.e. differentiable) functions $\psi_i : [0,1] \to [a_{i-1}, a_i]$, where $\psi_i(x) = a_{i-1}$ if $x \le \inf_{y\in(a_{i-1},a_i)} T_i(y)$; $\psi_i(x) = a_i$ if $x \ge \sup_{y\in(a_{i-1},a_i)} T_i(y)$. Since we are assuming $m \circ T_i^{-1} << m$ for each $i = 1, 2, \ldots, N$, we have

$$\frac{dm \circ T_i^{-1}}{dm} = \psi_i', \ m - \text{a.e.}.$$

Throughout this article m denotes the Lebesgue measure on Borel subsets $\mathcal{B}$ of $[0,1]$.

We consider the *Perron–Frobenius* operator P on $L^1 = L^1(I, \mathcal{B}, m)$, uniquely defined by the identity

$$\int Pf\, g\, dm = \int f\, g \circ T\, dm; \ \forall f \in L^1, g \in L^\infty. \tag{1.1}$$

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In view of our setup we have the following pointwise representation for P, taking $g = \chi_{[0,x]}$. For each $x \in [0, 1]$,

$$\begin{aligned}\int_0^x Pf(t)dm(t) &= \sum_i \int_{T_i^{-1}[0,x]} f(s)dm(s)\\ &= \sum_i \int_0^x f \circ \psi_i(t) \frac{dm \circ T_i^{-1}}{dm}(t)\, dm(t)\\ &= \int_0^x \sum_i f(\psi_i(t))\psi_i'(t)\, dm(t)\end{aligned}$$

from which it follows that $Pf(x) = \sum_{i=1}^N \psi_i'(x) f(\psi_i(x))$ for almost every $x \in [0, 1]$. Our convexity assumption takes the following form.

(C) Assume that for $i = 1, 2, \dots, N$ there are measurable functions $F_i : [0, 1] \to \mathrm{R}$ with $F_i = \psi_i'$ $m-$a.e, and such that the family F_i satisfies both

(C1): For each $k = 1, 2, \dots, N$ the functions $F_1 + F_2 + \cdots + F_k$ are decreasing, and
(C2): $F_1(0) < 1$.

Observe that a branch T_i is convex iff its associated F_i may be chosen decreasing. Our assumption **(C)** therefore is strictly weaker than the requirement that each branch be convex.

Under assumption **(C)** we may again rewrite the pointwise version of (1.1) as

$$Pf(x) = \sum_{i=1}^N F_i(x) f(\psi_i(x));\ \forall f \in L^1. \tag{1.2}$$

Remark 1.1. Of course, the formula (1.2) requires the following interpretation. Given $f \in L^1$, any *version* of f used on the right hand side of (1.2) will produce a *version* of Pf.

Theorem 1.1. *Let T be piecewise monotone and continuous on I as above, and satisfy the convexity condition* **(C)**. *Then T admits an absolutely continuous invariant probability measure ν, whose density $g = \frac{d\nu}{dm}$ may be chosen to be a decreasing function on I.*

Remark 1.2. We note that no *continuity* assumption is made on the derivative T' so our result is not like most of the classical existence criterion depending on smoothness of the map. See, for example [6], [2] and references cited there. An extensive analysis by Rychlik [9] is closer in spirit to our present investigation, but the uniform expansiveness assumed there is replaced by the weaker form in **(C2)** which only implies that the branch T_1 is expanding.

Perhaps closest to our present investigation is an older result of Lasota and Yorke [7] for piecewise convex maps where the main result proved there should be compared to ours. There, *all* branches are assumed to be convex, and the leftmost branch T_1 is assumed to satisfy $T'(0) > 1$. Further, all branches satisfy $T_1(0) = T_2(a_1) = \cdots = T_N(a_{N-1}) = 0$. The assumption of nonsingularity is not necessary as the stronger convexity assumption implies it. Our convexity assumption **(C1)** is not only weaker than that of Lasota–Yorke, but is also more natural in the sense that, as we shall see, the condition **(C1)** is necessary and sufficient for the Perron–Frobenius operator P to preserve the class of non-negative, decreasing functions on I (Lemma 2.2). So, our condition is invariant under taking powers of T. Also, this observation leads to a very simple proof of a classical variational inequality, after which the existence of an invariant density can be deduced in the classical manner; see for example [6]. This is discussed in §2, providing the proof of Theorem 1.1 above.

In [7], Lasota–Yorke convex maps are shown to have the property that there is a unique invariant probability density g and that the a.c.i.p.m. $d\mu = gdx$ is exact for T. This is not the case for maps satisfying our weaker convexity condition. In §4 and §5 we identify a dominant component for a given decreasing invariant density and prove uniqueness and exactness of this dominant component (an interval). Also, in §4 we prove exponential decay of correlations, the uniform expanding property and the Bernoulli property when the dominant component is not normalized Lebesgue measure. This restriction is equivalent to the requirement that T be expanding at the rightmost endpoint of the dominant component, which we call condition **(E)**, for expanding. We note that exponential decay of correlations and the uniform expanding property were proved for Lasota-Yorke convex maps in [3].

The remaining question of when there exists a unique a.c.i.m for T is discussed in §6. We identify a mixing conditon **(M)** which ensures that there is exactly one invariant density in BV and the resulting system (T, gdx) is exact. Again, if the expanding condition **(E)** is also satified, then some power of T is expanding and T is Bernoulli with respect to its unique a.c.i.m.. Lasota-Yorke convex maps satisfy our condition **(M)**, but **(M)** does not imply that some power of T is uniformly expanding, so this final section identifies a proper extension of the results in [7].

Much of our argument depends on the identification of suitable invariant cones for the operator P and the construction of norms equivalent to the bounded-variation norm from these cones. We give a brief discussion of these matters in §3. The reader looking for more complete background on this method should consult [8] where many of the omitted details may be found.

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2. Existence of an Invariant Density

Throughout this section T is assumed to satisfy the conditions of Theorem 1.1.

Lemma 2.1. *Without loss of generality, we may assume that for $k = 1, 2, \dots, N$ the functions $F_1 + F_2 + \cdots + F_k$ are decreasing, and* **upper-semicontinuous** *on* $[0, 1]$.

Proof. Every decreasing function on $[0,1]$ can be modified on a set of measure zero to be upper-semicontinuous and decreasing. Apply this inductively for $k = 1, 2, \dots, N$. Changing each of the functions F_i on sets of measure zero will not change the operator P. □

Remark 2.1. The simple result above leads to a kind of uniqueness in the pointwise representation of the Perron-Frobenius operator. Suppose $Ff(x) = \sum_{i=1}^{N} F_i(x)f(\psi_i(x))$ while $Gf(x) = \sum_{i=1}^{N} G_i(x)f(\psi_i(x))$ for all $f \in L^1$ are two Perron-Frobenius operators with weight functions $\{F_i\}$, $\{G_i\}$ both satifying **(C1)**. Suppose $Ff = Gf$ for all $f \in L^1$. If both sets of weight functions have upper-semicontinuous sums as in the above lemma, then $F_i = G_i$ on $(0,1]$ since two upper-semicontinuous, decreasing functions which agree almost everywhere must be identical except possibly at zero. Now it is a simple matter to change the definition of the F_I at the single point zero, still maintaining condition **(C1)** so that $F_i = G_i$ on $[0,1]$ We will have a number of opportunities in the following arguments to make use of this simple observation.

Remark 2.2. In a similar vein, suppose $c \in (a_{i-1}, a_i)$ for some $i = 1, 2, \dots, N$. Define

$$T_j(x) = \begin{cases} T_j(x) & \text{if } 1 \le j < i \\ T_i(x) & \text{if } j = i \text{ and } x \in [a_{i-1}, c] \\ T_i(x) & \text{if } j = i+1 \text{ and } x \in [c, a_i] \\ T_{j-1}(x) & \text{if } i+1 < j \le N+1 \end{cases}$$

in other words, we split the i-th branch into two sub-branches at the point c. Then this new T with $N+1$ branches still satisfies the convexity condition **(C)** and generates the same operator P, although the pointwise representation (1.2) will be changed.

Let $\mathcal{J} = \{f : [0,1] \to [0,\infty) | f(x) \ge f(y) \text{ whenever } x \le y\}$ be the cone of nonnegative, decreasing functions on I. As a further consequence of the convexity condition **(C1)** we have

Lemma 2.2. *A necessary and sufficient condition for an operator of the form*

$$Pf(x) = \sum_{i=1}^{N} F_i(x)f(\psi_i(x)); \;\; F_i \ge 0$$

to satisfy $P : \mathcal{J} \to \mathcal{J}$ is that the F_i's satisfy condition **(C1)**.

Since $T(0) = 0$ is implied by condition **(C1)** with $k = 1$ the above lemma leads to

Corollary 2.3. *Convexity condition* **(C)** *is preserved under powers of T*

Proof of Lemma 2.2. Let $f \in \mathcal{J}$ and $x \leq y$. Define $x_i = \psi_i(x)$ and $y_i = \psi_i(y) \geq x_i$ for $i = 1, 2, \ldots, N$.

$$\begin{aligned} Pf(x) - Pf(y) &= \sum_{i=1}^{N} [F_i(x)f(x_i) - F_i(y)f(y_i)] \\ &\geq \sum_{i=1}^{N} (F_i(x) - F_i(y))f(x_i) \end{aligned} \tag{2.1}$$

since $F_i \geq 0$ and $f(x_i) \geq f(y_i)$.

Define the following N-dimensional vectors.

$$\begin{aligned} \overrightarrow{\Delta F} &= \langle F_i(x) - F_i(y)\rangle \in \mathbf{R}^N \\ \vec{f} &= \langle f(x_i)\rangle \in \text{interior of positive cone of } \mathbf{R}^N \\ \overrightarrow{b_j} &= \langle \underbrace{1, 1, \ldots, 1}_{j \text{ times}}, 0, 0, \ldots, 0\rangle;\ j = 1, 2, \ldots, N \end{aligned}$$

With this notation we rewrite (2.1) simply as

$$\overrightarrow{\Delta F} \cdot \vec{f}. \tag{2.2}$$

The convexity assumption **(C1)** implies

$$\overrightarrow{\Delta F} \cdot \overrightarrow{b_j} \geq 0;\ j = 1, 2, \ldots, N. \tag{2.3}$$

Furthermore,

$$\vec{f} = \sum_{j=1}^{N-1} (f(x_j) - f(x_{j+1}))\overrightarrow{b_j} + f(x_N)\overrightarrow{b_N}, \tag{2.4}$$

with all coefficients in this expression nonnegative since $f \in \mathcal{J}$. Using (2.3), (2.4), linearity in (2.2) implies $Pf \in \mathcal{J}$.

As for the converse, note that the inequality at (2.1) is sharp on $\mathcal{J}$, that is, if the F_i fail to satisfy the convexity assumption **(C1)**, then there exists an f in $\mathcal{J}$ with $Pf \notin \mathcal{J}$. □

If $f : [0, 1] \to \mathbf{R}$ we denote the variation of f by $\bigvee_I f$ and let BV denote the Banach space of bounded variation functions (with norm $\|f\|_{BV} = \bigvee_I f + \|f\|_1$). As in the classical situation, we seek a variational inequality for P in order to establish compactness of the sequence of iterates of a function in BV. We have been unable to prove such an inequality on all of BV, but following below is the inequality on the sub-cone $\mathcal{J} \subset BV$, and this will turn out to be sufficient for our purposes.

Lemma 2.4. *For each a satisfying $F_1(0) < a < 1$ there exists a constant $b = b(a) < \infty$ so that for all $f \in \mathcal{J}$,*

$$\bigvee_I Pf \le a \bigvee_I f + b\|f\|_1.$$

Proof. We first note that there exists a weak variational inequality: there exist positive constants $A, B < \infty$ such that

$$\bigvee_I Pf \le A \bigvee_I f + B\|f\|_1, \ \forall f \in \mathcal{J}.$$

For $f \in \mathcal{J}$,

$$\begin{aligned} Pf(0) - Pf(1) &= \sum_{i=1}^{N} F_i(0)f(a_{i-1}) - \sum_{i=1}^{N} F_i(1)f(a_i) \\ &\le (\sum_{i=1}^{N} F_i(0))f(0) - (\sum_{i=1}^{N} F_i(1))f(1) \\ &\stackrel{\text{def}}{=} \Gamma f(0) - \gamma f(1) \end{aligned}$$

Note, by condition **(C1)**, $\Gamma \ge \gamma$, and for $f \in \mathcal{J},\ f(1) \le \|f\|_1$ so we obtain

$$\bigvee_I Pf \le \Gamma \bigvee_I f + (\Gamma - \gamma)\|f\|_1.$$

This shows we may set $A = \Gamma$ and $B = \Gamma - \gamma$ in our first variational inequality. It also shows that the lemma is true on the the subspace $\{c1\}_{c\ge 0}$, so we may restrict our attention to $\mathcal{J} - \{c1\}_{c\ge 0}$.

Suppose the lemma is false. Then there is a number $\hat{a}$, with $F_1(0) < \hat{a} < 1$, and sequences $0 \ne f_n \in \mathcal{J},\ \bigvee_I f_n \ne 0$ and $A_n > 0,\ \lim_n A_n = +\infty$ satisfying

$$\frac{\bigvee_I Pf_n - \hat{a}\bigvee_I f_n}{\|f_n\|_1} = A_n. \tag{2.5}$$

Since the left hand side of (2.5) is invariant by $f \to cf;\ c > 0$, we may assume $\bigvee_I f_n = 1$ for all n. Using the weak variational inequality established above, we have

$$\frac{A + B\|f_n\|_1 - \hat{a}}{\|f_n\|_1} \ge A_n \to \infty \tag{2.6}$$

The left hand side above is uniformly bounded on sets where $\|f_n\|_1 \ge \delta > 0$, so we must have $\|f_n\|_1 \to 0$. Dropping to a subsequence if necessary, we may assume $f_n \to 0$ for $m-$a.e. $x \in I$. Choose $x_0 \in (0, a_1)$ satisfying $f_n(x_0) \to 0$. Let $\delta > 0$ be fixed, and pick n_0 so $f_{n_0}(x_0) < \delta$.

