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Integrable lattice realizations of conformal twisted boundary conditions

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Abstract

We construct integrable lattice realizations of conformal twisted boundary conditions for $\widehat{s\ell}(2)$ unitary minimal models on a torus. These conformal field theories are realized as the continuum scaling limit of critical A–D–E lattice models with positive spectral parameter. The integrable seam boundary conditions are labelled by $(r, s, \zeta) \in (A_{g-2}, A_{g-1}, \Gamma)$ where Γ is the group of automorphisms of the graph G and g is the Coxeter number of $G = A, D, E$. Taking symmetries into account, these are identified with conformal twisted boundary conditions of Petkova and Zuber labelled by $(a, b, \gamma) \in (A_{g-2} \otimes G, A_{g-2} \otimes G, \mathbb{Z}_2)$ and associated with nodes of the minimal analog of the Ocneanu quantum graph. Our results are illustrated using the Ising (A_2, A_3) and 3-state Potts (A_4, D_4) models. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

There has been much recent progress [1–10] on understanding integrable boundaries in statistical mechanics, conformal boundary conditions in rational conformal field theories and the intimate relations between them on both the cylinder and the torus. Indeed it appears that, for certain classes of theories, all of the conformal boundary conditions on a cylinder can be realized as the continuum scaling limit of integrable boundary conditions for the associated integrable lattice models. For $\widehat{s\ell}(2)$ minimal theories, a complete classification has been given [1–3] of the conformal boundary conditions on a cylinder. These are labelled by nodes (r, a) of a tensor product graph $A \otimes G$ where the pair of graphs (A, G), with G of A–D–E type, coincide precisely with the pairs in the A–D–E classification of Cappelli et al. [11]. Moreover, the physical content of the boundary conditions on the cylinder has been ascertained [4,8] by studying the related integrable boundary conditions of the associated A–D–E lattice models [12] for both positive and negative spectral parameters, corresponding to *unitary minimal theories* and *parafermionic theories*, respectively. Recently, the lattice realization of integrable and conformal boundary conditions for $N = 1$ *superconformal theories*, which correspond to the *fused* A–D–E lattice models with positive spectral parameter, has also been understood in the diagonal case [10].

In this Letter, we use fusion to construct integrable realizations of conformal twisted boundary conditions on the torus [6,7]. Although the methods are very general we consider $\widehat{s\ell}(2)$ unitary minimal models for concreteness. The key idea is that fused blocks of elementary face weights on the lattice play the role of the local operators in the theory. The integrable and conformal boundary conditions on the cylinder are con-

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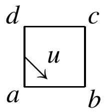
structed [4] by acting with these fused blocks on the simple integrable boundary condition representing the vacuum. By the generalized Yang–Baxter equations, these fused blocks or seams can be propagated into the bulk without changing the spectrum of the theory. The seams so constructed provide integrable and conformal boundary conditions on the torus. One subtlety with this approach is that fixed boundary conditions $a \in G$ on the edge of the cylinder are propagated into the bulk by the action $a = \zeta(1)$ of a graph automorphism ζ, which preserves the Yang–Baxter structure, on the distinguished (vacuum) node $1 \in G$. In general, for rational conformal field theories on the torus, we expect the fusions supplemented by the automorphisms to generate all of the integrable and conformal seams. We illustrate our approach in this Letter by using the Ising (A_2, A_3) and 3-state Potts (A_4, D_4) models as examples. A detailed consideration of the A–D–E unitary minimal models will be given in a forthcoming paper [13].

2. Lattice realization of twisted boundary conditions

2.1. A–D–E lattice models and integrable seam weights

The $\widehat{s\ell}(2)$ unitary minimal theories [14] are realized as the continuum scaling limit of critical A–D–E lattice models [12]. The spin states a, b, c, d are nodes of a graph $G = A, D, E$ with Coxeter number g. The bulk face weights are

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) =$$



$$= s_1(-u)\delta_{ac} + s_0(u)\frac{\sqrt{\psi_a\psi_c}}{\psi_b}\delta_{bd}, \quad (2.1)$$

where u is the spectral parameter with $0 < u < \lambda$, $\lambda = \pi/g$ is the crossing parameter, $s_k(u) = \sin(u + k\lambda)/\sin\lambda$ and the weights vanish if the adjacency condition is not satisfied on any edge. The crossing factors ψ_a are the entries of the Perron–Frobenius eigenvector of the adjacency matrix G. Note that if $\lambda - \pi/2 < u < 0$, the continuum scaling limit describes the principal $\mathbb{Z}_k$ parafermions with $k = g - 2$ [8].

