

# Decay of correlations for piecewise invertible maps in higher dimensions

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## Abstract

We study the mixing properties of equilibrium states  $\mu$  of non-Markov piecewise invertible maps  $T : X \rightarrow X$ , especially in the multidimensional case. Assuming mainly Hölder continuity and that the topological pressure of the boundary is smaller than the total topological pressure, we establish exponential decay of correlations, i.e.:

$$\left| \int_X \varphi \cdot \psi \circ T^n d\mu - \int_X \varphi d\mu \cdot \int_X \psi d\mu \right| \leq C \cdot e^{-\alpha n}$$

for all Hölder functions  $\varphi, \psi : X \rightarrow \mathbb{R}$ , all  $n \geq 0$  and some  $C < \infty$ ,  $\alpha > 0$ . We also obtain a Central Limit Theorem. Weakening the smoothness assumption, we get subexponential rates of decay.

## Résumé

Nous étudions les propriétés de mélange des états d'équilibre  $\mu$  d'applications  $T : X \rightarrow X$  inversibles par morceaux et non-markoviennes, plus particulièrement dans le cas multi-dimensionnel. En supposant (essentiellement) la continuité Hölder et que la pression topologique du bord est strictement majorée par la pression totale, nous montrons la décroissance exponentielle des corrélations:

$$\left| \int_X \varphi \cdot \psi \circ T^n d\mu - \int_X \varphi d\mu \cdot \int_X \psi d\mu \right| \leq C \cdot e^{-\alpha n}$$

pour toutes les fonctions Hölder  $\varphi, \psi : X \rightarrow \mathbb{R}$ , tout  $n \geq 0$  et des constantes  $C < \infty$ ,  $\alpha > 0$ . Nous obtenons également un Théorème Central Limite. Une diminution de la régularité fait apparaître des vitesses sous-exponentielles.

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## INTRODUCTION

Our goal is to generalize ergodic and statistical properties of equilibrium states, which are well-known in dimension one [LY, HK, R, Ke1, Sc, Go], to a natural *multidimensional* setting. Several questions in this regard have already been considered (existence and characterization of equilibrium states [Bu2], construction of conformal measures [BPS], absolutely continuous invariant measures [Bu0, Bu1, Bu3, Co, GB, Sau] or zeta functions [BuKe]), sometimes with surprising results [Bu4, T3].

In this paper we study the speed of mixing of equilibrium states and prove that it is exponential. This implies, e.g., the Central Limit Theorem.

We first consider Hölder continuous weights for simplicity and then move on to the more general case of summable moduli of continuity [Sc] (which was the setting for the previous works [Bu3, BPS]) relying on an abstract result to be proved in a companion paper [BuMa].

Our setting will be the following.  $(X, \mathcal{Z}, T, g)$  will be a *weighted piecewise invertible map*, i.e.:

- $X = \overline{\bigcup_{Z \in \mathcal{Z}} Z}$  is a locally connected compact metric space.
- $\mathcal{Z}$  is a finite collection of pairwise disjoint, bounded and open subsets of  $X$ . Let  $Y = \bigcup_{Z \in \mathcal{Z}} Z$ .
- $T : Y \rightarrow X$  is a map such that each restriction  $T|_Z$ ,  $Z \in \mathcal{Z}$ , coincides with the restriction of a homeomorphism  $T_Z : U \rightarrow V$  with  $U, V$  open sets such that  $U \supset \bar{Z}$ ,  $V \supset \overline{T(Z)}$ .
- $g : X \rightarrow \mathbb{R}$ .

$T$  will be assumed to be *non-contracting*, i.e., such that for all  $x, y$  in the same element  $Z \in \mathcal{Z}$ ,  $d(Tx, Ty) \geq d(x, y)$ .

Also  $\mathcal{Z}$  will be assumed to be *generating*, i.e.,  $\lim_{n \rightarrow \infty} \text{diam}(\mathcal{Z}^n) = 0$  where  $\mathcal{Z}^n$  denotes the set of  $n$ -cylinders, i.e., the non-empty sets of the form:

$$[A_0 \dots A_{n-1}] := A_0 \cap \dots \cap T^{-n+1} A_{n-1}$$

for  $A_0, \dots, A_{n-1} \in \mathcal{Z}$ .

Finally the *boundary of the partition*,  $\partial\mathcal{Z} = \bigcup_{Z \in \mathcal{Z}} \partial Z$ , will play an important role in our analysis. In particular, we shall assume “small boundary pressure” (see below), a fundamental condition which already appeared in [Bu2, Bu3, BPS].

A basic example is given by the *multidimensional  $\beta$ -transformations* [Bu0], i.e., maps  $T : [0, 1]^d \rightarrow [0, 1]^d$ ,  $T(x) = B.x \bmod \mathbb{Z}^d$  with  $B$  an expanding affine map on  $\mathbb{R}^d$ . An interesting choice of weight is the constant  $g(x) = |\det B|^{-1}$ .

These systems usually have plenty of invariant probability measures, some of them rather irrelevant such as those supported on periodic orbits. A classical

way to select “interesting” measures is through the following variational principle [DGS]. One considers *equilibrium states*, i.e., invariant probability measures which maximize the *measure-theoretic pressure*:

$$h(\mu, T) - \int_Y \log g \, d\mu$$

( $h(\mu, T)$  is the entropy of  $\mu$  — see [DGS]).

In our *piecewise, multidimensional and expanding* setting, existence and uniqueness of these measures have been studied in [Bu2] and [BPS]. Furthermore in [Bu2] such measures have been given the alternative and more “geometric” characterization of being exactly the measures absolutely continuous w.r.t. a “conformal measure”, i.e., a not necessarily invariant, probability measure on  $X$ , such that the Jacobian of  $T$  w.r.t. this measure is equal to  $e^{-P(X, T)} g^{-1}$ . This conformal measure can be given (like Lebesgue measure if  $g = |\det T'|^{-1}$ ) or constructed from the weight  $g$  [BPS].