Then we have $f_{n_0}(a_i) < \delta,\ i = 1, 2, \ldots, N,\ f_{n_0}(0) < 1 + \delta$, and we may make the estimate.

$$\begin{aligned} Pf_{n_0}(0) - Pf_{n_0}(1) &\leq Pf_{n_0}(0) \\ &\leq \sum_{i=1}^{N} F_i(0) f_{n_0}(a_{i-1}) \\ &\leq F_1(0)(1+\delta) + (\sum_{i=2}^{N} F_i(0))\delta \\ &= F_1(0) + (\sum_{i=1}^{N} F_i(1))\delta \\ &< \hat{a} \end{aligned}$$

provided δ was chosen sufficiently small. Thus $\bigvee_I Pf_{n_0} < \hat{a} \bigvee_I f_{n_0}$, which contradicts the sign in (2.5). This completes the proof of the lemma. □

Proof of Theorem 1.1. Applying the previous lemma iteratively to the function 1 we obtain

$$\bigvee_I P^s 1 \leq b(1 + a + a^2 + \cdots + a^{s-1}) \leq b(1-a)^{-1}.$$

Now consider the sequence $g_n = \frac{1}{n} \sum_{s=0}^{n-1} P^s 1$. The following properties are now evident.

(1) $g_n \in \mathcal{J}$ and $\|g_n\|_1 = \int g_n = 1;\ n \geq 1$.
(2) $\|Pg_n - g_n\|_1 \overset{n\to\infty}{\to} 0$ since $\sup \|P^s 1\|_\infty < \infty$.
(3) The sequence $\{g_n\}$ is relatively compact in L^1 by Helley's Theorem.
(4) If $g \overset{L^1-\text{norm}}{=} \lim_k g_{n_k}$ for some subsequence, then $g \geq 0,\ Pg = g,$ and $\|g\|_1 = 1$.

Finally, since g may be obtained as the L^1 limit of a sequence of decreasing functions, we may, by an elementary argument, find a version of g which is decreasing. This completes the proof of the Theorem 1.1. □

3. Preliminaries about Cones and Norms Equivalent to $||\cdot||_{BV}$

Recall that $\mathcal{J} = \{f : [0,1] \to [0,\infty) | f \text{ is decreasing}\}$. Given a function f on $I = [0,1]$, we will simply denote $\bigvee_I f$ by $\bigvee f$. BV_0 denotes the (Banach) subspace of bounded variation functions which integrate to zero. We continue to reserve $||\cdot||$ for the L^1-norm $||\cdot||_1$. Of course for a given $f \in BV$, there exist $f^1, f^2 \in \mathcal{J}$ such that $f = f^1 - f^2$. In fact the following is also true.

Lemma 3.1. *Given $f \in BV$, there exist $f^1, f^2 \in \mathcal{J}$ such that*

(1) $f = f^1 - f^2$;
(2) $\bigvee f = \bigvee f^1 + \bigvee f^2$;

(3) *if* $f = g - h \in BV$ *and* $g, h \in \mathcal{J}$, *then* $\bigvee f^1 \le \bigvee g$ *and* $\bigvee f^2 \le \bigvee h$ *and* $\|f^1\| + \|f^2\| \le \|g\| + \|h\|$. *Furthermore, if* $g \ne f^1$ *(so* $h \ne f^2$*), then* $\|f^1\| + \|f^2\| < \|g\| + \|h\|$.

Proof. For each $x \in I$, let $T_f(x) = \bigvee_0^x f$. Define f^1 and f^2 by

$$f^1 = \frac{1}{2}(\bigvee f + |f(1)| + f - T_f) \quad \text{and} \quad f^2 = \frac{1}{2}(\bigvee f + |f(1)| - f - T_f).$$

so clearly $f = f^1 - f^2$. It is easy to check that f^1 and f^2 are decreasing. Since $f^1(1) = \frac{1}{2}\{|f(1)| + f(1)\} \ge 0$, we have $f^1 \ge 0$ and similarly $f^2 \ge 0$. Thus $f^1, f^2 \in \mathcal{J}$. Also

$$\begin{aligned} \bigvee f^1 + \bigvee f^2 &= f^1(0) - f^1(1) + f^2(0) - f^2(1) \\ &= \frac{1}{2}(f(0) - f(1) + \bigvee f) + \frac{1}{2}(f(1) - f(0) + \bigvee f) \\ &= \bigvee f. \end{aligned}$$

Suppose $f = g - h$ with $g, h \in \mathcal{J}$. Then

$$\begin{aligned} \bigvee g + \bigvee h &= \bigvee g + [h(0) - h(1)] \\ &= \bigvee g + [g(0) - f(0) - g(1) + f(1)] \\ &= 2\bigvee g - f(0) + f(1). \end{aligned}$$

In particular, $\bigvee f^1 + \bigvee f^2 = 2\bigvee f^1 - f(0) + f(1) = 2\bigvee f^1 - \bigvee g + \bigvee h$. This implies that $\bigvee f^1 \le \bigvee g$, since $\bigvee f^1 + \bigvee f^2 = \bigvee f \le \bigvee g + \bigvee h$. Similarly, we have $\bigvee f^2 \le \bigvee h$.

To see $\|f^1\| + \|f^2\| \le \|g\| + \|h\|$, notice that for each $x \in I$,

$$\begin{aligned} 0 \le \bigvee f - T_f(x) &= \bigvee_0^1 f - \bigvee_0^x f \le \bigvee_x^1 f \le \bigvee_x^1 g + \bigvee_x^1 h \\ &= g(x) - g(1) + h(x) - h(1) \end{aligned} \tag{3.1}$$

and hence

$$\bigvee f - \|T_f\| = \|\bigvee f - T_f\| \le \|g\| + \|h\| - g(1) - h(1).$$

Since $|f(1)| \le g(1) + h(1)$, it follows that

$$\begin{aligned} \|f^1\| + \|f^2\| &= \frac{1}{2}\Big[\int(\bigvee f + |f(1)| + f - T_f) + \int(\bigvee f + |f(1)| - f - T_f)\Big] \\ &= \bigvee f + |f(1)| - \|T_f\| \le \|g\| + \|h\|. \end{aligned} \tag{3.2}$$

Finally suppose $\|f^1\| + \|f^2\| = \|g\| + \|h\|$. It follows from (3.1) and (3.2) that $|f(1)| = g(1) + h(1)$, which implies that for each $x \in I$,

$$\begin{aligned} f^1(x) + f^2(x) &= \bigvee f + |f(1)| - T_f(x) \\ &= g(x) + h(x) - g(1) - h(1) + |f(1)| \\ &= g(x) + h(x), \end{aligned}$$

i.e., $f^1 + f^2 = g + h$. Since $f^1 - f^2 = g - h$, we have $g = f^1$ and $h = f^2$. □

Definition 3.1. For a given $f \in BV$, with f^1, f^2 defined as above, we will call $f = f^1 - f^2$ *the variational decomposition* of f.

Notice that $\mathcal{J}$ is an $\mathbb{R}^+$–module, i.e., if $f, g \in \mathcal{J}$, then $f + g \in \mathcal{J}$ and for any $c \in \mathbb{R}^+$, $cf \in \mathcal{J}$. We now introduce a class of submodules (or cones) of $\mathcal{J}$.

Definition 3.2. For a given $K > 0$, define $\mathcal{C}_K$ by

$$\mathcal{C}_K = \{f \in BV | f \in \mathcal{J} \text{ and } \bigvee f \leq K\|f\|\}.$$

From each $\mathcal{C}_K$, $K > 0$, we may construct a vector space Γ_K of functions on I via

$$\Gamma_K = \{f \in BV | \text{ there exist } f^1, f^2 \in \mathcal{C}_K \text{ such that } f = f^1 - f^2 \text{ and } \|f^1\| = \|f^2\|\}.$$

On Γ_K define $\| \cdot \|_{\Gamma_K}$ as follows.

$$\|f\|_{\Gamma_K} = \inf\{\|f^1\| \mid f = f^1 - f^2, \text{ where } f^1, f^2 \in \mathcal{C}_K \text{ and } \|f^1\| = \|f^2\|\}.$$

We collect some basic facts about these objects.

Lemma 3.2. *For each $K > 0$, the following hold.*

(1) Γ_K with $|| \cdot ||_{\Gamma_K}$ is a normed vector space. (In fact it is a Banach space.)

(2) $f \in \Gamma_K$ if and only if $f \in BV_0$.

(3) If $f = f^1 - f^2$ is the variational decomposition of $f \in BV_0$, then there exists $c \geq 0$ such that $f^1 + c, f^2 + c \in \mathcal{C}_K$ and $\|f\|_{\Gamma_K} = \|f^1 + c\|$.

(4) For a given $f \in BV_0$,

$$\min\{1, K\}\|f\|_{\Gamma_K} \leq \|f\|_{BV} \leq 2(K + 1)\|f\|_{\Gamma_K}.$$

In particular, all the norms $\| \cdot \|_{\Gamma_K}$ are equivalent, and equivalent to $|| \cdot ||_{BV}$ on BV_0.

Proof. (1) It is an elementary check that Γ_K is a vector space and that $|| \cdot ||_{\Gamma_K}$ is a norm on it. Completeness follows from (2) and (4) below.

(2) If $f \in \Gamma_K$, then by definition there exist $f^1, f^2 \in \mathcal{C}_K$ such that $f = f^1 - f^2$ and $\|f^1\| = \|f^2\|$. Since $\int f = \|f^1\| - \|f^2\| = 0$, it follows that $f \in BV_0$.

Conversely, suppose $f \in BV_0$ and $f = f^1 - f^2$ is the variational decomposition of f. Since $0 = \int f = \|f^1\| - \|f^2\|$, we have $\|f^1\| = \|f^2\|$. Let

$$\alpha = \frac{1}{K} \max\{\bigvee f^1, \bigvee f^2\} - \|f^1\|.$$

If $\alpha < 0$, then $\bigvee f^1 \le K\|f^1\|$ and $\bigvee f^2 \le K\|f^1\| = K\|f^2\|$, so that $f^1, f^2 \in \mathcal{C}_K$. Thus $f \in \Gamma_K$. If $\alpha \ge 0$, then it is easy to check that $f^1+\alpha, f^2+\alpha \in \mathcal{C}_K$. Since $f = (f^1+\alpha)-(f^2+\alpha)$ and $\|f^1 + \alpha\| = \|f^2 + \alpha\|$, it follows that $f \in \Gamma_K$.

(3) Let $f \in BV_0$ and $f = f^1 - f^2$ the variational decomposition of f. Let α be given as above and $c = \max\{\alpha, 0\}$. It follows from the proof of (2) that $f^1 + c, f^2 + c \in \mathcal{C}_K$. If $\alpha < 0$, i.e., $c = 0$, then it is clear that $\|f\|_{\Gamma_K} = \|f^1\| = \|f^1 + c\|$ (see Lemma 3.1). Now suppose $c = \alpha \ge 0$. If $f = g - h$ and $g, h \in \mathcal{C}_K \subset \mathcal{J}$, then it follows from Lemma 3.1 that $\bigvee f^1 \le \bigvee g$ and $\bigvee f^2 \le \bigvee h$. Thus

$$\begin{aligned} \|f^1 + c\| = \|f^1\| + c = \frac{1}{K} \max\{\bigvee f^1, \bigvee f^2\} &\le \frac{1}{K} \max\{\bigvee g, \bigvee h\} \\ &\le \frac{1}{K} \max\{K\|g\|, K\|h\|\} = \|g\|. \end{aligned}$$

Since $f^1 + c, f^2 + c \in \mathcal{C}_K$, it follows that $\|f\|_{\Gamma_K} = \|f^1 + c\|$. Note that there is at most one value of c which can satisfy the previous identity.

(4) Let $f = f^1 - f^2$ be the variational decomposition of f and let c be as in (3) so that $\|f\|_{\Gamma_K} = \|f^1 + c\|$. Assume that $c = \alpha \ge 0$ and make the estimate

$$\begin{aligned} \|f^1 + c\| = \|f^1\| + c &= \frac{1}{K} \max\{\bigvee f^1, \bigvee f^2\} \\ &\le \frac{1}{K} \bigvee f \\ &\le \frac{1}{K} ||f||_{BV}. \end{aligned}$$

On the other hand, if $c = 0$ and $\alpha < 0$ we proceed as follows. Notice that either $f^1(1) = 0$ or $f^2(1) = 0$ for otherwise we could subtract a small multiple of the identity from both, contradicting Lemma 3.1 (3). Assuming the first case (the other is identical) estimate

$$\begin{aligned} \|f^1 + c\| = \|f^1\| &\le ||f||_\infty \\ &\le \bigvee f^1 \\ &\le \bigvee f \\ &\le ||f||_{BV}. \end{aligned}$$

This proves the first inequality in (4). For the other inequality, by the proof of (3) with c defined as above

$$\begin{aligned}\|f\|_{BV} = \bigvee f + \|f\| &= \bigvee (f^1 + c) + \bigvee (f^2 + c) + \|f\| \\ &\le K\|f^1 + c\| + K\|f^2 + c\| + \|f^1\| + \|f^2\| \\ &\le 2(K+1)\|f^1 + c\| \\ &= 2(K+1)\|f\|_{\Gamma_K}.\end{aligned}$$

This completes the proof. □

Remark 3.1. We remark that the above construction follows closely the setup in [8], although our choice of the basic cones $\mathcal{C}_K$ is different, leading to some changes in the proofs and to some of the constants in the estimates.