The A–D–E lattice models are Yang–Baxter integrable [15] on a cylinder in the presence of a boundary [16] with boundary conditions labelled [3] by $(r, a) \in (A_{g-2}, G)$. A general expression for the (r, a) boundary weights is given in [4]. They are constructed by starting at the edge of the lattice with a fixed node $a \in G$ and adding a fused block of $r - 1$ columns. Strictly speaking, this construction on the cylinder is implemented with double row transfer matrices. Nevertheless, our idea is to propagate these boundary conditions into the bulk using a description in terms of single row transfer matrices. The expectation, which we confirm numerically, is that these integrable boundary conditions continue to be "conformal" in the bulk.

For $(r, s, \zeta) \in (A_{g-2}, A_{g-1}, \Gamma)$, we define integrable seam weights

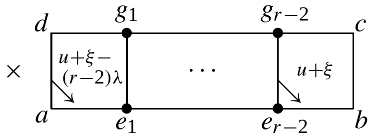
$$W^{(r,s,\zeta)}\left(\begin{matrix} d & \gamma & c \\ a & \alpha & b \end{matrix}\,\middle|\, u, \xi\right),$$

by taking the fusion product of three seams of types $(r, s, \zeta) = (r, 1, 1),\ (1, s, 1),\ (1, 1, \zeta)$, respectively. The $(r, 1, 1)$ seam is obtained by fusing $r - 1$ columns or faces

$$W^{(r,1,1)}\left(\begin{matrix} d & \gamma & c \\ a & \alpha & b \end{matrix}\,\middle|\, u, \xi\right) = \prod_{k=1}^{r-2} s_{-k}(u+\xi)^{-1}$$

$$U^r(d, c)_{\gamma,(d,g_1,\ldots,g_{r-2},c)}$$

$$\times$$



$$U^r(a, b)_{\alpha,(a,e_1,\ldots,e_{r-2},b)} \quad (2.2)$$

These weights depend on the external spins $a, b, c, d \in G$ and on the internal *bond variables* α, γ labelling the fused edges [17]. The remaining internal spins indicated with solid dots are summed out. Here $\alpha = 1, 2, \ldots, F^r_{ab}$ and $\gamma = 1, 2, \ldots, F^r_{cd}$ where the fused adjacency matrices F^r with $r = 1, 2, \ldots, g - 2$ are defined recursively in terms of the adjacency matrix G by the $s\ell(2)$ fusion rules

$$F^1 = I, \quad F^2 = G, \quad F^r = GF^{r-1} - F^{r-2}, \quad (2.3)$$

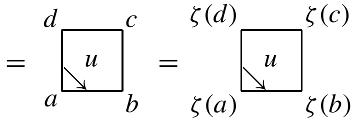
where the superscript is an index and not a matrix power. The fused adjacency matrices F^r are related to the usual intertwiners V_r and operator content n_r on the cylinder by $F^r_{ab} = {V_{ra}}^b = {n_{ra}}^b$. The seam weights vanish if $F^r_{ab} F^2_{bc} F^r_{cd} F^2_{da} = 0$ and the fusion is implemented via the fusion vectors U^r listed in [4]. The inhomogeneity or seam field ξ is arbitrary and can be taken complex but ξ must be specialized appropriately for the seam boundary condition to be conformal. Although we usually take $\xi = -3\lambda/2$ at the isotropic point $u = \lambda/2$, in fact we obtain the same twisted partition function for ξ in a suitable real interval.