In the example given above (multidimensional  $\beta$ -maps), the conformal measure is just Lebesgue measure and thus the equilibrium states are the absolutely continuous invariant probability measures, which can indeed be considered interesting.

## Statement of Results

To formulate the crucial “small boundary pressure” condition, we need first some definitions.

The *topological pressure* [DGS] of a subset  $S$  of  $X$  is:

$$P(S, T) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\substack{A \in \mathcal{Z}^n \\ A \cap S \neq \emptyset}} g^{(n)}(A)$$

where  $g^{(n)}(A) = \sup_{x \in A} g(x)g(Tx) \dots g(T^{n-1}x)$ .

The *small boundary pressure* condition is:

$$P(\partial \mathcal{Z}, T) < P(X, T).$$

This inequality is satisfied in many cases. In particular, if  $T$  is expanding and  $X$  is a Riemannian manifold and the weight is  $|\det T'(x)|^{-1}$  or close to it, then it is satisfied: (i) in dimension 1, in all cases; (ii) in dimension 2, if  $T$  is piecewise real analytic [Bu3, T1]; (iii) in arbitrary dimension, for all piecewise affine  $T$  [T2] or for generic  $T$  [Bu1, Co]. See however [Bu4] for an expanding counter-example in dimension 2.

In our favorite example (multidimensional  $\beta$ -transformations, see above), the inequality is satisfied as soon as:

$$\frac{\sup g}{\inf g} < \left( \frac{\Lambda_-}{\Lambda^+} \right)^d \Lambda_+$$

with  $\Lambda_+$ , resp.  $\Lambda_-$ , the largest, resp. the smallest, modulus of the eigenvalues of the linear map corresponding to the  $\beta$ -transformation. Clearly, this is satisfied for  $g = |\det B|^{-1}$ .

## 0.1 The Expanding and Hölder continuous case

Recall that a *conformal measure* for  $(X, T, g)$  is a probability measure  $\nu$  such that  $\frac{d\nu \circ T}{d\nu} = e^{-P(X, T)} g$ .

**MAIN THEOREM.** *Let  $(X, \mathcal{Z}, T, g)$  be a weighted piecewise invertible dynamical system. Assume that:*

*H1.  $T$  is expanding, i.e., there is some  $\lambda > 1$  such that for all  $x, y$  in the same element of  $\mathcal{Z}$ ,  $d(Tx, Ty) \geq \lambda \cdot d(x, y)$ ;*

*H2.  $g$  is Hölder continuous with exponent  $\gamma$  and is positively lower bounded.*

*Let  $K(f) = \max_{Z \in \mathcal{Z}} \sup_{x \neq y \in Z} \frac{|f(x) - f(y)|}{d(x, y)^\gamma}$  where  $\gamma$  is some Hölder exponent of  $g$ .*

*H3. the boundary pressure is small:  $P(\partial\mathcal{Z}, T) < P(X, T)$ .*

*H4. there is a conformal measure  $\nu$  such that  $T^n X \subset \text{supp}(\nu) \pmod{0}$  for large enough  $n$ .*

*Then,  $T$  admits a finite number of ergodic and invariant measures,  $\mu_1, \dots, \mu_r$ , absolutely continuous w.r.t.  $\nu$ . Each  $\mu_i$  is exponentially mixing up to a period  $p_i$ , i.e.,  $\mu_i$  can be written  $\frac{1}{n} \sum_{j=0}^{p_i-1} T_*^j \mu_i^0$  with  $T_*^{p_i} \mu_i^0 = \mu_i^0$  and, for all  $n \geq 1$ ,*

$$\left| \int_X \varphi \circ T^{np_i} \cdot \psi d\mu_i^0 - \int_X \varphi d\mu_i^0 \int_X \psi d\mu_i^0 \right| \leq C \cdot (\sup |\varphi| + K(\varphi)) \cdot \|\psi\|_{L^1(\mu_i^0)} \kappa^n.$$

*with constants  $C < \infty$  and  $\kappa < 1$  depending only on  $(X, \mathcal{Z}, T, g)$ , for any measurable functions  $\varphi, \psi : X \rightarrow \mathbb{R}$  such that  $\psi$  is bounded and  $\varphi$  is  $\gamma$ -Hölder continuous.*

We also obtain:

**CENTRAL LIMIT THEOREM** *Under the same conditions, if  $\varphi$  is  $\gamma$ -Hölder continuous and satisfies  $\int_X \varphi d\mu_i^0 = 0$ , then setting:*

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \left( \sum_{k=0}^{n-1} \varphi \circ T^k \right)^2 d\mu_i^0$$

*we have  $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ T^k \implies \mathcal{N}(0, \sigma)$ , where  $\implies$  is the convergence in law and  $\mathcal{N}(0, \sigma)$  is the normal distribution with mean zero and variance  $\sigma$  (the Dirac measure at 0 if  $\sigma = 0$ ). Moreover,  $\sigma = 0$  iff  $\varphi = \psi - \psi \circ T$  for some  $\psi \in L^2(\mu_i^0)$ .*

## 0.2 The general case

We shall control the modulus of continuity of  $g$  by the following, dynamically defined, sequence:

$$\omega_n(g) = \sup_{Z \in \mathcal{Z}^n} \sup_{x, y \in Z} \log \frac{g(x)}{g(y)}.$$

In fact, it is convenient just to assume that  $\omega_n(g)$  is not less than the right hand side of the above equation.

Such a sequence define the following functional space. It is the set of functions  $\varphi$  on  $X$  such that:

$$\sup_{Z \in \mathcal{Z}^p} \sup_{x, y \in Z} |\varphi(x) - \varphi(y)| \leq K \cdot \sum_{q > p} \omega_q(p)$$

We then set  $K(\varphi)$  to be the infimum of all numbers  $K$  such that the above equation holds for all  $p \geq 1$ .