Using Lemma 2.4, choose a and $b < \infty$ so that $F_1(0) < a < 1$ and for any $f \in \mathcal{J}$, $\bigvee Pf \le a \bigvee f + b\|f\|$. It inductively follows that for any given $f \in \mathcal{J}$ and for each $m \ge 1$,

$$\bigvee P^m f \le a^m \bigvee f + \frac{b(1-a^m)}{1-a}\|f\|. \tag{3.3}$$

Lemma 3.3. *For a given $K \ge b/(1-a)$, P preserves $\mathcal{C}_K$, i.e., P maps $\mathcal{C}_K$ into $\mathcal{C}_K$.*

Proof. Let $K \ge b/(1-a)$. If $f \in \mathcal{C}_K$, then

$$\begin{aligned}\bigvee Pf \le a \bigvee f + b\|f\| &\le aK\|f\| + (1-a)K\|f\| \\ &= K\|f\| = K\|Pf\|,\end{aligned}$$

which shows $Pf \in \mathcal{C}_K$. □

Remark 3.2. Let S_1 denote the unit sphere in L^1. Each subset $\mathcal{C}_K \cap S_1$ is a convex and compact subset of L^1. Using Lemma 3.3 and the Markov property for P, for all sufficiently large K each of these subsets is preserved by P. By the Schauder–Tychonov Theorem, P will have a fixed point in $\mathcal{C}_K \cap S_1$. This gives another proof of the existence of a decreasing invariant probability density as was already derived at the end of §2.

4. Ergodic Properties of an Invariant Measure: Case I

The techniques developed in the previous section will now be used to study the question of ergodic properties. Throughout this section T is assumed to satisfy the conditions of Theorem 1.1. For each $k = 1, \cdots, N$, let $\mathcal{F}_k$ denote $\mathcal{F}_k = \sum_{i=1}^{k} F_i$. From Lemma 2.1, without loss of generality, we may assume that for each $k = 1, \cdots, N$, $\mathcal{F}_k$ is upper semicontinuous. Also, for a given closed interval $[c, d] \subseteq [0, 1]$ $(c < d)$, we define $T[c, d]$ to be

$$T[c, d] = \overline{T(c, d)} = \bigcup_{j=1}^{N} T_j\big(\overline{[a_{j-1}, a_j] \cap (c, d)}\big).$$

Consequently, $T[c,d]$ is a finite union of non-trivial closed intervals. Under this notation, the following holds.

Lemma 4.1. *For each $k = 0, 1, 2 \ldots$, $T^k[0, a_1]$ is a closed interval containing $[0, a_1]$. Moreover, $T^k[0, a_1] \subseteq T^{k+1}[0, a_1]$ so that we have an increasing sequence of closed intervals.*

Proof. The first statement is obviously true for $k = 0$. Suppose $T^k[0, a_1] = [0, b_k]$. Choose $l \geq 1$ such that $a_{l-1} < b_k \leq a_l$. Then

$$T^{k+1}[0, a_1] = T_1[0, a_1] \cup \cdots \cup T_{l-1}[a_{l-2}, a_{l-1}] \cup T_l[a_{l-1}, b_k],$$

where each of these sets is a closed interval. If the union is not connected, then there exists an interval (c, d) with $d < \sup\{T(x)|x \in [0, b_k]\}$ and $\mathcal{F}_l \equiv 0$ on (c, d) since the $F_i = 0$ almost everywhere on this interval for $1 \leq i \leq l$ and the sum $\mathcal{F}_l$ is upper semicontinuous. But also, there exists a point $y > d$ such that $\mathcal{F}_l(y) > 0$ which contradicts our convexity condition **(C1)**. So $T^{k+1}[0, a_1]$ is an interval. Now note that since T_1 is convex and $F_1(0) < 1$ we have $[0, a_1] \subseteq T_1[0, a_1]$, so $T^{k+1}[0, a_1] \supseteq [0, a_1]$. Finally, since $[0, a_1] \subseteq T[0, a_1]$, it follows that for all k, $T^k[0, a_1] \subseteq T^{k+1}[0, a_1]$. □

Figure 4.1

Let $\beta \in (0, 1]$ be determined by $\bigcup_{k=0}^{\infty} T^k[0, a_1] = [0, \beta]$. Then, by Lemma 4.1, $a_1 < \beta \leq 1$ and $T[0, \beta] = [0, \beta]$. Following Remark 2.2, without loss of generality, we may assume that $\beta = a_{N_*}$ for some $N_* \in \{2, \cdots, N\}$. Then $\mathcal{F}_{N_*}(x) = 0$ on $(\beta, 1]$ and for any $c \in (0, \beta)$, $T[0, c] \nsubseteq [0, c]$, which leads to the following (see Figure 4.1).

Lemma 4.2. *For each $c \in (0, \beta)$, $T[c, \beta] \nsubseteq [c, \beta]$.*

Proof. Suppose $0 < c < \beta$ and $T[c,\beta] \subseteq [c,\beta]$. Again, in view of Remark 2.2, without loss of generality, we may assume that $c = a_s$ for some $s, 1 \le s < N_*$. It follows that $\sum_{i=s+1}^{N_*} F_i(x) = 0$ on $[0,c)$, i.e., $(\mathcal{F}_{N_*} - \mathcal{F}_s)(x) = 0$ on $[0,c)$. Also

$$c = \int_0^1 P\chi_{[0,c]}(t)dt = \int_0^1 \mathcal{F}_s(t)dt \ge \int_0^c \mathcal{F}_s(t)dt \ge c \cdot \mathcal{F}_s(c),$$

which means $\mathcal{F}_s(c) \le 1$. Since $\mathcal{F}_s(x) = \mathcal{F}_{N_*}(x)$ on $[0,c)$, this implies that $\lim_{x\to c^-} \mathcal{F}_{N_*}(x) \le 1$, and so $\mathcal{F}_{N_*}(c) \le 1$. Thus

$$\int_c^\beta \mathcal{F}_{N_*}(t)dt \le (\beta - c)\cdot \mathcal{F}_{N_*}(c) \le \beta - c.$$

Meanwhile,

$$\begin{aligned}\beta - c &= \int_0^1 P\chi_{[c,\beta]}(t)dt = \int_0^1 (\mathcal{F}_{N_*} - \mathcal{F}_s)(t)dt \\ &= \int_c^\beta (\mathcal{F}_{N_*} - \mathcal{F}_s)(t)dt \le \int_c^\beta \mathcal{F}_{N_*}(t)dt.\end{aligned}$$

Therefore $\int_c^\beta \mathcal{F}_{N_*}(t)dt = \beta - c$ and $\int_c^\beta \mathcal{F}_s(t)dt = 0$. Then $T[0,c] \subseteq [0,c]$, which is a contradiction. □

Lemma 4.3. *There exists an integer $r \ge 1$ such that $\bigcup_{k=0}^{r} T^k[0,a_1] = [0,\beta]$.*

Proof. Let $d = \max\{T_i(a_i) | 1 \le i \le N_* - 1\} (\le \beta)$. Then $d > a_{N_*-1}$. For, otherwise, $\bigcup_{k=0}^{\infty} T^k[0,a_1] \subseteq [0,a_{N_*-1}] \subsetneq [0,\beta]$, which is a contradiction. We claim $d = \beta$, in which case it is easy to see that there exists $r \ge 1$ such that $\bigcup_{k=0}^{r} T^k[0,a_1] = [0,\beta]$. To prove the claim, first suppose $T_{N_*}(d) > d$. Then $T[d,\beta] \subseteq [d,\beta]$ and so by Lemma 4.2, $d = \beta$. If $T_{N_*}(d) \le d$, then $T[0,d] \subseteq [0,d]$, which implies $d = \beta$. □

Lemma 4.4. *Let $g \in \mathcal{J}$ be an invariant density for T and $A = \int_0^\beta g dm$. Define $g_\beta : [0,1] \to \mathbb{R}^+$ by*

$$g_\beta(x) = \begin{cases} g(x)/A & \text{if } 0 \le x \le \beta \\ 0 & \text{if } \beta < x \le 1. \end{cases}$$

Then $g_\beta \in \mathcal{J}$ is an invariant density of T, i.e., $Pg_\beta = g_\beta$.

Proof. Let $g_1 = g \cdot \chi_{[0,\beta]}$ and $g_2 = g - g_1$. Then $g_1 + g_2 = Pg_1 + Pg_2$. Since $T[0,\beta] \subseteq [0,\beta]$, we have $g_1 \ge Pg_1$. From the fact that $\|Pg_1\| = \|g_1\|$, it follows that $Pg_1 = g_1$. Thus $g_\beta = g_1/\|g_1\|_1 \in \mathcal{J}$ is an invariant density of T. □

Remark 4.1. For the rest of this and the following section, we will study the ergodic properties of $(T, g_\beta dm)$. In effect we will study $T_\beta = T|_{[0,\beta]}$ an N_*-branched convex map on $[0, \beta]$. The Perron Frobenius operator P_β on $L^1(\beta) = L^1([0, \beta])$ is defined according to (1.2) using T_β. The connection between the two operators follows

$$P_\beta = P|_{L^1(\beta)}$$

This is easily seen from the representation (1.2). Similarly, we will adopt the notation $BV(\beta)$, $BV_0(\beta)$, $\mathcal{J}(\beta)$ and $\mathcal{C}_K(\beta), K > 0$, for each $K > 0$ to denote the function spaces and cones on the restricted domain $[0, \beta]$. For example it easily follows from Lemma 3.3 that for a given $K \geq b/(1-a)$, P_β preserves $\mathcal{C}_K(\beta)$, i.e., P_β maps $\mathcal{C}_K(\beta)$ into $\mathcal{C}_K(\beta)$.

Recall that the condition **(C2)** implies only that T_β is uniformly expanding on $[0, a_1]$, however, it need not be the case that T_β (or even some power of it) must be uniformly expanding on $[0, \beta]$. Suprisingly, if in addition, T_β is assumed to be expanding at β, then we will prove that some power of T_β is uniformly expanding. This motivates the terminology in the following:

We say T_β satisfies the expanding condition **(E)** if

(E) $F_{N_*}(\beta^-) < 1$.

We say (T, μ) has *exponential decay of correlations* if there exist $C < \infty$ and $\lambda < 1$ such that for any $h \in L^1(\mu)$, $f \in BV(\mu)$, and for each $k \geq 1$,

$$\Big| \int (h \circ T^k) f d\mu - \int h d\mu \int f d\mu \Big| \leq C \cdot \lambda^k \cdot \|h\|_1 \cdot \|f\|_{BV}.$$

We first show the following.

Theorem 4.5. *Suppose T_β satisfies the conditions in Theorem 1.1 and the expanding condition* **(E)***. Let g_β be an invariant density of T_β as defined in Lemma 4.4. Then $(T_\beta, g_\beta dm)$ is exact. Moreover it has exponential decay of correlations.*

The proof of Theorem 4.5 results from a series of lemmas which will be proved later. Similar methods may be found in the work of Bowen [1].

To simplify notation, and in view of the above correspondences, we will generally refrain from including the subscript β from T and P with the underlying assumption in this and the next section that the domain has been restricted to $[0, \beta]$.

Lemma 4.6. *Let $r \geq 1$ be given as in Lemma 4.3. Let $K > 0$ be given and choose $s \geq 0$ so that $\psi_1^s(a_1) < 1/K$. Then the following hold.*

(1) For each $x \leq a_1$, there is $\lambda_0 = \lambda_0(x) > 0$ such that for any $f \in \mathcal{C}_K$, $(P^s f)(x) \geq \lambda_0 \|f\|$.

(2) For each $x < \beta$, there is $\delta_0 = \delta_0(x) > 0$ such that for any $f \in \mathcal{C}_K$, $(P^{r+s} f)(x) \geq \delta_0 \|f\|$.

Lemma 4.7. *Let $K > b/(1-a)$. Then there exist $l = l(K) \geq 1$ and $h \in \mathcal{J}(\beta)$ such that $\int h > 0$ and for any $f \in \mathcal{C}_K(\beta)$, $P^l f - \|f\| \cdot h \in \mathcal{J}(\beta)$.*

Lemma 4.8. *Let $K > b/(1-a)$. Then there exist $n = n(K) \geq 1$ and $\hat{h} \in \mathcal{J}(\beta)$ such that $\int \hat{h} > 0$ and for any $f \in \mathcal{C}_K(\beta)$, $P^n f - \|f\| \cdot \hat{h} \in \mathcal{C}_K(\beta)$.*

Proposition 4.9. *Let $K > b/(1-a)$. Then there exist $n = n(K) \geq 1$ and $\delta = \delta(K) > 0$ such that for any given $f \in BV_0(\beta)$, for each $k \geq 1$,*

$$\|P^{kn} f\|_{\Gamma_K} \leq (1-\delta)^k \|f\|_{\Gamma_K}.$$

Proof of Theorem 4.5. Fix $K > b/(1-a)$. It follows from Proposition 4.9 that there exist $n = n(K) \geq 1$ and $\delta = \delta(K) > 0$ such that for any given $f \in BV_0(\beta)$, for each $k \geq 1$,

$$\|P^{kn} f\|_{\Gamma_K} \leq (1-\delta)^k \|f\|_{\Gamma_K}.$$

Since the right hand side converges to 0 as $k \to \infty$, the left hand side converges to 0 as $k \to \infty$ and so by Lemma 3.2 (4) in $\|\cdot\|_{BV}$, which implies

$$\|P^{kn} f\|_1 \to 0 \quad \text{as } k \to \infty.$$

Since P is a contraction in $\|\cdot\|_1$,

$$\|P^m f\|_1 \to 0 \quad \text{as } m \to \infty.$$

This will be enough for the exactness of the a.c.i.m. (see, for example [5] where the term **asymptotic stability** is used). In fact, let $\phi \in BV(\beta)$ and $\phi \geq 0$. Noticing that $f = \phi - (\int \phi) g_\beta \in BV_0(\beta)$, we have $\|P^m f\|_1 \to 0$ as $m \to \infty$, i.e., $\lim_{m\to\infty} P^m \phi = (\int \phi) g_\beta$ in L_1. Therefore, $(T, g_\beta dm)$ is exact.