The $(1, s, 1)$ seam weights are independent of u and ξ and are given by the *braid limit* $\xi \to i\infty$ of the $(r, 1, 1)$ seam weights divided by $i^{s-1} s_0(u + \xi)$

$$W^{(1,s,1)}\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix} = \left(-ie^{-i\frac{\lambda}{2}}\right)^{s-1} \times \lim_{\xi\to i\infty} s_0(u+\xi)^{-1} W^{(s,1,1)}\left(\begin{matrix} d & \gamma & c \\ a & \alpha & b \end{matrix}\middle| u, \xi\right). \tag{2.4}$$

The automorphisms $\zeta \in \Gamma$ of the adjacency matrix, satisfying $G_{a,b} = G_{\zeta(a),\zeta(b)}$, leave the face weights invariant

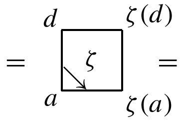
$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) =$$



$$= W\left(\begin{matrix} \zeta(d) & \zeta(c) \\ \zeta(a) & \zeta(b) \end{matrix}\middle| u\right), \tag{2.5}$$

and act through the special seam [18]

$$W^{(1,1,\zeta)}\begin{pmatrix} d & c \\ a & b \end{pmatrix} =$$



$$= \begin{cases} 1, & b = \zeta(a),\ c = \zeta(d), \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}$$

Notice that the $(r, s, \zeta) = (1, 1, 1)$ seam, where $\zeta = 1$ denotes the identity automorphism, is the empty seam corresponding to periodic boundary conditions

$$W^{(1,1,1)}\begin{pmatrix} d & c \\ a & b \end{pmatrix} = \delta_{ab}\delta_{cd} F^2_{bc}. \tag{2.7}$$

The A–D–E face and seam weights satisfy the generalized Yang–Baxter and boundary Yang–Baxter equations ensuring commuting row transfer matrices and integrability with an arbitrary number of seams. Also by the generalized Yang–Baxter equation, the $(r, 1, 1)$ and $(1, s, 1)$ seams can be propagated along a row and even pushed through one another without changing the spectrum. The conformal partition functions with multiple seams are described by the fusion algebra. Explicit expressions can be obtained for the (r, s, ζ) seam weights. The $(2, 1, 1)$-seam corresponds to a single column with spectral parameter $u + \xi$ and the $(1, 2, 1)$ seam, given by the braid limit, has weights

$$W^{(1,2,1)}\begin{pmatrix} d & c \\ a & b \end{pmatrix} = ie^{i\frac{\lambda}{2}}\delta_{ac} - ie^{-i\frac{\lambda}{2}}\frac{\sqrt{\psi_a\psi_c}}{\psi_b}\delta_{bd}. \tag{2.8}$$

More generally, we find that

$$W^{(r,1,1)}\left(\begin{matrix} d & \gamma & c \\ a & \alpha & b \end{matrix}\middle| u, \xi\right) = S_{r-1} s_1(u+\xi) U^r\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix} - S_r s_0(u+\xi) V^r\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix}, \tag{2.9}$$

$$W^{(1,s,1)}\left(\begin{matrix} d & \gamma & c \\ a & \alpha & b \end{matrix}\middle| u, \xi\right) = i^{s-1}\left[S_{s-1} e^{i\frac{\lambda}{2}} U^s\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix} - S_s e^{-i\frac{\lambda}{2}} V^s\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix}\right], \tag{2.10}$$

where $S_k = s_k(0)$ and, in terms of the fusion vectors U^r listed in [4], the elementary fusion faces are

$$U^r\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix} = \sum_{(d,a,e_1,\ldots,e_{r-3},c,b)} U^r_\gamma(d,c)_{(d,a,e_1,\ldots,e_{r-3},c)} \times U^r_\alpha(a,b)_{(a,e_1,\ldots,e_{r-3},c,b)}, \tag{2.11}$$

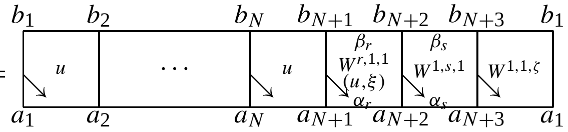
$$V^r\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix} = \sum_{(d,a,e_1,\dots,e_{r-3},c,b)} U^r_\gamma(d,c)_{(d,a,e_1,\dots,e_{r-3},c)} \times U^r_\alpha(a,b)_{(a,e_1,\dots,e_{r-3},c,b)} \times \sum_{\beta=1}^{F^{r+1}_{bd}} U^{r+1}_\beta(d,b)_{(d,a,e_1,\dots,e_{r-3},c,b)}. \tag{2.12}$$