This condition together with this functional space were considered by B. Schmitt [Sc, KMS].

We may now state:

**MAIN THEOREM (general version)** *Let  $(X, \mathcal{Z}, T, g)$  be a weighted piecewise invertible dynamical system. Assume that:*

- H1.  $T$  is non-contracting;
- H2.  $g$  satisfies  $\sum_{n \geq 1} \omega_n(g) < \infty$  and that it is positively lower bounded.
- H3.  $P(\partial \mathcal{Z}, T) < P(X, T)$ .
- H4. there is a conformal measure  $\nu$  such that  $T^n X \subset \text{supp}(\nu) \pmod{0}$  for large enough  $n$ .

Then,  $T$  admits a finite number of ergodic and invariant measures,  $\mu_1, \dots, \mu_r$ , absolutely continuous w.r.t.  $\nu$ . Each  $\mu_i$  is mixing up to a period  $p_i$ , i.e.,  $\mu_i$  can be written  $\frac{1}{n} \sum_{j=0}^{p_i-1} T_*^j \mu_i^0$  with  $T_*^{p_i} \mu_i^0 = \mu_i^0$  and, for all  $n \geq 1$ ,

$$\left| \int_X \varphi \circ T^{np_i} \cdot \psi d\mu_i^0 - \int_X \varphi d\mu_i^0 \int_X \psi d\mu_i^0 \right| \leq (\sup |\varphi| + K(\varphi)) \cdot \|\psi\|_{L^1(\mu_i^0)} u_n.$$

for any measurable functions  $\varphi, \psi : X \rightarrow \mathbb{R}$  such that  $\psi$  is bounded and  $K(\varphi) < \infty$

The sequence  $(u_n)_{n \geq 1}$  depends on  $(T, g)$  and goes to zero with a speed which can be explicited:

1. if  $\omega_n(g) = \mathcal{O}(\rho^n)$  for some  $\rho < 1$ , then  $u_n = \mathcal{O}(\kappa^n)$  for some  $\kappa < 1$ ;
2. if  $\omega_n(g) = \mathcal{O}(n^{-\alpha})$  for some  $\alpha > 1$ , then  $u_n = \mathcal{O}(n^{-(\alpha-1)})$ .

We also obtain:

CENTRAL LIMIT THEOREM (*general version*) Assuming additionally that  $\sum_{n \geq 1} u_n < \infty$ , if  $K(\varphi) < \infty$  and  $\int_X \varphi d\mu_i^0 = 0$ , then setting:

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \left( \sum_{k=0}^{n-1} \varphi \circ T^k \right)^2 d\mu_i^0$$

we have  $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ T^k \implies \mathcal{N}(0, \sigma)$ , where  $\implies$  is the convergence in law and  $\mathcal{N}(0, \sigma)$  is the normal distribution with mean zero and variance  $\sigma$  (the Dirac measure at 0 if  $\sigma = 0$ ). Moreover,  $\sigma = 0$  iff  $\varphi = \psi - \psi \circ T$  for some  $\psi \in L^2(\mu_i^0)$ .

## Outline of the paper

The rest of the paper is devoted to the proof of the above Theorems.

First we define a Markov extension –or tower–  $(\hat{X}, \hat{T})$  à la Hofbauer-Keller. Then we check that the return time  $\hat{R}$  with respect to a slightly enlarged basis  $\hat{X}_*$  of this tower satisfies an exponential estimate with rate  $\exp -(P(X, T) - P(\partial Z, T))$ . Finally we build from this Markov extension an abstract tower in the spirit of L.-S. Young [Y0]. We show that the a.c.i.m.’s on  $X$  lifts to a.c.i.m.’s on this tower. From this, the Theorems above follow from an abstract result which is proved in a companion paper [BuMa] dealing with the non-Markov case. In the expanding and Hölder continuous case, they also follow from a slight adaptation of [Y0].

## Comments

- Our Main Theorem generalizes results well-known in the *globally expanding* case (see, e.g., [Bo]) or in the *piecewise expanding* one-dimensional setting (see [Ke1, LSV] and the references therein). The non-Hölder (piecewise invertible) case is new, even in the one-dimensional setting with  $g = |\det T'|$  —we remark that the claim in [Go] that (in dimension 1 and with  $g = |\det T'|^{-1}$ ) the decay of correlations is always exponential rests on a faulty lemma.

The special case  $g = |\det T'|^{-1}$  Hölder continuous has been considered in a multidimensional setting (under a slightly more restrictive condition than the small boundary pressure condition) in [Bu3, Sau].

- The assumption of small boundary pressure (H3) is natural. Indeed it reduces, for instance, in dimension 1, to a standard “spectral gap” condition. It also implies the conditions appearing in [Bu3, Sau] by a remark of M. Tsujii.
- The existence of a conformal measure (H4) is automatic in the case with  $g = |\det T'|^{-1}$  (Lebesgue measure is enough). In the general case, it was proved in a similar setting by J. Buzzi, F. Paccaut, B. Schmitt [BPS].

- L.S. Young [Y2] has asked how one could *control the upper floors* of the tower for equilibrium states. We prove here that they are controlled in thermodynamical terms: their conformal measure decreases like  $e^{-(P(X,T)-P(\partial Z,T))n}$  —see Proposition 1.1.
- By going to a Markov extension, we avoid spaces of discontinuous functions, i.e., spaces of functions with bounded *multi-dimensional variation*. This is already a significant technical gain in the case of Lebesgue measure ( $g = |\det T'|^{-1}$ ) and seems necessary for more general weights as the classical functional spaces are no longer adapted and tailored ones (see [Ke1, Sau]) become too difficult to control ([P] studies however a promising functional space).
- Our result are deduced from similar statements about a tower extension the spirit of L.-S. Young (see [Y0, Y1]). These statements, quoted at the end of our proof, are proved in [BuMa].