To prove the second statement, fix $K > b/(1-a)$. Notice that P is also a contraction in $\|\cdot\|_{\Gamma_K}$. It follows from Lemma 3.2 and Proposition 4.9 that there exist $C_1 = C_1(K) < \infty$ and $\lambda = \lambda(K) < 1$ such that for any given $f \in BV_0(\beta)$, for each $k \geq 1$,

$$\|P^k f\|_{BV} \leq C_1 \cdot \lambda^k \cdot \|f\|_{BV}.$$

Let $h \in L^1$ and $f \in BV$. Then with $d\mu = g_\beta dm$, $\hat{f} = f - \int f d\mu \in BV$ and $\hat{f} \cdot g_\beta \in BV_0(\beta)$. Since $\bigvee(\hat{f} \cdot g_\beta) \leq ||g_\beta||_\infty \bigvee(f)$ and $||\hat{f} \cdot g_\beta|| \leq ||g_\beta||_\infty ||\hat{f}|| \leq ||g_\beta||_\infty(||f|| + ||g_\beta||_\infty ||f||) = ||g_\beta||_\infty(1+||g_\beta||_\infty)||f||$, we have $||\hat{f} \cdot g_\beta||_{BV} \leq (1+||g_\beta||_\infty)||g_\beta||_\infty ||f||_{BV}$. Thus for each $k \geq 1$,

$$\begin{aligned}
&\Big|\int (h \circ T^k) f d\mu - \int h d\mu \int f d\mu\Big| \\
&\qquad = \Big|\int (h \circ T^k) \hat{f} d\mu\Big| = \Big|\int h \cdot P^k(\hat{f} \cdot g_\beta) dm\Big| \\
&\qquad \leq \|h\|_1 \cdot \|P^k(\hat{f} \cdot g_\beta)\|_\infty \leq \|h\|_1 \cdot \|P^k(\hat{f} \cdot g_\beta)\|_{BV} \\
&\qquad \leq \|h\|_1 \cdot C_1 \cdot \lambda^k \cdot \|\hat{f} \cdot g_\beta\|_{BV} \\
&\qquad \leq \|h\|_1 \cdot C_1 \cdot \lambda^k \cdot (1 + ||g_\beta||_\infty) \cdot \|g_\beta\|_\infty \cdot \|f\|_{BV} \\
&\qquad \leq C \cdot \lambda^k \cdot \|h\|_1 \cdot \|f\|_{BV}
\end{aligned}$$

where $C = C_1 \cdot (1 + ||g_\beta||_\infty) \cdot \|g_\beta\|_\infty < \infty$. Therefore $(T, g_\beta dm)$ has exponential decay of correlations. □

Now in order to prove Lemma 4.6, we will use the following notations. For each $n \geq 1$, $\{a_i^{(n)}\}_{i=0}^{N_n}$ denotes the partition for T^n; for each $i = 1, \cdots, N_n$, $T_i^{(n)}$, $\psi_i^{(n)}$, and $F_i^{(n)}$ are similarly defined.

Proof of Lemma 4.6. (1) Let $y < 1/K$. We will first show that there exists $\epsilon = \epsilon(y) > 0$ such that for any $f \in \mathcal{C}_K$,

$$f(y) \geq \epsilon \|f\|. \tag{4.1}$$

In fact, define $\epsilon = \epsilon(y)$ by

$$\epsilon = \frac{1 - Ky}{1 + K(1 - y)} > 0.$$

Then for a given $f \in \mathcal{C}_K$,

$$\begin{aligned} f(0) - f(y) &\leq f(0) - f(1) = \bigvee f \leq K\|f\| \\ &\leq K[f(0)y + f(y)(1 - y)] \end{aligned}$$

which leads to

$$[1 + K(1 - y)]f(y) \geq (1 - Ky)f(0) \geq (1 - Ky)\|f\|,$$

i.e., $f(y) \geq \epsilon\|f\|$.

Let $x \leq a_1$ be given. Then $F_1^{(s)}(x) > 0$ and $\psi_1^{(s)}(x) \leq \psi_1^{(s)}(a_1) < 1/K$. Let $\lambda_0 = \lambda_0(x) = F_1^{(s)}(x) \cdot \epsilon(\psi_1^{(s)}(x)) > 0$. It follows from (4.1) that for any $f \in \mathcal{C}_K$,

$$\begin{aligned} (P^s f)(x) &= \sum_{i=1}^{N_s} F_i^{(s)}(x) f(\psi_i^{(s)}(x)) \\ &\geq F_1^{(s)}(x) f(\psi_1^{(s)}(x)) \\ &\geq F_1^{(s)}(x) \cdot \epsilon(\psi_1^{(s)}(x))\|f\| \\ &= \lambda_0\|f\|. \end{aligned}$$

(2) Note that for any $x < \beta$, $F_1^{(r)}(x) > 0$. Fix $x < \beta$. It follows from (1) that $\lambda_0 = \lambda_0(\psi_1^{(r)}(x)) > 0$ is well defined, since $\psi_1^{(r)}(x) \leq a_1$. Let $\delta_0 = \delta_0(x) = F_1^{(r)}(x) \cdot \lambda_0(\psi_1^{(r)}(x)) > 0$.

From (1), for any given $f \in \mathcal{C}_K$,

$$\begin{aligned}(P^{r+s}f)(x) &= \sum_{i=1}^{N_r} F_i^{(r)}(x)(P^s f)(\psi_i^{(r)}(x)) \\ &\geq F_1^{(r)}(x)(P^s f)(\psi_1^{(r)}(x)) \\ &\geq F_1^{(r)}(x) \cdot \lambda_0(\psi_1^{(r)}(x))\|f\| \\ &= \delta_0\|f\|,\end{aligned}$$

which completes the proof. □

Proof of Lemma 4.7. Given a function ϕ on I, for each $x \in (0,1)$, $\phi(x^-)$ will denote $\lim\limits_{t\to x^-} \phi(t)$ provided the limit exists; similarly for $\phi(x^+)$. Let r, s be given as in Lemma 4.6 and let $l = 1 + r + s \geq 2$.

The first guess for a choice of h would be $h = ||f|| \cdot P\chi_{[0,\beta]}$ for then $h \in \mathcal{J}$ and $P^l f - h = P(P^{l-1}f - ||f||\chi_{[0,\beta]})$ which is decreasing on $[0,\beta]$. However, it is not the case that we can always choose l large enough, independent of f such that this function is positive on $[0,\beta]$. A slight modification is required.

We break the analysis into three cases.

Case 1. $d \equiv T(\beta^-) < \beta$.

By Lemma 4.6 (2), $\delta_0 = \delta_0(a_{N_*-1}) > 0$ is well defined. Noticing that $(P\chi_{[0,\beta]})(d^+) = \mathcal{F}_{N_*-1}(d^+) > 0$, let $t = \delta_0 \cdot (P\chi_{[0,\beta]})(d^+) > 0$ and define $h \in \mathcal{J}(\beta)$ by

$$h = t \cdot \chi_{[0,d]} + \delta_0 \cdot P\chi_{[0,\beta]} \cdot \chi_{(d,\beta]}.$$

Then $\int h > 0$. For a given $f \in \mathcal{C}_K(\beta)$, let $\hat{f} = P^{r+s}f$ and observe

$$(P\hat{f} - \|f\| \cdot h)(x) = \begin{cases} (P\hat{f})(x) - t\|f\| & \text{if } 0 \leq x \leq d \\ (P\hat{f})(x) - \delta_0\|f\|(P\chi_{[0,\beta]})(x) & \text{if } d < x \leq \beta \\ 0 & \text{if } \beta < x \leq 1. \end{cases}$$

It is clear that $P\hat{f} - \|f\| \cdot h$ is decreasing on $[0,d]$. Let $d < x \leq \beta$. For each $i = 1, \cdots, N_* - 1$,

$$\hat{f}(\psi_i(x)) \geq \hat{f}(\psi_{N_*-1}(x)) \geq \hat{f}(a_{N_*-1}) \geq \delta_0\|f\|.$$

Thus

$$(P\hat{f} - \|f\| \cdot h)(x) = \sum_{i=1}^{N_*-1} F_i(x)[\hat{f}(\psi_i(x)) - \delta_0\|f\|] \geq 0$$

and it is decreasing on $(d,\beta]$. Using the fact that h is continuous at d, we conclude that $P^l f - \|f\| \cdot h \geq 0$ is decreasing on $[0,1]$, i.e., $P^l f - \|f\| \cdot h \in \mathcal{J}(\beta)$.

Case 2. $T(\beta^-) = \beta$ and $P\chi_{[0,\beta]} \neq \chi_{[0,\beta]}$.

Notice that there exist c, d, $c < d < \beta$, such that $(P\chi_{[0,\beta]})(c^+) > (P\chi_{[0,\beta]})(d^-)$. Since $\psi_{N_*}(d^+) < \beta$, it follows from Lemma 4.6 (2) that $\delta_0 = \delta_0(\psi_{N_*}(d^+)) > 0$ is well defined. Let

$$t = \delta_0[(P\chi_{[0,\beta]})(c^+) - (P\chi_{[0,\beta]})(d^-)] > 0$$

and define $h \in \mathcal{J}(\beta)$ by

$$h = t \cdot \chi_{[0,c]} + \delta_0 \cdot [P\chi_{[0,\beta]} - (P\chi_{[0,\beta]})(d^-)] \cdot \chi_{(c,d]}.$$

Then $\int h > 0$. Let $f \in \mathcal{C}_K(\beta)$ and $\hat{f} = P^{r+s}f$. Letting $s = \delta_0 \cdot P\chi_{[0,\beta]}(d^-) \cdot \|f\|$, we have

$$(P\hat{f} - \|f\| \cdot h)(x) = \begin{cases} (P\hat{f})(x) - t\|f\| & \text{if } 0 \le x \le c \\ (P\hat{f})(x) - \delta_0\|f\|(P\chi_{[0,\beta]})(x) + s & \text{if } c < x \le d \\ (P\hat{f})(x) & \text{if } d < x \le 1. \end{cases}$$

Observe that $P\hat{f} - \|f\| \cdot h$ is decreasing on each of the three intervals $[0, c]$, $(c, d]$ and $(d, 1]$. Since h is continuous at c and d, then $P\hat{f} - \|f\| \cdot h$ is decreasing on $[0, 1]$. Since $(P\hat{f} - \|f\| \cdot h)(1) = (P\hat{f})(1) \ge 0$, it follows that $P^l f - \|f\| \cdot h \in \mathcal{J}(\beta)$.

Case 3. $T(\beta^-) = \beta$ and $P\chi_{[0,\beta]} = \chi_{[0,\beta]}$ (and $F_{N_*}(\beta) < 1$).

Since $a_{N_*-1} < \beta$, it follows from Lemma 4.6 (2) that $\delta_0 = \delta_0(a_{N_*-1}) > 0$ is well defined. Let $t = (1 - F_{N_*}(\beta)) \cdot \delta_0 > 0$ and define $h \in \mathcal{J}(\beta)$ by $h = t \cdot \chi_{[0,\beta]}$. Then $\int h > 0$. We show that for a given $f \in \mathcal{C}_K(\beta)$, $P^l f - \|f\| \cdot h \in \mathcal{J}(\beta)$. Let $\hat{f} = P^{r+s}f$. It is clear that $P\hat{f} - \|f\| \cdot h$ is decreasing. Using the fact that $\mathcal{F}_{N_*}(\beta) = 1$, we have

$$\begin{aligned} (P\hat{f})(\beta) &= \sum_{i=1}^{N_*} F_i(\beta)\hat{f}(\psi_i(\beta)) \ge \sum_{i=1}^{N_*-1} F_i(\beta)\hat{f}(\psi_{N_*-1}(\beta)) \\ &\ge \sum_{i=1}^{N_*-1} F_i(\beta) \cdot \hat{f}(a_{N_*-1}) \\ &\ge (1 - F_{N_*}(\beta)) \cdot \delta_0\|f\| = t\|f\|. \end{aligned}$$

Thus $P^l f - \|f\| \cdot h \in \mathcal{J}(\beta)$. ☐

Proof of Lemma 4.8. Let $l = l(K) \ge 1$ and $h \in \mathcal{J}(\beta)$ be given as in Lemma 4.7. Choose $\delta > 0$ so that

$$0 < \delta < \frac{K - b/(1-a)}{K + b/(1-a)} < 1.$$

Note that $K > b(1+\delta)/[(1-\delta)(1-a)] > b(1+\delta)/(1-a)$. Let $\epsilon = \bigvee h \ge 0$ and $\lambda = \min\{\delta/(\int h), 1\} > 0$. Since $a < 1$ one may choose $m \ge 0$ so that

$$m \ge \log_a \Big[\frac{K(1-\delta)(1-a) - b(1+\delta)}{(K+\lambda\epsilon)(1-a) - b(1+\delta)}\Big].$$

A simple computation shows

$$0 < \Big(K + \lambda\epsilon - \frac{b(1+\delta)}{1-a}\Big)a^m + \frac{b(1+\delta)}{1-a} \le K(1-\delta). \tag{4.2}$$

Let $n = m+l \ge 1$ and $\hat{h} = P^m(\lambda \cdot h) \in \mathcal{J}(\beta)$. For a given $f \in \mathcal{C}_K(\beta)$, let $\zeta = P^n f - \|f\| \cdot \hat{h}$ and $\phi = \lambda\|f\| \cdot h \in \mathcal{J}$. Note that

$$\begin{aligned}\zeta &= P^m(P^l f - \lambda\|f\| \cdot h)\\ &= P^m(P^l f - \|f\| \cdot h) + (1-\lambda)P^m(\|f\| \cdot h)\end{aligned}$$

from which $\zeta \in \mathcal{J}$, since $P^l f - \|f\| \cdot h \in \mathcal{J}$ and $\|f\| \cdot h \in \mathcal{J}$. Also $\|\phi\| \le \delta\|f\|$. Thus

$$\begin{aligned}\|\zeta\| &= \|P^m(P^l f - \lambda\|f\| \cdot h)\| = \|P^n f\| - \|P^m\phi\|\\ &= \|f\| - \|\phi\| \ge (1-\delta)\|f\|.\end{aligned}$$