2.2. Transfer matrices and symmetries

The entries of the transfer matrix with an (r,s,ζ) seam are given by

$$\langle \boldsymbol{a}, \alpha_r, \alpha_s | \boldsymbol{T}_{(r,s,\zeta)}(u,\xi) | \boldsymbol{b}, \beta_r, \beta_s \rangle$$

=



$$= \prod_{j=1}^N W\left(\begin{matrix} b_j & b_{j+1} \\ a_j & a_{j+1}\end{matrix}\middle| u\right) \times W^{(r,1,1)}\left(\begin{matrix} b_{N+1} & \beta_r & b_{N+2} \\ a_{N+1} & \alpha_r & a_{N+2}\end{matrix}\middle| u,\xi\right) \times W^{(1,s,1)}\begin{pmatrix} b_{N+2} & \beta_s & b_{N+3} \\ a_{N+2} & \alpha_s & a_{N+3}\end{pmatrix} \times W^{(1,1,\zeta)}\begin{pmatrix} b_{N+3} & b_1 \\ a_{N+3} & a_1\end{pmatrix}. \tag{2.13}$$

By the generalized Yang–Baxter equations these form a one-parameter family of commuting transfer matrices. The transfer matrices with a seam are not translationally invariant, however using the generalized Yang–Baxter equations and a similarity transformation, the seam can be propagated along the row leaving the spectrum invariant. This is the analog of the property that the twisted partition functions are invariant under deformation of the inserted defect lines.

The seam weights and transfer matrices satisfy certain symmetries as a consequence of the usual crossing and reflection symmetries of the face weights. For the $(1,s,1)$ seam we find the crossing symmetry

$$W^{(1,s,1)}\begin{pmatrix} d & \gamma & c \\ a & \alpha & b \end{pmatrix} = \sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\, W^{(1,s,1)}\begin{pmatrix} a & \alpha & b \\ d & \gamma & c \end{pmatrix}^*, \tag{2.14}$$

so that for real u

$$\boldsymbol{T}^{(1,s,1)}(\lambda-u) = \boldsymbol{T}^{(1,s,1)}(u)^\dagger, \tag{2.15}$$

and the $(1,s,1)$ transfer matrices are normal matrices. In particular, at the isotropic point $u=\lambda/2$ the $(1,s,1)$ transfer matrices are Hermitian and the eigenvalues are real. In contrast, for the $(r,1,1)$ seam, $\boldsymbol{T}^{(r,1,1)}(u,\xi)$ is not normal in general due to the parameter ξ. However, at $\xi=\xi_k=\frac{\lambda}{2}(r-2+kg)$ with k even we find

$$\boldsymbol{T}^{(r,1,1)}(\lambda-u,\xi_k) = \boldsymbol{T}^{(r,1,1)}(u,\xi_k)^T, \tag{2.16}$$

so for $\xi=\xi_k$ the transfer matrices are normal. In this case the transfer matrices are real symmetric at the isotropic point $u=\lambda/2$ and the eigenvalues are again real.

2.3. Finite-size corrections

In the scaling limit the A–D–E lattice models reproduce the conformal data of the unitary minimal models through finite-size corrections to the eigenvalues of the transfer matrices. If we fix ξ to a conformal value and write the eigenvalues of the row transfer matrix $\boldsymbol{T}^{(r,s,\zeta)}(u,\xi)$ as

$$T_n(u) = \exp(-E_n), \quad n = 0,1,2,\dots, \tag{2.17}$$

then to order $o(1/N)$ the finite-size corrections to the energies E_n take the form

$$E_0 = Nf(u) + f_{r,s}(u) - \frac{\pi c}{6N}\sin\vartheta, \tag{2.18}$$

$$E_n - E_0 = \frac{2\pi}{N}\big[\big(\Delta_n+\bar\Delta_n+k_n+\bar k_n\big)\sin\vartheta + i\big(\Delta_n-\bar\Delta_n+k_n-\bar k_n\big)\cos\vartheta\big], \tag{2.19}$$

where $f(u)$ is the bulk free energy, $f_{r,s}(u,\xi)$ is the seam free energy, c is the central charge, $\Delta_n, \bar\Delta_n$ are conformal weights, $\vartheta = gu$ is the anisotropy angle and $k_n, \bar k_n \in \mathbb{N}$. The bulk and seam free energies depend on G only through the Coxeter number g and are independent of s and ζ. To determine $f(u)$ and $f_r(u)$ it is therefore sufficient to consider A-type graphs.