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## 1 THE MARKOV EXTENSION

We define a Markov tower extension  $(\hat{X}, \hat{T})$  in the vein of F. Hofbauer [Ho] and G. Keller [Ke2] and then prove that the pressure of the boundary provides an exponential estimate on the tail statistics of the return times to the basis.

### 1.1 Construction

Let  $(X, Z, T, g, \nu)$  be as in the Main Theorem. Let  $\hat{X}_0 = \{(x, Z) : x \in Z \text{ and } Z \in \mathcal{Z}\}$ . Recall that  $Y = \bigcup_{Z \in \mathcal{Z}} Z$  is the domain of  $T$ . For  $x \in Y \cap T^{-1}Y$  and  $A \subset X$ , set:

$$\hat{T}(x, A) = (Tx, T(A) \cap \mathcal{Z}[Tx])$$

where  $\mathcal{Z}[x]$  is that element of  $\mathcal{Z}$  which contains  $x$ .

Set  $\hat{X} = \bigcup_{n \geq 0} \hat{T}^n(\hat{X}_0)$ . Remark that

$$\hat{X} = \bigcup_{D \in \mathcal{D}} \hat{D} \text{ with } D \times \{D\} \text{ and } \mathcal{D} = \{T^n Z : n \geq 0 \text{ and } Z \in \mathcal{Z}^{n+1}\}.$$

Thus,  $\hat{X}$  is a countable, disjoint union of (sets naturally isomorphic to) subsets of  $X$ .

$\mathcal{D}$  has a natural graph structure:  $D \rightarrow D'$  iff  $D' = T(D) \cap Z$  for some  $Z \in \mathcal{Z}$ .

## 1.2 The statistics of returns

We shall consider returns to an enlarged basis of the Markov extensions:  $\hat{X}_* = T^{N_*} \hat{X}_0$  for some  $N_*$ .  $\hat{X}_*$  can also be seen as the disjoint union of  $\mathcal{D}_*$ , the collection of the sets  $T_*^N Z$ ,  $Z \in \mathcal{Z}^{N_*+1}$ .

The figure below illustrate why points not too close to the boundary return to  $\hat{X}_0$ , explaining the idea behind the next proposition.

figure=tower.epsf,width=15cm

*How points return to  $\hat{X}_0$ .*

*$\mathcal{Z}$  is a grid of small squares and we have drawn some set  $A$  and its image and pointed out the behavior of  $\hat{T}(x, A)$  for  $x \in A$ .*

PROPOSITION 1.1 *For each  $\delta > 0$ , if  $N_* < \infty$  is large enough, then for*

$$\hat{X}_* = \hat{T}^{N_*} \hat{X}_0,$$

*the return time to  $\hat{X}_*$ ,  $\hat{R}(\hat{x}) = \inf\{n \geq 1 : \hat{T}^n(\hat{x}) \in \hat{X}_*\}$ , satisfies, for each  $D \in \mathcal{D}$ ,*

$$\nu\left(\{x \in D : \hat{R}((x, D)) > n\}\right) \leq \text{const} \cdot \exp[-n(P(X, T) - P(\partial\mathcal{Z}, T) - \delta)]$$

*for some positive number  $\text{const} < \infty$ . This number depends on  $D$ .*

We first prove that if a point is close to  $\partial\mathcal{Z}$  then it remains bound to  $\partial\mathcal{Z}$  for a long time. Remark that this is obvious in dimension 1.

LEMMA 1.2 *Given  $n < \infty$ , there exists  $\varepsilon > 0$  such that, for all  $x \in X$  such that  $\mathcal{Z}^n[x]$  is defined,*

$$d(x, \partial\mathcal{Z}[x]) < \varepsilon \implies \overline{\mathcal{Z}^n[x]} \cap \partial\mathcal{Z} \neq \emptyset.$$

*Proof:* We prove that for any  $k \geq 1$ , for every  $\varepsilon > 0$  there exists  $\delta > 0$  depending only on  $\mathcal{Z}^{k+1}[x]$  such that:

$$d(x, \partial\mathcal{Z} \cap \overline{\mathcal{Z}^k[x]}) < \delta \implies d(x, \partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}) < \varepsilon. \quad (1)$$

Here we use the convention  $d(x, \emptyset) = \infty$ . Thus, the above implication assumes that  $\partial\mathcal{Z} \cap \overline{\mathcal{Z}^k[x]} \neq \emptyset$  and concludes that  $\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]} \neq \emptyset$ .

The Lemma will follow by an obvious induction and using that  $\mathcal{Z}^{n+1}$  is finite. For arbitrary  $\varepsilon > 0$ , let

$$\delta := \begin{cases} d(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^k[x]}, \mathcal{Z}^{k+1}[x] \setminus B(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}, \varepsilon)) \\ \text{if } \mathcal{Z}^{k+1}[x] \setminus B(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}, \varepsilon) \neq \emptyset \\ 1 + \varepsilon \end{cases} \quad \text{otherwise.}$$

Observe that  $\delta$  is finite (the sets are not empty) and positive: otherwise

$$\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x] \setminus B(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}, \varepsilon)} \neq \emptyset$$

which is absurd. Let us prove (1).

If  $\mathcal{Z}^{k+1}[x] \setminus B(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}, \varepsilon) = \emptyset$ , (1) is obvious. Otherwise,  $d(x, \partial\mathcal{Z} \cap \overline{\mathcal{Z}^k[x]}) < \delta$  implies  $x \notin \mathcal{Z}^{k+1}[x] \setminus B(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}, \varepsilon)$ . This implies that  $x \in B(\partial\mathcal{Z} \cap \overline{\mathcal{Z}^{k+1}[x]}, \varepsilon)$ , proving (1) and the Lemma.  $\blacksquare$

We turn to the proof of the Proposition.