Using (3.3) and (4.2), observe

$$\begin{aligned}\bigvee \zeta &\le \bigvee P^m(P^l f) + \bigvee P^m\phi\\ &\le \Big(a^m \bigvee P^l f + \frac{b(1-a^m)}{1-a}\|P^l f\|\Big) + \Big(a^m \bigvee \phi + \frac{b(1-a^m)}{1-a}\|\phi\|\Big)\\ &\le \Big[a^m K + \frac{b(1-a^m)}{1-a} + a^m\lambda\epsilon + \frac{b(1-a^m)}{1-a}\delta\Big]\|f\|\\ &= \Big[\Big(K + \lambda\epsilon - \frac{b(1+\delta)}{1-a}\Big)a^m + \frac{b(1+\delta)}{1-a}\Big]\|f\|\\ &\le K(1-\delta)\|f\|\\ &\le K\|\zeta\|.\end{aligned}$$

Clearly $\zeta(x) = 0$ on $(\beta, 1]$. Therefore, $\zeta = P^n f - \|f\| \cdot \hat{h} \in \mathcal{C}_K(\beta)$. □

Proof of Proposition 4.9. Fix $K > b/(1-a)$. Let $n = n(K) \ge 1$ and $\hat{h} \in \mathcal{J}(\beta)$ be given as in Lemma 4.8. Let $\delta = \int \hat{h} > 0$. Suppose $f = f^1 - f^2 \in BV_0(\beta)$, where $f^1, f^2 \in \mathcal{C}_K(\beta)$ and $\|f\|_{\Gamma_K} = \|f^1\| = \|f^2\|$ (such f^1, f^2 exist from Lemma 3.1). It follows from Lemma 4.8 that $P^n f^1 - \|f^1\| \cdot \hat{h} \in \mathcal{C}_K(\beta)$ and $P^n f^2 - \|f^2\| \cdot \hat{h} \in \mathcal{C}_K(\beta)$. Note that $P^n f = (P^n f^1 - \|f^1\| \cdot \hat{h}) - (P^n f^2 - \|f^2\| \cdot \hat{h})$ and $\|P^n f^1 - \|f^1\| \cdot \hat{h}\| = \|P^n f^2 - \|f^2\| \cdot \hat{h}\|$. Thus

$$\begin{aligned}\|P^n f\|_{\Gamma_K} &\le \|P^n f^1 - \|f^1\| \cdot \hat{h}\|\\ &= (1-\delta)\|f^1\| = (1-\delta)\|f\|_{\Gamma_K}.\end{aligned}$$

Since $P^n f \in BV_0(\beta)$, it follows that

$$\|P^n(P^n f)\|_{\Gamma_K} \le (1-\delta)\|P^n f\|_{\Gamma_K} \le (1-\delta)^2\|f\|_{\Gamma_K}.$$

Repeating this process, we obtain that for each $k \ge 1$,

$$\|P^{kn} f\|_{\Gamma_K} \le (1-\delta)^k\|f\|_{\Gamma_K}.$$

□

We will show some other ergodic properties of $(T, g_\beta dm)$ and use a notation such as $i_1 \cdots i_n$, $n \geq 1$, to denote an index with $i_k \in \{1, \cdots, N\}$ for each $k = 1, \cdots, n$. This notation is particularly involved in the map T^n; for a given index $i_1 \cdots i_n$, $I^{(n)}_{i_1 \cdots i_n}$ will denote the interval that is the domain of $T_{i_n} \circ \cdots \circ T_{i_1}$, and define

$$F^{(n)}_{i_1 \cdots i_n} = F_{i_1}(\psi_{i_2} \circ \cdots \circ \psi_{i_n}) F_{i_2}(\psi_{i_3} \circ \cdots \circ \psi_{i_n}) \cdots F_{i_{n-1}}(\psi_{i_n}) F_{i_n}.$$

Then, for almost every x

$$F^{(n)}_{i_1 \cdots i_n}(x) = (\psi^{(n)}_{i_1 \cdots i_n})'(x) = (\psi_{i_1} \circ \cdots \circ \psi_{i_n})'(x),$$

so the $F^{(n)}_{i_1 \cdots i_n}$ form a consistent set of weights for the Perron-Frobenius operator for T^n.

Let $\mathcal{I}$ denote the set of all finite strings of indices such as $i_1 \cdots i_n$ with $i_k \in \{1, \cdots, N\}$ for each $k = 1, \cdots, n$, and $\mathcal{I}_\beta = \{i_1 \cdots i_n \in \mathcal{I} | 1 \leq i_k \leq N_* \text{ for } k = 1, \cdots, n\}$. Recall that $\beta = a_{N_*}$.

Theorem 4.10. *Suppose T satisfies the conditions in Theorem 1.1 and the expanding condition* **(E)**. *Then*

$$\limsup_{n \to \infty} \max_{i_1 \cdots i_n \in \mathcal{I}_\beta} \left\| F^{(n)}_{i_1 \cdots i_n} \right\|_\infty < 1.$$

Corollary 4.11. *If T satisfies the conditions in Theorem 1.1 and the expanding condition* **(E)***, then the partition of $I = [0, \beta]$ into monotonicity intervals for T is weak-Bernoulli for $(T_\beta, g_\beta\, dx)$. The natural extension of this system is therefore isomorphic to a Bernoulli shift.*

Proof. One can easily deduce that

$$\lim_{n \to \infty} \max_{i_1 \cdots i_n \in \mathcal{I}_\beta} \left\| F^{(n)}_{i_1 \cdots i_n} \right\|_\infty^{\frac{1}{n}} < 1,$$

Choose a power $s > 0$ such that T^s_β is uniformly expanding on $[0, \beta]$. Combining this with the exponential rate of decay in Theorem 4.5 (applied to T^s_β), it is not difficult to show directly that the monotonicity partition for T^s_β is weak-Bernoulli, and by elementary argument, so is the original monotonicity partition for T_β. Alternatively, the article of Rychlik [9] may be invoked. A few comments are in order. First the assumptions in [9] may appear to be incompatible with our convexity assumption **(C)**, however, note that the latter implies the weight function g (in the notation of [9]) is of bounded variation. The condition that $g|_S \equiv 0$ is not generally satisfied by our maps, but can be obtained with a measure-zero perturbation of our weight function, so the operator P in that work is identified with our Perron-Frobenius operator (1.2). The proof of the weak-Bernoulli property in §3 of [9] depends only on the uniform expanding condition and the fact that the peripheral spectrum of the operator P consists of one simple eigenvalue at 1. These follow from our Theorem 4.5 applied to T^s_β. □

Remark 4.2. The convexity condition **(C)** always guarantees that

$$\max_{i_1\cdots i_n\in\mathcal{I}_\beta}\left\|F^{(n)}_{i_1\cdots i_n}\right\|_\infty \le 1$$

for all n large enough. This is obvious from Theorem 4.10 when $F_{N_*}(\beta)<1$, i.e., T satisfies the expanding condition **(E)**. Suppose $F_{N_*}(\beta)=1$. It follows that $P\chi_{[0,\beta]}=\chi_{[0,\beta]}$, or equivalently, $\mathcal{F}_{N_*}=\chi_{[0,\beta]}$, which means that for each $j=1,\cdots,N_*$, $F_j(x)\le 1$ for all $x\in[0,1]$. Thus for all $n\ge 1$

$$\max_{i_1\cdots i_n\in\mathcal{I}_\beta}\left\|F^{(n)}_{i_1\cdots i_n}\right\|_\infty \le 1.$$

In fact, in case where $F_{N_*}(\beta)=1$, we have $T_{N_*}(\beta)=\beta$, so that $\psi_{N_*}(\beta)=\beta$. Thus for each $n\ge 1$,

$$\begin{aligned}\max_{i_1\cdots i_n\in\mathcal{I}_\beta}\left\|F^{(n)}_{i_1\cdots i_n}\right\|_\infty &\ge F^{(n)}_{N_*\cdots N_*}(\beta)\\ &=\prod_{k=0}^{n-1}F_{N_*}(\psi^k_{N_*}(\beta))=\prod_{k=0}^{n-1}F_{N_*}(\beta)=1\end{aligned}$$

which implies

$$\max_{i_1\cdots i_n\in\mathcal{I}_\beta}\left\|F^{(n)}_{i_1\cdots i_n}\right\|_\infty = 1.$$

Thus $F_{N_*}(\beta)<1$ if and only if some power of T is expanding on $[0,\beta]$ (see [3]).

To prove Theorem 4.10, we first present two simple observations which require the convexity only.

Lemma 4.12. *There exists $M\ge 1$ such that for all $m\ge 1$, $\|P^m\mathbf{1}\|_\infty\le M$.*

Proof. Recall (the paragraph preceding Lemma 3.3) that there exist $a\in(F_1(0),1)$ and $b<\infty$ such that for any given $f\in\mathcal{J}$ and for any $m\ge 1$,

$$\bigvee P^m f\le a^m\bigvee f+\frac{b(1-a^m)}{1-a}\cdot\|f\|_1.$$

With $f=\mathbf{1}\in\mathcal{J}$, we have $\bigvee f=0$ and $\|f\|_1=1$. Thus for any $m\ge 1$,

$$\bigvee P^m\mathbf{1}\le\frac{b}{1-a}.$$

Since $(P^m\mathbf{1})(1)\le 1$, it follows that

$$\bigvee P^m\mathbf{1}=(P^m\mathbf{1})(0)-(P^m\mathbf{1})(1)\ge(P^m\mathbf{1})(0)-1,$$

so that

$$\|P^m\mathbf{1}\|_\infty=(P^m\mathbf{1})(0)\le\bigvee P^m\mathbf{1}+1\le\frac{b}{1-a}+1.$$

Letting $M=b/(1-a)+1$ completes the proof. ∎

Lemma 4.13. *Let $n \geq 1$ and $F^{(n)}_{i_1\cdots i_n}(x^*) \geq B > 0$ for some index $i_1 \cdots i_n$ and $x^* \in [0,1]$. Then for any given p, q, $0 \leq p < q \leq n$, letting $x_{q+1} = \psi_{i_{q+1}} \circ \psi_{i_{q+2}} \circ \cdots \circ \psi_{i_n}(x^*)$ ($x_{n+1} = x^*$), we have*

$$F^{(q-p)}_{i_{p+1}\cdots i_q}(x_{q+1}) \geq \frac{B}{M^2}.$$

Proof. Let $x_{p+1} = \psi_{i_{p+1}} \circ \psi_{i_{p+2}} \circ \cdots \circ \psi_{i_n}(x^*)$. If $1 \leq p < q \leq n-1$, then it immediately follows from Lemma 4.12 that

$$\begin{aligned} B \leq F^{(n)}_{i_1\cdots i_n}(x^*) &= F^{(p)}_{i_1\cdots i_p}(x_{p+1}) \cdot F^{(q-p)}_{i_{p+1}\cdots i_q}(x_{q+1}) \cdot F^{(n-q)}_{i_{q+1}\cdots i_n}(x^*) \\ &\leq M \cdot F^{(q-p)}_{i_{p+1}\cdots i_q}(x_{q+1}) \cdot M, \end{aligned}$$

which implies $F^{(q-p)}_{i_{p+1}\cdots i_q}(x_{q+1}) \geq B/M^2$.

If $p = 0$ or $q = n$, then similarly we have

$$B \leq F^{(n)}_{i_1\cdots i_n}(x^*) \leq M \cdot F^{(q-p)}_{i_{p+1}\cdots i_q}(x_{q+1})$$

which will imply that $F^{(q-p)}_{i_{p+1}\cdots i_q}(x_{q+1}) \geq B/M \geq B/M^2$. □

Lemma 4.14. *Suppose the assumption in Theorem 4.10 holds. Then for each $j = 1, \cdots, N_*$, there exists $\alpha(j) \in \mathbb{N}$ such that*

$$\left\| \prod_{k=1}^{\alpha(j)} F_j(\psi_j^{k-1}) \right\|_\infty < \frac{1}{2M^2}.$$

Proof. For $j = 1$, since $\|F_1\|_\infty < 1$, there exists $m \geq 1$ such that $(\|F_1\|_\infty)^m < 1/(2M^2)$. With $\alpha(1) = m$, it is clear that

$$\left\| \prod_{k=1}^{\alpha(1)} F_1(\psi_1^{k-1}) \right\|_\infty \leq \Big(\|F_1\|_\infty \Big)^{\alpha(1)} < \frac{1}{2M^2}.$$

Fix j, $2 \leq j \leq N$. First suppose either $T_j(x) < x$ for all $x \in I_j = [a_{j-1}, a_j]$, or that $T_j(x) > x$ for all $x \in I_j$. Then either $T_j(a_j) < a_j$ or $T_j(a_{j-1}) > a_{j-1}$. It is easy to see that there exists $m \geq 1$ such that either for any $k \geq m$,

$$\psi_j^k(x) > \frac{a_j + T_j(a_j)}{2} > T_j(a_j) \qquad \text{for all } x$$

or for any $k \geq m$,

$$\psi_j^k(x) < \frac{a_{j-1} + T_j(a_{j-1})}{2} < T_j(a_{j-1}) \qquad \text{for all } x,$$

which implies in either case that for any $k \geq m$, we have $F_j(\psi_j^k(x)) = 0$ for all x. Letting $\alpha(j) = m + 1$, we get

$$\left\| \prod_{k=1}^{\alpha(j)} F_j(\psi_j^{k-1}) \right\|_\infty = 0,$$

which completes the proof in this case.

Now assume $2 \leq j \leq N_*$ and $T_j(z^*) = z^*$ (so $\psi_j(z^*) = z^*$) for some $z^* \in [a_{j-1}, a_j]$ ($0 < z^* \leq \beta$ if such z^* exists). Then

$$\begin{aligned} z^* = \psi_j(z^*) &= \int_0^{z^*} F_j(x)dx + a_{j-1} \\ &= \int_0^{z^*} \mathcal{F}_j(x)dx + \int_{z^*}^1 \mathcal{F}_{j-1}(x)dx \\ &\geq \int_0^{z^*} \mathcal{F}_j(x)dx \geq z^* \cdot \mathcal{F}_j(z^*), \end{aligned} \tag{4.3}$$

since $\mathcal{F}_j = \sum_{i=1}^{j} F_i$ is decreasing. Thus $\mathcal{F}_j(z^*) \leq 1$ and hence for any $x \geq z^*$, $\mathcal{F}_j(x) \leq 1$.