For $G = A_L$, the transfer matrices $\boldsymbol{T}(u) = \boldsymbol{T}^{(r,s,\zeta)}(u,\xi)$ satisfy universal TBA functional equations [19] independent of the boundary conditions (r,s,ζ). It follows that the eigenvalues satisfy the inversion relation

$$T(u)T(u+\lambda) = (-1)^{s-1} s_1(u+\xi) s_{1-r}(u+\xi) \left[s_1(u) s_{-1}(u)\right]^N. \tag{2.20}$$

Using appropriate analyticity properties, it is straightforward to solve [20–22] this relation to obtain closed formulas for the bulk and seam free energies but we do not give the formulas here. Removing the bulk and boundary seam contributions to the partition function on a torus allows us to obtain numerically the conformal twisted partion functions $Z_{(r,s,\zeta)}(q)$ for an $M \times N$ lattice where $q = \exp(2\pi i \tau)$ is the modular parameter and $\tau = (M/N)\exp[i(\pi - \vartheta)]$.

3. Conformal twisted partition functions

The conformal twisted partition functions for a rational CFT with a seam $x = (a,b,\gamma)$ have been given by Petkova and Zuber [7] as

$$Z_x(q) = \sum_{i,j} \widetilde{V}_{ij^*;1}{}^{x} \chi_i(q) \chi_j(q)^*,$$
$$\widetilde{V}^{(G)\,x}_{ij^*;1} = \sum_{c \in T_\gamma} n_{ic}{}^{a} n_{jc}{}^{b}, \tag{3.1}$$

where $a, b \in G$ and T_γ, possibly depending on a $\mathbb{Z}_2$ automorphism $\gamma = 0, 1$, is a specified subset of nodes of G. For the D_4 example discussed below, $b = 1$ and T_γ is the set of nodes of $\mathbb{Z}_2$ grading equal to γ so $T_0 = \{2\}$ and $T_1 = \{1, 3, 4\}$. For the A–D–E WZW theories $n_{ia}{}^{b} = F^i_{ab}$, however, since the minimal models are WZW cosets there is an additional tensor product structure of the graphs $A \otimes G$ giving [7]

$$\widetilde{V}_{(r,s)(r',s');1}{}^{(r'',a,b,\zeta)} = N_{rr'}^{(A_{g-2})\,r''} \widetilde{V}^{(G)\,x}_{ij^*;1}, \tag{3.2}$$

where $N_r = F^r$ are the A-type Verlinde fusion matrices of $\widehat{sl}(2)_{g-2}$. The integers $\widetilde{V}^{(G)\,x}_{ij^*;1} \in \mathbb{N}$ encode [23–26] the Ocneanu quantum graphs and the fusion algebra of the WZW models with a seam. For the minimal models the fusion algebra and quantum graphs are encoded by (3.2).

For diagonal A-type theories the conformal twisted partition functions simplify to

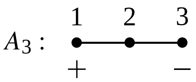
$$Z_k(q) = \sum_{i,j} N_{ij}{}^{k} \chi_i(q) \chi_j(q)^*. \tag{3.3}$$

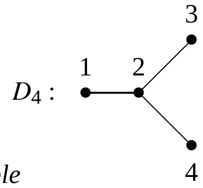
In particular, for (A_{L-1}, A_L) unitary minimal models the seams are labelled by the Kac labels (r,s) and, since the $\mathbb{Z}_2$ diagram automorphism is included in the fusion algebra, we can take $\zeta = 1$. In this case the partition functions are given in terms of Virasoro characters by [1]

$$Z_{(r,s)}(q) = \sum_{(r',s'),(r'',s'')} N_{r'r''}^{(A_{g-2})\,r} N_{s's''}^{(A_{g-1})\,s} \times \chi_{r',s'}(q) \chi_{r'',s''}(q)^*. \tag{3.4}$$

These results were verified numerically for $L = 3, 4, 5$ and 6 and matrix size given by $N = 22, 16, 14$ and 12, respectively.

The integrable seam weights of the lattice models give the physical content of the conformal twisted boundary conditions. This physical content is not at all clear from the conformal labels alone. We illustrate this by discussing the Ising and 3-state Potts models as examples.