*Proof:* Let  $\delta > 0$  and  $D \in \mathcal{D}$ . Observe that  $D$  being of the form  $T^k(A_0 \cap \dots \cap T^{-k}A_k)$ ,  $P(\partial D, T) \leq P(\partial\mathcal{Z}, T)$ .

Pick  $N$  so large that for all  $n \geq 0$ ,

$$\sum_{\substack{Z \in \mathcal{Z}^n \\ Z \cap (\partial D \cup \partial\mathcal{Z}) \neq \emptyset}} g^{(n)}(Z) \leq e^{\delta N} \cdot e^{(P(\partial\mathcal{Z}, T) + \delta)n} \text{ and } \binom{2n/N}{n} \leq e^{\delta n}. \quad (2)$$

Let  $\varepsilon_1 = \varepsilon(N) > 0$  be given by Lemma 1.2. Let  $\varepsilon_2 > 0$  be such that for all  $x \in X$  there is a connected  $\Gamma \subset X$  with  $B(x, \varepsilon_2) \subset \Gamma \subset B(x, \varepsilon_1)$ . This exists because of the local connexity and the compactness of  $X$ .

For  $x \in D$  and  $n \geq 0$ , let

$$\rho_n(x) := \sup\{r > 0 : B(T^n x, r) \subset T^n(D \cap \mathcal{Z}^{n+1}[x])\}.$$

and set

$$R(x) := \min\{n \geq 0 : \rho_n(x) \geq \varepsilon_2\}.$$

We choose  $N_* < \infty$  so large that the diameter of the partition  $\mathcal{Z}^{N_*}$  is less than  $\varepsilon_2$ . Let us observe that

$$\hat{R}((x, D)) \leq R(x) + N_*.$$

Indeed,

$$\begin{aligned} \hat{T}^{R(x)+N_*}(x, D) &= \left( T^{R(x)+N_*}x, T^{N_*}(\mathcal{Z}^{N_*}[T^{R(x)}x] \cap T^{R(x)}(D \cap \mathcal{Z}^{R(x)}[x])) \right) \\ &= (T^{R(x)+N_*}x, T^{N_*}\mathcal{Z}^{N_*}[T^{R(x)}x]) \\ &= \hat{T}^{N_*}(T^{R(x)}x, \mathcal{Z}[T^{R(x)}x]) \in T^{N_*}\hat{X}_0 = \hat{X}_*. \end{aligned}$$

It is therefore enough to prove the exponential estimate for  $\nu(\{x \in D : R(x) > n\})$ .

**Claim.** For every  $x \in D$  such that  $R(x) > n$ , there exists a finite sequence of times  $n_0 = 0 < n_1 < \dots < n_r = n$  such that, for all  $0 \leq i < r$ ,

$$r \leq \frac{2n}{N} + 1 \quad (3)$$

$$\overline{\mathcal{Z}^{n_{i+1}-n_i}(T^{n_i}x)} \cap (\partial\mathcal{Z} \cup \partial D) \neq \emptyset \quad (4)$$

where  $N$  was defined in (2).

Let us see that the claim implies the proposition. Let  $C = \sup_n \sup_{Z \in \mathcal{Z}^n} \sup_{x, y \in Z} \frac{g^{(n)}(x)}{g^{(n)}(y)}$ .  $C < \infty$  because  $g$  is Hölder and positively lower-bounded, and  $T$  is expanding. We compute:

$$\begin{aligned}
\nu(\{R > n\}) &\leq \sum_{\substack{Z \in \mathcal{Z}^n \\ \exists x \in Z \text{ s.t. } R(x) > n}} \nu(Z) \\
&\leq \sum_{\substack{Z \in \mathcal{Z}^n \\ \exists x \in Z \text{ s.t. } R(x) > n}} C \cdot e^{-nP(X,T)} g^{(n)}(x) \cdot \nu(T^n Z) \\
&\text{(we used that } \nu \text{ is conformal)} \\
&\leq C \cdot e^{-nP(X,T)} \sum_{r=1}^{(2n/N)+1} \sum_{n_0, \dots, n_r} \\
&\prod_{i=0}^{r-1} \sum_{\substack{Z \in \mathcal{Z}^{n_{i+1}-n_i} \\ \bar{Z} \cap (\partial Z \cup \partial D) \neq \emptyset}} g^{(n_{i+1}-n_i)}(Z) \\
&\leq C \cdot e^{-nP(X,T)} \left( \frac{2n}{N} + 1 \right) \binom{\frac{2n}{N}}{n} e^{\delta N \frac{2n}{N}} e^{n(P(\partial Z, T) + \delta)} \\
&\leq C \cdot e^{-n(P(X,T) - P(\partial Z, T) - 5\delta)}
\end{aligned}$$

which is the statement of the Proposition (up to substituting  $\delta/5$  to  $\delta$ ). Thus, we only need to prove the claim by building the required times  $n_i$ .

Define  $p(k) = \max\{p \geq k : \overline{\mathcal{Z}^{p-k+1}[T^k x]} \cap \partial \mathcal{Z} \neq \emptyset\} + 1$  for  $k > 0$  and  $p(0)$  by the same formula with  $\partial D$  instead of  $\partial \mathcal{Z}$ .

We set  $n_0 = 0$  and define the rest of the  $n_i$ 's inductively.

If  $p(n_i) \geq n$ , then set  $r := i$  and stop. Otherwise proceed as follows.