We claim that such z^* is unique. In fact, if $\psi_j(z_0) = z_0$, $\psi_j(z_1) = z_1$, and $0 < z_0 < z_1 \leq \beta$, then $F_j(x) \leq 1$ on $[z_0, 1]$ so that $z_1 - z_0 = \int_{z_0}^{z_1} F_j(t)dt$ and hence $F_j(x) = 1$ on $[z_0, z_1]$. Then $\mathcal{F}_{j-1}(x) = 0$ on $[z_0, z_1]$. Thus $\beta \leq z_0$, which is a contradiction.

We will show that there exist $\epsilon > 0$ and $w^* < z^*$ such that for any $y \in [w^*, 1]$, $F_j(y) \leq 1-\epsilon < 1$. First suppose $\mathcal{F}_j(z^*) < 1$. The assumption that $\mathcal{F}_j$ is upper semicontinuous implies then that there exist $\epsilon > 0$ and $w^* < z^*$ such that for any $y \in [w^*, 1]$, $\mathcal{F}_j(y) \leq 1 - \epsilon$ and so $F_j(y) \leq 1-\epsilon < 1$. Next in case $\mathcal{F}_j(z^*) = 1$, it follows from (4.3) that $\mathcal{F}_j(x) = 1$ on $[0, z^*]$ and $\mathcal{F}_{j-1}(x) = 0$ on $(z^*, 1]$. Thus $\beta \leq z^*$, which means $z^* = \beta$ and so $j = N_*$. Then $\mathcal{F}_{N_*}(x) = 1$ on $[0, \beta]$, which indicates that F_{N_*} is increasing on $[0, \beta]$. Since $F_{N_*}(\beta) < 1$, it easily follows that there exists $\epsilon(= 1 - F_{N_*}(\beta)) > 0$ such that for any $y \in [0, 1]$, $F_j(y) = F_{N_*}(y) \leq 1 - \epsilon < 1$. Therefore in either case, there exist $\epsilon > 0$ and $w^* < z^*$ such that for any $y \in [w^*, 1]$, $F_j(y) \leq 1 - \epsilon < 1$.

Note that for any $x < z^*$, $\psi_j^k(x) \nearrow z^*$ as $k \to \infty$ and for any $x > z^*$, $\psi_j^k(x) \searrow z^*$ as $k \to \infty$. Thus there exists $m \geq 1$ such that for any given $k \geq m$, for any $x \in [0, 1]$, $\psi_j^k(x) > w^*$ and hence $F_j(\psi_j^k(x)) \leq 1 - \epsilon$. Let

$$\alpha(j) = \left[m - \frac{\ln(2M^3)}{\ln(1-\epsilon)} \right] + 1.$$

By Lemma 4.12,

$$\begin{aligned}\left\|\prod_{k=1}^{\alpha(j)} F_j(\psi_j^{k-1})\right\|_\infty &\le \left\|\prod_{k=1}^{m} F_j(\psi_j^{k-1})\right\|_\infty \cdot \left\|\prod_{k=m+1}^{\alpha(j)} F_j(\psi_j^{k-1})\right\|_\infty \\ &\le \left\|F^{(m)}_{\underbrace{j\cdots j}_{m}}\right\|_\infty \cdot \prod_{k=m+1}^{\alpha(j)} \left\|F_j(\psi_j^{k-1})\right\|_\infty \\ &\le M\cdot(1-\epsilon)^{\alpha(j)-m} < \frac{1}{2M^2},\end{aligned}$$

which completes the proof. □

We now introduce some notations and definitions.

Definition 4.1. (1) For a given $n \ge 3$, an index $i_1\cdots i_n$ is said to be *bowl-shaped* if there exists $r, 2 \le r \le n-1$, for which $i_1 > i_r$ and

$$i_1 \ge i_2 \ge \cdots \ge i_r < i_{r+1} < \cdots < i_n.$$

(2) For a given $n \ge 1$, an index $i_1\cdots i_n$ is said to be *increasing* if $i_1 \le i_2 \le \cdots \le i_n$.
(3) For a given $n \ge 1$, an index $i_1\cdots i_n$ is said to be *decreasing* if $i_1 \ge i_2 \ge \cdots \ge i_n$.
(4) For a given $n \ge 1$, an index $i_1\cdots i_n$ is said to be *monotone* if it is either increasing or decreasing.

Under the notations in Lemma 4.14, let $\alpha = \sum_{j=1}^{N_*} \alpha(j)$.

Lemma 4.15. *Suppose the assumption in Theorem 4.10 holds. Let $n \ge 1$ and an index $i_1\cdots i_n \in \mathcal{I}_\beta$ be given so that $\|F^{(n)}_{i_1\cdots i_n}\|_\infty \ge 1$. If for some s, t, $1 \le s < t \le n$, a subindex $i_s\cdots i_t$ is monotone, then $t-s<\alpha$.*

Proof. Suppose $t-s\ge\alpha$. Since either $i_s \le \cdots \le i_t$ or $i_s \ge \cdots \ge i_t$, there exist j and r such that $1 \le j \le N_*$ and $s-1 \le r \le t-\alpha(j)$ and

$$i_{r+1} = \cdots = i_{r+\alpha(j)} = j.$$

Using Lemma 4.12 and Lemma 4.14, we obtain

$$\begin{aligned}
1 \leq \left\| F^{(n)}_{i_1\cdots i_n} \right\|_\infty &\leq \left\| F^{(r)}_{i_1\cdots i_r} \right\|_\infty \cdot \left\| F^{(\alpha(j))}_{i_{r+1}\cdots i_{r+\alpha(j)}} \right\|_\infty \cdot \left\| F^{(n-r-\alpha(j))}_{i_{r+\alpha(j)+1}\cdots i_n} \right\|_\infty \\
&\leq M \cdot \Big\| F^{(\alpha(j))}_{\underbrace{j\cdots j}_{\alpha(j)}} \Big\|_\infty \cdot M \\
&= M^2 \cdot \left\| \prod_{k=1}^{\alpha(j)} F_j(\psi_j^{k-1}) \right\|_\infty \\
&< M^2 \cdot \frac{1}{2M^2} = \frac{1}{2},
\end{aligned}$$

which is a contradiction. □

Remark 4.3. The proof of Lemma 4.15 directly shows that especially when $i_s \cdots i_t = \underbrace{j\cdots j}_{t-s+1}$ for some $j \in \{1, \cdots, N_*\}$, we have $t - s + 1 < \alpha(j)$.

It is not difficult to show the following result.

Lemma 4.16. *For any $n \geq 1$ and for any given index $i_1 \cdots i_n$, there exists a (unique) partition of $\{1, \cdots, n\}$ such that*

1. *$1 = n_0 \leq n_1 < n_2 < \cdots < n_k \leq n_{k+1} = n$ for some $k \geq 0$;*
2. *either $n_1 = 1$ or $i_1 \cdots i_{n_1}$ is increasing with $i_{n_1-1} < i_{n_1}$;*
3. *for each $l = 1, \cdots, k-1$, $i_{n_l} \cdots i_{n_{l+1}}$ is bowl-shaped;*
4. *$i_{n_k} \cdots i_n$ is decreasing.*

Definition 4.2. For a given index $\omega = i_1 \cdots i_n$, $n \geq 3$, $b(\omega)$ denotes the number of bowl-shaped subindices contained in ω, i.e., $b(\omega) = \max\{k-1, 0\}$ with the notation as above.

For each $n \geq 1$, define $\mathcal{U}_n$ to be

$$\mathcal{U}_n = \left\{ i_1 \cdots i_n \in \mathcal{I}_\beta \middle| \left\| F^{(n)}_{i_1\cdots i_n} \right\|_\infty \geq 1 \right\}$$

and let $\mathcal{U} = \bigcup\limits_{n=1}^{\infty} \mathcal{U}_n$.

Lemma 4.17. *Suppose the assumption in Theorem 4.10 holds and that there exists $\{n_k\}_{k=1}^{\infty}$ for which $n_k \to \infty$ as $k \to \infty$ and $\mathcal{U}_{n_k} \neq \emptyset$. Then for any given $L \geq 1$, there exists $\omega \in \mathcal{U}$ such that $b(\omega) \geq L$.*

Proof. We first show that there exists $D \geq 1$ for which for any $n \geq 1$ and for any given $\omega = i_1 \cdots i_n \in \mathcal{U}_n$, there exists a (unique) partition of $\{1, \cdots, n\}$ such that

1. $1 = n_0 \leq n_1 < n_2 < \cdots < n_k \leq n_{k+1} = n$ for some $k \geq 0$;
2. either $n_1 = 1$ or $i_1 \cdots i_{n_1}$ is increasing with $i_{n_1-1} < i_{n_1}$;
3. for each $l = 1, \cdots, k-1$, $i_{n_l} \cdots i_{n_{l+1}}$ is bowl-shaped;

(4) $i_{n_k}\cdots i_n$ is decreasing;
(5) for each $l=0,1,\cdots,k$, $n_{l+1}-n_l\le D$.

By Lemma 4.16, it suffices to show (5). In fact, it immediately follows from Lemma 4.15 that $n_1-n_0=n_1-1<\alpha$ and $n_{k+1}-n_k=n-n_k<\alpha$, since each corresponding index is monotone. Also, for each $l=1,\cdots,k-1$, we have $n_{l+1}-n_l<\alpha+N_*$, since each index $i_{n_l}\cdots i_{n_{l+1}}$ consists of one decreasing index and one strictly increasing index. Letting $D=\alpha+N_*-1$, we obtain $n_{l+1}-n_l\le D$ for each $l=0,1,\cdots,k$.

Now choose $n\ge(L+3)D$ and $\omega=i_1\cdots i_n\in\mathcal{U}_n$. Using the notations as above, we get

$$(L+3)D\le n=\sum_{l=0}^{k}(n_{l+1}-n_l)+1\le(k+1)\cdot D+1,$$

which implies that $b(\omega)=k-1\ge L$. □

Remark 4.4. In case $k\ge1$, it is easy to see that

$$b(\omega)>\frac{n}{D}-2.$$

The following is true for any interval map that satisfies the conditions **(C1)**.

Lemma 4.18. *Let $j_1\cdots j_m$, $m\ge3$, be bowl-shaped, i.e., there exists r, $2\le r\le m-1$ such that $j_1>j_r$ and*

$$j_1\ge j_2\ge\cdots\ge j_r<j_{r+1}<\cdots<j_m.$$

Let $z_{m+1}\in(0,1]$ be given and for each $s=1,\cdots,m$, let

(1) $z_s=\psi_{j_s}\circ\cdots\circ\psi_{j_m}(z_{m+1})=\psi_{j_s}(z_{s+1}),(z_s\in I_{j_s})$;
(2) $A_s=F_{j_s}(z_{s+1})$.

Then for each $s=1,\cdots,m$,

$$A_s\le\frac{z_s}{z_{s+1}}.$$

Moreover, if for some $t\in\{1,\cdots,r-1\}$,

$$j_1\ge\cdots\ge j_t>j_{t+1}=\cdots=j_r<j_{r+1}$$

(such t always exists), then

$$A_t\le\frac{z_t}{z_{t+1}}-A_r.$$

Proof. First, note that for each $s = 1, \cdots, m$, we have $z_{s+1} > 0$. Then for a fixed $s, 1 \le s \le m$,

$$\begin{aligned} z_s = \psi_{j_s}(z_{s+1}) &= \int_0^{z_{s+1}} F_{j_s}(x)dx + a_{j_s-1} \\ &= \int_0^{z_{s+1}} \mathcal{F}_{j_s}(x)dx + \int_{z_{s+1}}^1 \mathcal{F}_{j_s-1}(x)dx \\ &\ge \int_0^{z_{s+1}} \mathcal{F}_{j_s}(x)dx \\ &\ge z_{s+1} \cdot \mathcal{F}_{j_s}(z_{s+1}) \\ &\ge z_{s+1} \cdot F_{j_s}(z_{s+1}) \\ &= z_{s+1} \cdot A_s, \end{aligned}$$

which implies

$$A_s \le \frac{z_s}{z_{s+1}}.$$

To prove the second statement, observe that $z_{t+1} \in I_{j_{t+1}}$ and $z_{r+1} \in I_{j_{r+1}}$, so that $z_{t+1} \le z_{r+1}$, since $j_{t+1} < j_{r+1}$. Noticing that $j_t > j_r$, we have $\mathcal{F}_{j_t} \ge \mathcal{F}_{j_r} + F_{j_t}$, which means

$$\begin{aligned} \mathcal{F}_{j_t}(z_{t+1}) &\ge \mathcal{F}_{j_r}(z_{t+1}) + F_{j_t}(z_{t+1}) \\ &\ge \mathcal{F}_{j_r}(z_{r+1}) + F_{j_t}(z_{t+1}) \\ &\ge A_r + A_t. \end{aligned}$$

Thus similarly

$$\begin{aligned} z_t = \psi_{j_t}(z_{t+1}) &\ge \int_0^{z_{t+1}} \mathcal{F}_{j_t}(x)dx \\ &\ge z_{t+1} \cdot \mathcal{F}_{j_t}(z_{t+1}) \\ &\ge z_{t+1}(A_r + A_t). \end{aligned}$$

Therefore

$$A_t \le \frac{z_t}{z_{t+1}} - A_r,$$

which completes the proof. □

Proof of Theorem 4.10. Let $\omega = i_1 \cdots i_n \in \mathcal{U}_n$, $n \ge 3\alpha$, and choose $x^* \in [0,1]$ so that $F^{(n)}_{i_1 \cdots i_n}(x^*) \ge 1/2$. Denote the partition of $\{1, \cdots, n - \alpha(1)\}$ given from Lemma 4.16 by

$$1 = n_0 \le n_1 < n_2 < \cdots < n_k \le n_{k+1} = n - \alpha(1),\ k \ge 2.$$

Note that $b(\omega) \le k - 1 + \alpha(1)$. Let $x_{n+1} = x^*$ and for each $s = 1, \cdots, n$, let

$$x_s = \psi_{i_s} \circ \cdots \circ \psi_{i_n}(x^*) = \psi_{i_s}(x_{s+1}).$$

Since $i_{n-\alpha(1)+1}\cdots i_n \neq 1\cdots 1$ (see Remark 4.3), we have $x_s > 0$ for each $s = 1, \cdots, n - \alpha(1) + 1$. Also for each $l = 2, \cdots, k$, $x_{n_l} \geq a_1$, since $i_{n_l} > i_{n_l - 1} \geq 1$.