3.1. *Ising (A_2, A_3) example*

The three twisted partition functions $Z_{(r,s)}$ for the Ising model are

$$Z_P = Z_0 = Z_{(1,1)} = |\chi_0(q)|^2 + |\chi_{1/2}(q)|^2 + |\chi_{1/16}(q)|^2, \tag{3.5}$$

$$Z_A = Z_{1/2} = Z_{(1,3)} = \chi_0(q)\chi_{1/2}(q)^* + \chi_0(q)^*\chi_{1/2}(q) + |\chi_{1/16}(q)|^2, \tag{3.6}$$

[1] This formula for the case $r = r' = r'' = 1$ was obtained by Orrick and Pearce and explained in private communication to Zuber in September 1999 at the time of the A–D–E conference in Warwick.

$$Z_{1/16} = Z_{(1,2)} = [\chi_0(q) + \chi_{1/2}(q)]\chi_{1/16}(q)^* + [\chi_0(q) + \chi_{1/2}(q)]^*\chi_{1/16}(q). \tag{3.7}$$

The $(r,s) = (1,1)$ and $(1,3)$ seams reproduce the periodic P and antiperiodic A boundary conditions, respectively, and their associated partition functions [27]. The weights giving the physical content of the third seam are

$$\begin{aligned}
&W^{(1,2)}\begin{pmatrix} \pm & \\ & \pm \end{pmatrix} = ie^{i\frac{\pi}{8}} - \sqrt{2}\, ie^{-i\frac{\pi}{8}},\\
&W^{(1,2)}\begin{pmatrix} \pm & \\ & \mp \end{pmatrix} = ie^{i\frac{\pi}{8}},\\
&W^{(1,2)}\begin{pmatrix} & \pm \\ \pm & \end{pmatrix} = ie^{i\frac{\pi}{8}} - ie^{-i\frac{\pi}{8}}/\sqrt{2},\\
&W^{(1,2)}\begin{pmatrix} & \pm \\ \mp & \end{pmatrix} = -ie^{-i\frac{\pi}{8}}/\sqrt{2},
\end{aligned} \tag{3.8}$$

where the upper or lower signs are taken and for A_3 we identify $+ = 1$, $- = 3$ and the frozen state $0 = 2$ is omitted. The $(1,2)$ seam weights are complex but at the isotropic point $u = \pi/8$ the $(1,2)$ transfer matrix is Hermitian so its eigenvalues are real. In fact, after removing $\pm$ degeneracies, the eigenvalues are all positive.

The twisted Ising partition functions are obtained numerically to very high precision. The energy levels in the q series of the twisted partition functions are reproduced for at least the first 10 levels counting degeneracies to 4–8 digit accuracy. The conformal weights are obtained to 8 digit accuracy.

3.2. 3-state Potts (A_4, D_4) example

The conformal twisted partition functions of the 3-state Potts model have been listed by Petkova and Zuber [7]. This list extends the previously known twisted boundary conditions [28–30] corresponding to the automorphisms $\zeta = 1, \omega, \tau \in \Gamma(D_4)$ and corresponding to the periodic $P = (1,1,1)$, cyclic $C = (1,1,\omega)$ and twisted $T = (1,1,\tau)$ boundary conditions, respectively. Explicitly, the 3- and 2-cycles are given by the permutations $\omega = (1,3,4)$ and $\tau = (3,4)$.

The twisted partition functions are written most compactly in terms of the extended block characters

$$\hat\chi_{r,a}(q) = \sum_{s\in A_{g-1}} n_{s1}{}^a \chi_{r,s}(q) = \sum_{s\in A_{g-1}} F^s_{1a}\chi_{r,s}(q), \tag{3.9}$$

where $\chi_{r,s}(q)$ are the Virasoro characters. Considering all the seams (r,s,ζ) and taking into account symmetries, we find 8 distinct conformal twisted partition functions in complete agreement with Petkova and Zuber [7]