As  $p := p(n_i) < n \leq R(x)$ ,  $B(T^p x, \varepsilon_2) \not\subset T^p \mathcal{Z}^{p+1}[x]$ . By the choice of  $\varepsilon_2$ , there exists a connected  $\Gamma$  with  $B(T^p x, \varepsilon_2) \subset \Gamma \subset B(T^p x, \varepsilon_1)$ . As  $\Gamma$  is connected, there must be points of the boundary of  $T^p \mathcal{Z}^{p+1}[x]$  within  $\varepsilon_1$  of  $T^p x$ . Each such point is the image by  $T^{p-q}$  of  $\overline{\mathcal{Z}^{p-q+1}[T^q x]}$  of some point  $z_{i+1}$  of  $\partial \mathcal{Z}[T^q x]$  for some  $q \in [0, p]$ .  $d(z_{i+1}, \partial \mathcal{Z}[T^q x]) < \varepsilon_1$  as  $T$  is non-contracting. Select such a  $q$  maximizing  $p(q)$ . Finally set  $n_{i+1} = q$  and go on inductively.

Remark that  $p(n_{i+1}) > p(n_i)$  by construction (indeed,

$z_{i+1} \in \overline{\mathcal{Z}^{p(n_i)-n_{i+1}+1}[T^{n_{i+1}} x]} \cap \partial \mathcal{Z}$ ) so that  $r$  is well-defined. Also  $p(n_{i-1}) < n_{i+1}$  because otherwise, as  $p(n_{i+1}) > p(n_i)$ ,  $q = n_{i+1}$  would contradict the maximality of  $p(n_i)$  among the  $p(q)$ ,  $q \leq p(n_{i-1})$ .

Thus we have defined a finite sequence  $n_0 = 0 < n_1 < \dots < n_r$ . This sequence obviously satisfies eq. (4).

We conclude by proving that for  $1 \leq i \leq r - 2$ ,  $n_{i+2} - n_i \geq N$  so that eq. (3) is also satisfied.

Consider the point  $z_i$  defined above. As  $d(z_i, T^{n_i} x) < \varepsilon_1$ ,  $p(n_i) \geq n_i + N$  by Lemma 1.2. Now, we saw above that  $n_{i+2} \geq p(n_i)$ . This proves the claim and the proposition.  $\blacksquare$

## 2 THE INDUCING TOWER

Working from the previous Markov extension  $\hat{X}$ , we define another tower describing the map induced by  $\hat{T}$  on  $\hat{X}_*$  following L.-S. Young's philosophy [Y1].

Let  $\Delta_0 = \hat{X}_*$  be the enlarged basis of the Markov tower previously defined. We shall write  $x$  for a point of  $\Delta_0 \subset X \times \mathcal{D}$ .

Let  $(\Delta_{0,j})_{j \in \mathbb{N}}$  be the partition of  $\Delta_0$  obtained by dividing it first into level sets  $\{x : \hat{R}(x) = r\}$  and then according to  $\mathcal{Z}^{r+1}$  on the  $r$ th level set, i.e.,  $(\Delta_{0,j})_{j \in \mathbb{N}}$  is the coarsest partition of  $\Delta_0$  such that  $\hat{R}(x)$  and  $Z_{\hat{R}(x)}(x)$  are constant on atoms.

We define

$$\Delta_\ell = \{(x, \ell) / x \in \Delta_0 \text{ and } \hat{R}(x) > \ell\}.$$

The tower  $\Delta \subset X \times \mathcal{D} \times \mathbb{N}$  is the disjoint union of the  $\Delta_\ell$ 's. We will denote by  $\Delta_{\ell,j} \subset \Delta_\ell$  the set of  $(x, \ell)$  such that  $(x, 0)$  belongs to  $\Delta_{0,j}$ . The  $\Delta_{\ell,j}$ 's for  $\ell \in \mathbb{N}$ ,  $j \in \mathbb{N}$ , form a partition of  $\Delta$  which we will denote by  $\mathcal{P}$ .

Let  $F : \Delta \rightarrow \Delta$  be defined in the following way:

$$\begin{cases} F(x, \ell) &= (x, \ell + 1) \text{ if } \ell + 1 < \hat{R}(x) \\ &= (\hat{T}^{\hat{R}(x)} x, 0) \text{ otherwise.} \end{cases}$$

Remark that we do not assume that the images  $F^R \Delta_{0,j}$  are the whole basis  $\Delta_0$ . That is why we have to introduce the partition  $\{B_1, \dots, B_p\}$  of  $\Delta_0$  generated by  $\{F^R \Delta_{0,j} : j \in \mathbb{N}\}$ . It is finite because each  $F^R \Delta_{0,j}$  corresponds to some union of elements of  $\mathcal{D}_*$ , which is finite.

Finally, we lift the probability measure  $\nu$  on  $X$  to a measure  $\hat{\nu}$  on  $\Delta$  defined by:

$$\hat{\nu}(S \times \{D\} \times \{\ell\}) = \nu(S).$$

Since  $\Delta_0$  is a finite union of  $\hat{D}$ ,  $D \in \mathcal{D}$ ,  $\hat{\nu}(\Delta_0) < \infty$ . Also, by the choice of  $N_*$ ,  $\text{supp}(\hat{\nu}) = \Delta$ .

We define the following metric on  $\Delta$ :

$$d_0(x, y) = C \cdot \exp \sum_{j > s(x,y)} \omega_j(g).$$

where  $s(x, y) = \min\{k \geq 0 : F^k(x), F^k(y) \text{ are not in the same element of } \mathcal{P}\}$  and  $C = \exp \sum_{j \geq 1} \omega_j(g)$ . This last constant is introduced to have (A.III) below.

Let us summarize the crucial properties of the tower.

(A.I) **Exponentially small upper floors.** 1.1,

$$\hat{\nu}(\{x \in \Delta_0 / \hat{R}(x) > \ell\}) \leq \text{const} \cdot \gamma^\ell$$

for some  $0 < \gamma < 1$ . In particular,  $\hat{\nu}(\Delta) = \sum_\ell \hat{\nu}(\Delta_\ell) \leq \text{const} \sum_\ell \gamma^\ell < \infty$ . Moreover, the support of  $\hat{\nu}$  is  $\Delta$ : this follows from our construction and the fact that the support of  $\nu$  contains  $T^{N^*}X$ . In what follows, we assume that  $\hat{\nu}$  has been normalized i.e.  $\hat{\nu}(\Delta) = 1$ .