Fix $l, 1 \leq l \leq k-1$, and consider a bowl-shaped index $j_1 \cdots j_m = i_{n_l} \cdots i_{n_{l+1}}$, where $m = n_{l+1} - n_l + 1$. Let $z_{m+1} = x_{n_{l+1}+1}$ (> 0) and for each $s = 1, \cdots, m$, define $z_s = \psi_{j_s}(z_{s+1})$ $(= \psi_{i_{n_l+s-1}}(x_{n_l+s}))$ and $A_s = F_{j_s}(z_{s+1})$. Under the notations in Lemma 4.18, observe that for each $s = 1, \cdots, m$,

$$A_s \leq \frac{z_s}{z_{s+1}} \tag{4.4}$$

and

$$A_t \leq \frac{z_t}{z_{t+1}} - A_r = \Big(1 - \frac{z_{t+1}}{z_t} \cdot A_r\Big) \cdot \frac{z_t}{z_{t+1}}.$$

We now show that there exists $\delta > 0$, which depends on T only, such that

$$\frac{z_{t+1}}{z_t} \cdot A_r \geq \delta > 0.$$

First it follows from Lemma 4.13 that for each $s = 1, \cdots, m$,

$$A_s \geq \frac{1}{2M^2}.$$

Consequently,

$$\begin{aligned} z_{t+1} \geq A_{t+1} \cdot z_{t+2} \geq \cdots &\geq A_{t+1} \cdot A_{t+2} \cdot \cdots \cdot A_r \cdot z_{r+1} \\ &\geq \Big(\frac{1}{2M^2}\Big)^{r-t} \cdot z_{r+1}. \end{aligned}$$

Using the notation in the proof of Lemma 4.17, we have $r - t \leq m - 2 \leq D - 1$. Also using the fact that $z_{r+1} \in I_{j_{r+1}}$ and $j_{r+1} > j_r \geq 1$, we get $z_{r+1} \geq a_1$ and so

$$z_{t+1} \geq \big(\frac{1}{2M^2}\big)^{D-1} \cdot a_1.$$

Let $\delta = (1/2M^2)^D \cdot a_1 > 0$. Then

$$\frac{z_{t+1}}{z_t} \cdot A_r \geq \big(\frac{1}{2M^2}\big)^{D-1} \cdot a_1 \cdot \frac{1}{2M^2} = \delta.$$

This indicates that

$$A_t \leq (1 - \delta) \cdot \frac{z_t}{z_{t+1}}$$

which combined with (4.4) implies that for a fixed l, $1 \le l \le k-1$,

$$\begin{aligned} F^{(n_{l+1}-n_l)}_{i_{n_l}\cdots i_{n_{l+1}-1}}(x_{n_{l+1}}) &= F^{(m-1)}_{j_1\cdots j_{m-1}}(z_m) \\ &= \prod_{s=1}^{m-1} A_s \le (1-\delta)\prod_{s=1}^{m-1}\frac{z_s}{z_{s+1}} \\ &= (1-\delta)\cdot\frac{z_1}{z_m} \\ &= (1-\delta)\cdot\frac{x_{n_l}}{x_{n_{l+1}}}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2} \le F^{(n)}_{i_1\cdots i_n}(x^*) &= F^{(n_1-1)}_{i_1\cdots i_{n_1-1}}(x_{n_1})\cdot\left(\prod_{l=1}^{k-1} F^{(n_{l+1}-n_l)}_{i_{n_l}\cdots i_{n_{l+1}-1}}(x_{n_{l+1}})\right)\cdot F^{(n-n_k+1)}_{i_{n_k}\cdots i_n}(x^*) \\ &\le M\cdot\prod_{l=1}^{k-1}\left((1-\delta)\cdot\frac{x_{n_l}}{x_{n_{l+1}}}\right)\cdot M \\ &\le M^2\cdot\frac{x_{n_1}}{x_{n_k}}\cdot(1-\delta)^{k-1} \\ &\le M^2\cdot\frac{1}{a_1}\cdot(1-\delta)^{b(\omega)-\alpha(1)} \end{aligned}$$

where we have used the fact that for each $l = 2, \cdots, k$, $x_{n_l} \ge a_1$, since $i_{n_l} > i_{n_l-1} \ge 1$.

This shows a contradiction, if n is chosen so that

$$\frac{n}{D} > \frac{\ln(a_1/2M^2)}{\ln(1-\delta)} + 2 + \alpha(1),$$

which would imply that

$$b(\omega) - \alpha(1) > \frac{n}{D} - 2 - \alpha(1) > \frac{\ln(a_1/2M^2)}{\ln(1-\delta)}$$

(see Remark 4.4) or equivalently

$$M^2\cdot\frac{1}{a_1}\cdot(1-\delta)^{b(\omega)-\alpha(1)} < \frac{1}{2}.$$

Therefore there exists $L \ge 1$ such that for any $n \ge L$, $\mathcal{U}_n = \emptyset$. □

5. Ergodic Properties of an Invariant Measure: Case II

Throughout this section T is assumed to satisfy the conditions of Theorem 1.1. As mentioned in Remark 4.2, when $F_{N_*}(\beta) = 1$, every power of the map T fails to be expanding on $[0, \beta]$. However it turns out that $(T, g_\beta dm)$ is exact.

Theorem 5.1. *Suppose T satisfies the conditions in Theorem 1.1 and $F_{N_*}(\beta) = 1$. If $f \in BV(\beta), f \geq 0$, and $\|f\|_1 = \beta$, then*

$$\|P^n f - \chi_{[0,\beta]}\|_1 \to 0 \quad \text{as } n \to \infty.$$

Notice that the assumption $F_{N_*}(\beta) = 1$ implies $P\chi_{[0,\beta]} = \chi_{[0,\beta]}$. Let $g_* = \chi_{[0,\beta]}/\beta \in BV(\beta)$. Theorem 5.1 shows that $(T, g_* dm)$ is exact (see the proof of Theorem 4.5), so that $g_\beta = g_* = \chi_{[0,\beta]}/\beta$. Combined with Theorem 4.5, this concludes the following.

Corollary 5.2. *Suppose T satisfies the conditions in Theorem 1.1. Then g_β obtained from Lemma 4.4 is a unique invariant density of T in $BV(\beta)$ and $(T, g_\beta dm)$ is exact.*

In order to show Theorem 5.1, we establish convergence in Theorem 5.1 at the single point zero, after which the full result will follow easily.

Lemma 5.3. *Let $f \in \mathcal{J}(\beta)$. Then $(P^n f)(0) \to \|f\|$ as $n \to \infty$.*

Proof. Since we assume $P\chi_{[0,\beta]} = \chi_{[0,\beta]}$, using Remark 2.1, without loss of generality we may assume that $\mathcal{F}_{N_*} \equiv 1$.

For each $n \geq 0$, let $C_n = (P^n f)(0)$. Then

$$C_{n+1} = \sum_{i=1}^{N} F_i(0)(P^n f)(\psi_i(0)) \leq \sum_{i=1}^{N} F_i(0)(P^n f)(0) = C_n,$$

so that as $n \to \infty$, $C_n \searrow C_*$ for some $C_* \geq \|f\|$.

Let $b_0 = 0$. For each $k \geq 1$, let $l_k = \max\{i | 1 \leq i \leq N_*,\ F_i(b_{k-1}) > 0\}$ and $b_k = \psi_{l_k}(b_{k-1})$.

Next, we claim that for each k, $\lim_{n\to\infty} P^n f(b_k) = \lim_{n\to\infty} P^n f(b_0) = C_*$.

Once again we use **(C1)** for a fixed k to find

$$F_{l_k}(b_{k-1}) = 1 - \mathcal{F}_{l_k - 1}(b_{k-1}) > 0$$

while for each $n \geq 0$,

$$\begin{aligned}
(P^{n+1} f)(b_{k-1}) &= \sum_{i=1}^{N_*} F_i(b_{k-1})(P^n f)(\psi_i(b_{k-1})) \\
&= \sum_{i=1}^{l_k} F_i(b_{k-1})(P^n f)(\psi_i(b_{k-1})) \\
&\leq \sum_{i=1}^{l_k - 1} F_i(b_{k-1})(P^n f)(0) + F_{l_k}(b_{k-1})(P^n f)(\psi_{l_k}(b_{k-1})) \\
&= \sum_{i=1}^{l_k - 1} F_i(b_{k-1}) C_n + F_{l_k}(b_{k-1})(P^n f)(b_k).
\end{aligned}$$

Thus for each $k \geq 1$,

$$\liminf_{n\to\infty}(P^n f)(b_{k-1}) \leq (1 - F_{l_k}(b_{k-1}))C_* + F_{l_k}(b_{k-1}) \cdot \liminf_{n\to\infty}(P^n f)(b_k).$$

Noticing that $\lim_{n\to\infty}(P^n f)(b_0) = C_*$ and using Induction on k, we have for each $k \geq 0$,

$$C_* \leq \liminf_{n\to\infty}(P^n f)(b_k).$$

Meanwhile, fix $k \geq 0$. Since for any given $n \geq 0$, $P^n f$ is decreasing, it follows that $(P^n f)(b_k) \leq (P^n f)(0) = C_n$, which implies

$$\limsup_{n\to\infty}(P^n f)(b_k) \leq C_*.$$

Thus for each $k \geq 0$, $\lim_{n\to\infty}(P^n f)(b_k) = C_*$, as we have claimed.

Next, observe that for each $k \geq 0$, $b_k < \beta$ for if not, and if k is chosen to be minimal such that $b_k = \beta$ then obviously $l_k = N_*$ and $F_{N_*}(b_{k-1}) > 0$. However, $F_{N_*} = 1 - \mathcal{F}_{N_*-1}$ is increasing, so $F_{N_*} > 0$ on $[b_{k-1}, b_k] = [b_{k-1}, \beta]$, contradicting $\psi_{N_*} \equiv \beta$ on $[b_{k-1}, \beta]$ in the convexity condition **(C1)**.

However, we will show that

$$\sup\{b_k | k \geq 0\} = \beta, \tag{5.1}$$

in which case for any $x \in [0, \beta)$, $\lim_{n\to\infty}(P^n f)(x) = C_*$. Since $\lim_{n\to\infty}(P^n f)(x) = 0$ on $(\beta, 1]$, it easily follows that $C_* = \|f\|$, i.e., $\lim_{n\to\infty}(P^n f)(x) = \|f\|$ on $[0, \beta]$. In particular, $(P^n f)(0) \to \|f\|$ as $n \to \infty$, which completes the proof.

To see (5.1), let $r = \max\{l_k | k \geq 1\}$ and notice that $l_1 \geq 2$ so $r \geq 2$. If $r < N_*$ set $s = \inf\{T(x^+) | a_r \leq x < \beta\}$, otherwise set $s = \beta$. We want to show that the sequence b_k is contained in [0,s]. If for some k, $b_k > s$, then $r < N_*$ and $\sum_{i=r+1}^{N_*} F_i(b_k) > 0$. If this is not the case, i.e.; $\sum_{i=r+1}^{N_*} F_i(b_k) = 0$ then $\sum_{i=r+1}^{N_*} F_i \equiv 0$ on $[s, b_k]$ since the sum is an increasing function on $[0, \beta]$, which in view of convexity condition **(C1)** contradicts the definition of s above. But then clearly $l_{k+1} > r$, another contradiction.

Choose $m \geq 1$ so that $l_m = r$. Then $b_m = \psi_{l_m}(b_{m-1}) = \psi_r(b_{m-1}) \in [a_{r-1}, a_r]$. Since $0 < b_m < \beta$ and

$$T[b_m, \beta] \subseteq [T_r(b_m), \beta] \cup [s, \beta] \subseteq [T_r(b_m), \beta] \cup [b_m, \beta],$$

it follows from Lemma 4.2 that $b_{m-1} = T_r(b_m) < b_m$, i.e., $T_r(a_{r-1}) \leq T_r(b_m) < b_m \leq s$.

Now, on $[0, s]$ F_r is increasing, so

$$0 < F_r(b_{m-1}) = F_r(T_r(b_m)) \leq F_r(b_m)$$

which implies $l_{m+1} = r$. A simple induction shows that $l_{m+k} = r$ for all $k = 0, 1, \ldots$ and that the sequence b_{m+k} is increasing on $[a_{r-1}, a_r]$ for all $k = 0, 1, \ldots$.

Define $b^* = \lim_{k\to\infty} b_{m+k} = \lim_{l\to\infty} b_l$. Using $a_{r-1} \le b^* \le a_r$ and $b^* \le s$, combined with $T_r(b^*) = b^*$ yields

$$T[b^*, \beta] \subseteq [T_r(b^*), \beta] \cup [s, \beta] \subseteq [b^*, \beta]$$

Finally, Lemma 4.2 implies $b^* = \beta$ and (5.1) has been verified. □

Proof of Theorem 5.1. Let $f \in \mathcal{J}(\beta)$ and $\|f\| = \beta$. For each $n \ge 1$, let $d_n = \inf\{x \in [0,1] | (P^n f)(x) \le 1\}$. Clearly, for any given $n \ge 1$, $d_n (\le \beta)$ exists, and

$$\beta = \|P^n f\| = \int_0^{d_n} P^n f + \int_{d_n}^1 P^n f$$

which implies that

$$\begin{aligned} \int_{d_n}^1 |P^n f - \chi_{[0,\beta]}| &= \int_{d_n}^{\beta} [1 - (P^n f)(x)]\, dx \\ &= \int_0^{d_n} [(P^n f)(x) - 1]\, dx = \int_0^{d_n} |P^n f - \chi_{[0,\beta]}|. \end{aligned}$$

Since for each $n \ge 1$, $P^n f$ is decreasing, we have

$$\begin{aligned} \|P^n f - \chi_{[0,\beta]}\| &= 2\int_0^{d_n} |P^n f - \chi_{[0,\beta]}| \\ &\le 2 \cdot d_n[(P^n f)(0) - 1] \le 2[(P^n f)(0) - 1]. \end{aligned}$$

It follows from Lemma 5.3 that $\|P^n f - \chi_{[0,\beta]}\| \to 0$ as $n \to \infty$.