$$\begin{aligned}
Z_P = Z_{(1,1,1)} &= |\hat\chi_{1,1}(q)|^2 + |\hat\chi_{1,3}(q)|^2 + |\hat\chi_{1,4}(q)|^2 + |\hat\chi_{3,1}(q)|^2 + |\hat\chi_{3,3}(q)|^2 + |\hat\chi_{3,4}(q)|^2,\\
Z_{(1,2,1)} &= \hat\chi_{1,2}(q)\big[\hat\chi_{1,1}(q) + \hat\chi_{1,3}(q) + \hat\chi_{1,4}(q)\big]^* + \hat\chi_{3,2}(q)\big[\hat\chi_{3,1}(q) + \hat\chi_{3,3}(q) + \hat\chi_{3,4}(q)\big]^*,\\
Z_C = Z_{(1,1,\omega)} &= \hat\chi_{1,1}(q)\hat\chi_{1,3}(q)^* + \hat\chi_{1,1}(q)^*\hat\chi_{1,3}(q) + |\hat\chi_{1,3}(q)|^2 + \hat\chi_{3,1}(q)\hat\chi_{3,3}(q)^* + \hat\chi_{3,1}(q)^*\hat\chi_{3,3}(q) + |\hat\chi_{3,3}(q)|^2,\\
Z_T = Z_{(1,1,\tau)} &= |\hat\chi_{1,2}(q)|^2 + |\hat\chi_{3,2}(q)|^2,\\
Z_{(3,1,1)} &= \hat\chi_{1,1}(q)\hat\chi_{3,1}(q)^* + \hat\chi_{1,1}(q)^*\hat\chi_{3,1}(q) + |\hat\chi_{3,1}(q)|^2 + \hat\chi_{1,3}(q)\hat\chi_{3,3}(q)^* + \hat\chi_{1,3}(q)^*\hat\chi_{3,3}(q) + |\hat\chi_{3,3}(q)|^2 + \hat\chi_{1,4}(q)\hat\chi_{3,4}(q)^* + \hat\chi_{1,4}(q)^*\hat\chi_{3,4}(q) + |\hat\chi_{3,4}(q)|^2,\\
Z_{(3,2,1)} &= \hat\chi_{1,2}(q)\big[\hat\chi_{3,1}(q) + \hat\chi_{3,3}(q) + \hat\chi_{3,4}(q)\big]^* + \hat\chi_{3,2}(q)\big[\hat\chi_{1,1}(q) + \hat\chi_{1,3}(q) + \hat\chi_{1,4}(q) + \hat\chi_{3,1}(q) + \hat\chi_{3,3}(q) + \hat\chi_{3,4}(q)\big]^*,\\
Z_{(3,1,\omega)} &= \hat\chi_{3,3}(q)\big[\hat\chi_{1,1}(q) + \hat\chi_{3,1}(q) + \hat\chi_{1,4}(q)\big]^* + \hat\chi_{1,3}(q)\hat\chi_{3,1}(q)^* + \hat\chi_{3,3}(q)^*\big[\hat\chi_{1,1}(q) + \hat\chi_{3,1}(q) + \hat\chi_{1,4}(q)\big] + \hat\chi_{1,3}(q)^*\hat\chi_{3,3}(q) + |\hat\chi_{3,3}(q)|^2,\\
Z_{(3,1,1)} &= \hat\chi_{1,2}(q)\hat\chi_{3,2}(q)^* + \hat\chi_{3,2}(q)\hat\chi_{1,2}(q)^* + |\hat\chi_{3,2}(q)|^2.
\end{aligned} \tag{3.10}$$

Our construction labels (r,s,ζ) correspond with the labels (r,a,γ) of Petkova and Zuber with the obvious identifications $\zeta = 1 \mapsto \gamma = 1$, $\zeta = \tau \mapsto \gamma = 0$ and $(r,1,\omega) \mapsto (r,3,1)$.

The twisted 3-state Potts partition functions are obtained numerically with reasonable precision and, as expected, intertwine with those of A_5. The energy levels in the q series of the twisted partition functions are reproduced for at least the first 10 levels counting degeneracies to 2–4 digit accuracy. The conformal weights are obtained to at least 4 digit accuracy.

The other D and E cases are of much interest because of connections with Ocneanu quantum graphs. We will present the details of these cases in our subsequent paper [13]. Although we have emphasized the specialized conformal twisted boundary conditions in this Letter, we point out that the same fusion techniques can be used to construct integrable seams off-criticality for the elliptic A and D lattice models.

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