(A.II) **Generating Partition.** The partition  $\mathcal{P}$  generates under  $F$  i.e.: the partition  $\bigvee_{n=0}^{\infty} F^{-n}\mathcal{P}$  is the partition into points. In particular,  $d_0$  defines a metric on  $\Delta$ .

(A.III) **Bounded distortion.** Let  $JF$  be the Jacobian of  $F$  with respect to  $\hat{\nu}$ . Obviously,  $JF(x, \ell) = 1$  if  $\ell + 1 < \hat{R}(x)$  and  $JF(x, \hat{R}(x) - 1) = g_{\hat{R}(x)}(x)$ . If  $x, y$  are in the same  $B_j$ , we can defined their paired pre-images  $x'$  and  $y'$  as follows.  $F^n x' = x$ ,  $F^n y' = y$  and  $F^k(x')$  and  $F^k(y')$  belong to the same element of  $\mathcal{P}$  for  $0 \leq k < n$ . We have:

$$\left| \frac{JF^n(x')}{JF^n(y')} - 1 \right| \leq d_0(x, y).$$

Indeed,

$$\left| \frac{JF^n(x')}{JF^n(y')} - 1 \right| \leq C \cdot \left| \log \frac{JF^n(x')}{JF^n(y')} \right| \leq \sum_{j \geq n+s(x,y)} \omega_j(n).$$

(A.IV) **Large image and Markov properties.** *Large image:* there exists  $\eta > 0$  such that

$$\inf_{j \in \mathbb{N}} \hat{\nu}(F^{\hat{R}}(\Delta_{0,j})) \geq \eta.$$

*Markov property:*  $F^{\hat{R}}\Delta_{0,j}$  is a union of some  $\Delta_{0,p}$ .

## 2.1 Lifting a.c.i.m.'s

To deal with an ergodic a.c.i.m. on  $X$  we first have to lift it to the inducing tower  $\Delta$ , i.e.,

**PROPOSITION 2.1** *If  $\mu$  is an ergodic  $\nu$ -a.c.i.m. on  $X$ , then there exists an ergodic  $\hat{\nu}$ -a.c.i.m.  $\hat{\mu}$  on  $\Delta$  projecting to  $\mu$ .*

To prove this, we begin by finding a non zero, not necessarily invariant probability measure  $\hat{\mu}_0$  on  $\Delta$  with projection on  $X$  absolutely continuous w.r.t.  $\mu$ . We set  $\hat{\mu}_0 = 1_{\hat{X}_* \cap \pi^{-1}(S)} \cdot \hat{\nu}$  where  $S$  is invariant and such that  $\mu(S) = 1$  and  $\mu'(S) = 0$  for all other ergodic  $\nu$ -a.c.i.m.'s (there can be only countably many measures wich are pairwise singular and absolutely continuous w.r.t. a given measure).

We have to check that this  $\hat{\mu}_0$  can be normalized. Because of (A.I),  $\pi_*(\hat{\nu}) \leq C \cdot \nu$  for some constant  $C < \infty$ , so that  $\hat{\mu}_0$  is finite. We have to see that  $\hat{\mu}_0$  is not zero. But this follows from the invariance of  $S$  and the fact that  $\hat{\nu}$ -a.e. point of  $\hat{X}$  eventually enters  $\hat{X}_*$  by Proposition 1.1.

Thus we can normalize  $\hat{\mu}_0$ .

Now, we claim that we can take  $\hat{\mu}$  to be any weak\* limit point, say  $\hat{\mu}$ , of the sequence  $\frac{1}{n} \sum_{k=0}^{n-1} F_*^k \hat{\mu}_0$ .

Clearly  $\hat{\mu}$  is  $F$ -invariant. Let us show that  $\hat{\mu}$  is  $\hat{\nu}$ -absolutely continuous. For this we consider the sequence of the corresponding densities. Introducing the transfer operator for  $(F, \hat{\nu})$ ,

$$\mathcal{L}f(x) = \sum_{y \in F^{-1}(x)} \frac{f(y)}{JF(y)}$$

where  $JF$  is the Jacobian of  $F$  w.r.t.  $\hat{\nu}$ , these densities are  $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k \frac{d\hat{\mu}_0}{d\hat{\nu}}$  (recall that  $\frac{d\hat{\mu}_0}{d\hat{\nu}} = C$ ). The absolute continuity will therefore follow from:

**Claim.** There is a constant  $K < \infty$  such that for all  $x \in \Delta$ ,  $n \geq 0$ ,  $(\mathcal{L}^n 1)(x) \leq K$ .

The claim together with  $\hat{\nu}(\Delta) = 1$  allows the application of the Lebesgue dominated convergence theorem to see that the  $\mu$ -absolute continuity of the projections of the measures  $F_*^k \hat{\mu}_0$  passes to the limit  $\hat{\mu}$ . For the same reason,  $\hat{\mu}$  is a probability measure. Finally, as  $\pi_* \hat{\mu} \ll \nu$  and  $\pi_* \hat{\mu}(S) = 1$ ,  $\pi_* \hat{\mu} \ll \mu$  and, by ergodicity,  $\pi_* \hat{\mu} = \mu$ , proving the Proposition, except for the claim. We need some facts.

Recall that  $\mathcal{P}$  is the partition of  $\Delta$  obtained by translation from the partition of  $\Delta_0$  according to the return time  $\hat{R}$  and  $Z^{\hat{R}}$ -itinerary. For  $k \in \mathbb{N}$  and  $x \in \Delta$ ,  $C_k(x)$  denotes the  $k$ ,  $\mathcal{P}$ -cylinder which contains  $x$ .