Now suppose $f \in BV(\beta), f \ge 0$, and $\|f\| = \beta$. Then $f - \chi_{[0,\beta]} \in BV_0(\beta)$, and hence there exist $f^1, f^2 \in \mathcal{J}(\beta)$ such that $f - \chi_{[0,\beta]} = f^1 - f^2$ and $\|f^1\| = \|f^2\|$. Thus

$$\begin{aligned} \|P^n f - \chi_{[0,\beta]}\| &= \|P^n(f^1 - f^2)\| \\ &= \|(P^n f^1 - \|f^1\| \cdot \chi_{[0,\beta]}/\beta) - (P^n f^2 - \|f^2\| \cdot \chi_{[0,\beta]}/\beta)\| \\ &\le \Big[\|P^n(\beta f^1/\|f^1\|) - \chi_{[0,\beta]}\| + \|P^n(\beta f^2/\|f^2\|) - \chi_{[0,\beta]}\|\Big] \cdot \|f^1\|/\beta, \end{aligned}$$

where the last expression converges to 0 as $n \to \infty$. Therefore $\|P^n f - \chi_{[0,\beta]}\| \to 0$ as $n \to \infty$. □

6. Ergodic Properties on the Unit Interval

Suppose T satisfies the conditions of Theorem 1.1 and $\bigcup_{n=1}^{\infty} T^n[0, a_1] = T[0, 1]$ (which is the case, especially when $\beta = 1$). Corollary 5.2 indicates that $g \in BV$ obtained from Theorem 1.1 is a unique invariant density of T and (T, gdm) is exact. Furthermore, Theorem 4.5, combined with Theorem 4.10, shows that if $F_N(1) < 1$, then (T, gdm) has exponential decay of correlations and some power of T is expanding, hence Bernoulli. In this section, we will consider the case where $\bigcup_{n=1}^{\infty} T^n[0,a_1] \subsetneq T[0,1]$ (so $\beta < 1$) and investigate ergodic properties of (T, gdm) on the unit interval. Maps satisfying the Lasota–Yorke convexity condition are known to have the property that the invariant probability density is unique and the unique a.c.i.m. is **exact** for T [7]. It turns out that our weaker convexity condition **(C)** is not sufficient to imply exactness, or even to guarantee uniqueness of the invariant probability density as the following simple example shows.

FIGURE 6.1

Example 6.1. Let

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/4 \\ 2x - 1/2 & \text{if } 1/4 \le x < 3/4 \\ 2x - 1 & \text{if } x \ge 3/4 \end{cases}$$

(see Figure 6.1). Lebesgue measure is preserved and $\beta = 1/2$. However, T supports infinitely many a.c.i.m. on $[0,1]$. with densities $g_1 = 2\chi_{[0,1/2]}$, $g_2 = 2\chi_{[1/2,1]}$ and $g_\alpha = \alpha g_1 + (1-\alpha) g_2$, for $0 < \alpha < 1$, so T is certainly not exact. (However, T_β is exact, as required by the arguments in §5, and obviously $T|_{[1/2,1]}$ is also exact.) Consider the nontrivial invariant interval $[\frac{1}{2}, 1]$. If it is only noticed that $T[\frac{1}{2}, 1] \subseteq [\frac{1}{2}, 1]$ then $T^{-1}[\frac{1}{2}, 1] \supseteq [\frac{1}{2}, 1]$ from which it follows that the interval is invariant. This simple observation turns out to be the key to understanding exactness, even when Lebesgue measure is not invariant.

With this example in mind, we define our mixing condition denoted by **(M)**:

(M) For each $d \in (\beta, 1]$, $T[\beta, d] \not\subseteq [\beta, d]$.

Remark 6.1. We present two conditions related to the uniqueness of the invariant density.

(A) For any $c, d, 0 < c < d$, we have $0 \in \bigcap_{n\geq 1} \bigcup_{k\geq n} T^k[c,d]$.

(B) If D is a finite union of closed intervals, then either $0 \in D$ or $T(D) \nsubseteq D$.

It can be shown directly that the condition **(A)** is a necessary and sufficient condition for T to have a unique invariant density in BV. It is clear that the condition **(A)** implies the condition **(B)**, which is stronger than **(M)**. In general, neither of the converses is true. It, however, turns out that the condition **(M)**, together with the convexity condition **(C)**, does imply the condition **(A)**. In other words, it guarantees the uniqueness of invariant density and in fact the exactness also follows.

Throughout this section T is assumed to satisfy the conditions of Theorem 1.1 and the condition **(M)**.

Lemma 6.1. *Let g_β be the invariant density defined in Lemma 4.4. Then $g_\beta = g$.*

Proof. Let $g_2 = g \cdot \chi_{(\beta,1]} = g - g_\beta$. Since $Pg_\beta = g_\beta$, it follows that $Pg_2 = g_2$. Suppose $g_2 \not\equiv 0$. Since g_2 is decreasing on $(\beta, 1]$, there exists $\gamma \in (\beta, 1]$ such that $\overline{g_2^{-1}(\mathbb{R} \setminus \{0\})} = [\beta, \gamma]$. We claim that $T[\beta,\gamma] \subseteq [\beta,\gamma]$, which contradicts the mixing condition **(M)** and so concludes $g_2 \equiv 0$, i.e., $g_\beta = g$. To prove the claim, let $B = [0,\beta) \cup (\gamma, 1]$ and observe

$$0 = \int_B g_2 dm = \int Pg_2 \cdot \chi_B dm = \int_{T^{-1}B} g_2 dm = \int_{B^c \cap T^{-1}B} g_2 dm,$$

which implies that $B^c \cap T^{-1}B \subseteq \{\beta, \gamma\}$. Since $T(B^c) = T[\beta,\gamma]$ is a finite union of non-trivial closed intervals, it follows that $T(B^c) \subseteq B^c$, as we have claimed. □

Lemma 6.2. *For a given $f \in BV$,*

$$\|(P^n f) \cdot \chi_{(\beta,1]}\|_1 \to 0 \quad \text{as } n \to \infty.$$

Proof. For each $n \geq 1$, let $h_n = \frac{1}{n} \sum_{s=0}^{n-1} P^s \mathbf{1}$ and take a subsequence $\{n_k\}_{k=1}^\infty$ so that $h_{n_k} \to h$ in $\|\cdot\|_1$ as $n_k \to \infty$ where $h \in \mathcal{J}$ is an invariant density of T (see the proof of Theorem 1.1). It follows from Lemma 6.1 that $h(x) = 0$ on $(\beta, 1]$.

Let $f \geq 0$ and $f \in BV$. Since $T[0,\beta] \subseteq [0,\beta]$, we have $T^{-1}[\beta,1] \subseteq [\beta,1]$ and so

$$\int_\beta^1 Pf dm \leq \int_\beta^1 f dm.$$

Let $M = \|f\|_\infty < \infty$. It inductively follows that for a given $n \geq 1$,

$$\int_\beta^1 P^{n-1} f dm \leq \int_\beta^1 P^{n-2} f dm \leq \cdots \leq \int_\beta^1 f dm,$$

which implies

$$\begin{aligned}\int_\beta^1 P^{n-1} f dm &= \frac{1}{n}\sum_{s=0}^{n-1}\int_\beta^1 P^{n-1} f dm \le \frac{1}{n}\sum_{s=0}^{n-1}\int_\beta^1 P^s f dm \\ &\le \frac{1}{n}\sum_{s=0}^{n-1}\int_\beta^1 P^s (M\cdot \mathbf{1}) dm \\ &= M\cdot \frac{1}{n}\sum_{s=0}^{n-1}\int_\beta^1 P^s \mathbf{1} dm \\ &= M\cdot \|h_n\cdot \chi_{(\beta,1]}\|_1 \le M\cdot \|h_n - h\|_1.\end{aligned}$$

Since $\|h_{n_k} - h\|_1 \to 0$ as $n_k \to \infty$, it follows that

$$\|(P^n f)\cdot \chi_{(\beta,1]}\|_1 \to 0 \quad \text{as } n\to\infty.$$

For any given $f \in BV$, letting $f = f_+ - f_-$, where $f_+, f_- \ge 0$ and $f_+, f_- \in BV$, and applying the same argument to f_+, f_-, we have

$$\|(P^n f)\cdot \chi_{(\beta,1]}\|_1 \to 0 \quad \text{as } n\to\infty,$$

which completes the proof. □

Corollary 6.3. *Let g_β be the invariant density defined in Lemma 4.4. Then $g_\beta = g$ is a unique invariant density of T.*

Proof. Suppose $\phi \in BV, \phi \ge 0$, and $P\phi = \phi$. By Lemma 6.2

$$\int_\beta^1 \phi dm = \int_\beta^1 P^m \phi dm \to 0 \quad \text{as } m\to\infty,$$

which implies $\phi(x) = 0$ on $(\beta, 1]$, i.e., $\phi \in BV(\beta)$. From Corollary 5.2, we get $\phi = g$. Therefore $g = g_\beta$ is a unique invariant density of T. □

Theorem 6.4. *Suppose T satisfies the conditions in Theorem 1.1 and the condition* **(M)**. *Then (T, gdm) is exact.*

Proof. Let $f \in BV, f \ge 0$, and $\|f\|_1 = 1$. Let $\epsilon > 0$ be given. From Lemma 6.2, there exists $m \ge 1$ such that

$$\|(P^m f)\cdot \chi_{(\beta,1]}\|_1 < \frac{\epsilon}{3}.$$

Let $f_\beta = (P^m f)\cdot \chi_{[0,\beta]} \in BV(\beta)$ and $A = \|f_\beta\|_1$. Using Theorem 4.5, choose $n > m$ so that

$$\|P^{n-m}(f_\beta/A) - g\|_1 < \frac{\epsilon}{3}.$$

Since $A = 1 - \|(P^m f)\cdot\chi_{(\beta,1]}\|_1 \le 1$ and P is a contraction in $\|\cdot\|_1$,

$$\begin{aligned}\|P^n f - g\|_1 &\le \|P^{n-m} f_\beta - A\cdot g\|_1 + \|(A-1)\cdot g\|_1 + \|P^{n-m}(P^m f - f_\beta)\|_1\\ &\le \|P^{n-m}(f_\beta/A) - g\|_1 + (1-A) + \|(P^m f)\cdot\chi_{(\beta,1]}\|_1\\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore (T, gdm) is exact. □

Theorem 6.5. *Suppose T satisfies the conditions in Theorem 1.1 and the conditions* **(M)**, **(E)**. *If $\lim_{x\to\beta^+} F_{N_*+1}(x) < 1$, then some power of T is expanding, hence Bernoulli.*

Using the additional hypothesis: $\lim_{x\to\beta^+} F_{N_*+1}(x) < 1$ and mixing condition **(M)**, we modify the proof of Lemma 4.14 and obtain the following.

Lemma 6.6. *Suppose the assumption in Theorem 6.5 holds. Then for each $j = 1,\cdots,N$, there exists $\alpha(j)\in\mathbb{N}$ such that*

$$\Big\|\prod_{k=1}^{\alpha(j)} F_j(\psi_j^{k-1})\Big\|_\infty < \frac{1}{2M^2}.$$

Proof. From the proof of Lemma 4.14, it suffices to show that if $N_* < j \le N$ and $T_j(z^*) = z^*$ for some $z^* \in [a_{j-1}, a_j]$ $(z^* \ge \beta)$, then there exist $\epsilon > 0$ and $w^* < z^*$ such that for any $y \in [w^*, 1]$, $F_j(y) \le 1-\epsilon < 1$. In fact, in case where $\mathcal{F}_j(z^*) < 1$, similarly the fact that $\mathcal{F}_j$ is upper semicontinuous completes the proof of the claim. If $\mathcal{F}_j(z^*) = 1$, then it also follows from (4.3) that $\mathcal{F}_j(x) = 1$ on $[0, z^*)$. Using $T[0,\beta] = [0,\beta]$ and the markov property of P $\int_0^1 \mathcal{F}_{N_*} = \int_0^\beta \mathcal{F}_{N_*} = \beta$ so that $\mathcal{F}_{N_*}(x) = 1$ on $[0,\beta]$. Conclude that $(\mathcal{F}_j - \mathcal{F}_{N_*})(x) = 0$ on $[0,\beta]$. Thus $T[\beta, z^*] \subseteq [\beta, z^*]$. By the mixing condition **(M)**, we get $z^* = \beta$ and so $j = N_* + 1$. Note that $F_{N_*+1}(x) = 0$ on $[0,\beta]$ and F_{N_*+1} is decreasing on $(\beta, 1]$. Since $\lim_{x\to\beta^+} F_{N_*+1}(x) < 1$, it follows that there exist $\epsilon > 0$ and $w^* < z^* = \beta$ such that for any $y \in [w^*, 1]$, $F_j(y) = F_{N_*+1}(y) \le 1-\epsilon < 1$. We observe also that there can be at most one fixed point in each monotonicity interval. For if $a_{j-1} \le z_1 < z_2 \le a_j$ are two fixed points, then as in the proof of Lemma 4.14 we see that $F_j \equiv 1$ on $[z_1, z_2]$ from which it follows that $T[\beta, z_2] \subseteq [\beta, z_2]$, contradicting **(M)**. The remainder of the argument follows as in Lemma 4.14. We omit the details. □

For the completion of the proof of Theorem 6.5, the rest of the arguments in the proof of Theorem 4.10 can be applied only with a slight modification, e.g., replacing N_* and $\mathcal{I}_\beta$ with N and $\mathcal{I}$, respectively.

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