**LEMMA 2.2** *There exists  $C < \infty$  such that for any  $\ell \in \mathbb{N}$ ,  $x \in \Delta_\ell$  and  $k \in \mathbb{N}$  with  $F^k x \in \Delta_0$ ,*

$$C^{-1} \hat{\nu}(C_k(x)) \leq \frac{1}{JF^k(x)} \leq C \hat{\nu}(C_k(x)).$$

*Proof:* Let  $x \in \Delta_\ell$  such that  $F^k(x) \in \Delta_0$ . The Markov property and the large image property (A.IV) imply that  $\hat{\nu}(F^k C_k(x)) \geq \eta > 0$ . The bounded distortion property (A.III) gives:

$$\begin{aligned} C^{-1} \frac{\hat{\nu}(C_k(x))}{\hat{\nu}(F^k C_k(x))} &\leq \frac{1}{JF^k(x)} \leq C \frac{\hat{\nu}(C_k(x))}{\hat{\nu}(F^k C_k(x))} \\ C^{-1} \frac{\hat{\nu}(C_k(x))}{1} &\leq \frac{1}{JF^k(x)} \leq C \frac{\hat{\nu}(C_k(x))}{\eta} \end{aligned}$$

The Lemma is proved. ■

We now prove the claim.

The upper bound  $\mathcal{L}^n \mathbf{1} \leq K$  follows from Lemma 2.2, writing:

$$\mathcal{L}^n \mathbf{1}(x) = \sum_{x' \in F^{-n}x} \frac{1}{JF^n(x')} \leq C \sum_{x' \in F^{-n}x} \hat{\nu}(C_n(x')) \leq C \cdot \hat{\nu}(\hat{X}) \quad (1)$$

### 3 PROOF OF THE THEOREMS

Let us first quote a theorem with two corollaries that give our Main Theorem and Central Limit Theorem for an abstract tower. These are proved in a companion paper which considers a more general setting, in particular non-Hölder smoothness. However the reader may remark that in the Hölder setting of the present paper, these facts essentially reduce to L.-S. Young's abstract results [Y1], after taking care of the following technicalities:

- We have a “Markov rather than a Bernoulli picture”: the returns are not to  $\Delta_0$ , but to finitely many elements  $B_1, \dots, B_p \subset \Delta_0$ ;
- Our return times are not lower bounded by an arbitrary *a priori* constant;
- The tower is not aperiodic.

**THEOREM 3.1** [BuMa] *Let  $(\Delta, F, \hat{\nu}, d_0)$  be a tower system satisfying (A.I - IV). Let  $\mathcal{L}$  be the transfer operator associated to it.*

*Then, if  $\hat{\mu}$  is a  $\hat{\nu}$ -a.c.i.m., there exist an integer  $1 \leq p < \infty$  a decomposition  $\Delta = \bigcup_{k=0}^{p-1} T^k \Delta_*$  (mod  $\hat{\mu}$ ) ( $F^p(\Delta_*) = \hat{X}_*$ ) and a function  $h_* \in L^\infty(\Delta_*)$  such that, for any  $\varphi : \Delta_* \rightarrow \mathbb{R}$  bounded with  $K(\varphi) < \infty$ ,*

$$\|\mathcal{L}^n \varphi - \hat{\mu}_*(\varphi) \cdot h_*\|_\infty \leq (\|\varphi\|_\infty + K(\varphi)) \cdot u_n$$

*for all  $n \geq 0$ . Here  $\hat{\mu}_* = p \cdot \hat{\mu} \cdot 1_{\Delta_*}$ ,  $K(\varphi)$  is the supremum of the Lipschitz constants, w.r.t.  $d_0$ , of the restrictions  $\varphi|_{\Delta_{i,j}}$ .*

*The sequence  $(u_n)_{n \geq 1}$  depends on  $(F, \hat{\nu})$  and goes to zero with a speed which can be explicited:*

1. *if  $\omega_n(g) = \mathcal{O}(\rho^n)$  for some  $\rho < 1$ , then  $u_n = \mathcal{O}(\kappa^n)$  for some  $\kappa < 1$ ;*
2. *if  $\omega_n(g) = \mathcal{O}(n^{-\alpha})$  for some  $\alpha > 1$ , then  $u_n = \mathcal{O}(n^{-(\alpha-1)})$ .*

**COROLLARY 3.2** [BuMa] *In the same situation, we have that for all  $\varphi, \psi \in L^\infty(\Delta_*)$  with  $K(\varphi) < \infty$ ,*

$$\left| \int_{\Delta} \varphi \circ F^n \cdot \psi d\hat{\mu} - \int_{\Delta} \varphi d\hat{\mu} \int_{\Delta} \psi d\hat{\mu} \right| \leq (\|\varphi\|_\infty + K(\varphi)) \|\psi\|_{L^1(\hat{\mu}_*)} \cdot u_n$$

*for all  $n \geq 0$  for the same sequence  $(u_n)_{n \geq 1}$  as above.*

**COROLLARY 3.3** [BuMa] *Assuming additionally that  $\sum_{n \geq 1} u_n < \infty$ , we have the Central Limit Theorem for all functions  $\varphi$  bounded and  $d_0$ -Lipschitz.*

To deduce our Main Theorem and our Central Limit Theorem, it is enough to first lift the ergodic  $\nu$ -a.c.i.m.  $\mu$  to  $\Delta$  using Proposition 2.1 and make the following remark. If  $\varphi, \psi : X \rightarrow \mathbb{R}$  are as in the Main Theorem, then  $\varphi \circ \pi$  is bounded and  $\gamma$ -Hölder continuous and  $\psi \circ \pi \in L^\infty$ . Finally remark that the integrals over  $\Delta$  involving these lifted functions are equal to the integrals in the Main Theorem.

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