

Lecture # 4
REGULARITY THEORY

(a) Exercises for the November 17 session

Exercise 1. Let $N \geq 2$ and $u \in C^1(\mathbb{R}^N \setminus \{0\})$ be such that $\partial_1 u \in L^1_{loc}(\mathbb{R}^N)$. Prove that $u \in L^1_{loc}(\mathbb{R}^N)$ and that $\partial_1 u$ is the distributional derivative of u . What if $N = 1$?

Exercise 2. Let ω_N be the area of \mathbb{S}^{N-1} . Let E be “the” fundamental solution of $-\Delta$ in \mathbb{R}^N ,

$$E(x) := \begin{cases} -(1/\omega_2) \ln |x|, & \text{if } N = 2 \\ (1/[(N-2)\omega_N]) |x|^{2-N}, & \text{if } N \geq 3 \end{cases}.$$

1. Prove that, in the distributions sense,

$$\partial_j E = g_j, \text{ where } g_j(x) := -\frac{1}{\omega_N} \frac{x_j}{|x|^N}.$$

2. If $1 \leq p \leq \infty$ and $f \in L^p_c(\mathbb{R}^N)$, then, in the distributions sense,

$$\partial_j (f * E) = h_j, \text{ where } h_j(x) := \int_{\mathbb{R}^N} f(y) g_j(x-y) dy.$$

Exercise 3. Let $K \in \mathcal{D}'(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N \setminus \{0\})$. Let $f \in C^\infty_c(\mathbb{R}^N)$ and set $L := \text{supp } f$. Then:

$$(K * f)(x) = \int_{\mathbb{R}^N} f(y) K(x-y) dy = \int_L f(y) K(x-y) dy, \quad \forall x \notin \text{supp } L. \quad (1)$$

Exercise 4. Let (X, \mathcal{F}, μ) be a measured space. (Warning: μ is not supposed σ -finite.) If $f : X \rightarrow \mathbb{R}$ is measurable and $1 \leq p < \infty$, then

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \underbrace{\mu(\{|f| > t\})}_{:= F_f(t)} dt.$$

(b) Exercises for the November 24 session

Exercise 1. Let $u \in H^1_0(\Omega)$ be an eigenfunction of $-\Delta$. Prove that $u \in C^\infty(\Omega)$.

In what follows, ω is an open set in \mathbb{R}^N .

Exercise 2. Let $\Phi : \omega \rightarrow \Omega$ be a bi-Lipschitz C^1 -diffeomorphism. Prove that, with constants $0 < C_{1,p} < C_{2,p} < \infty$ depending only on $1 \leq p < \infty$ and on Φ , we have

$$C_1 \|f \circ \Phi\|_{L^p(\omega)} \leq \|f\|_{L^p(\Omega)} \leq C_2 \|f \circ \Phi\|_{L^p(\omega)}, \quad \forall \text{ measurable function } f : \Omega \rightarrow \mathbb{R}.$$

Slightly more difficult questions: (i) prove that the above constants depend only on p and on the Lipschitz constants of Φ and Φ^{-1} ; (ii) prove that the above still holds under the weaker assumption that Φ is a bi-Lipschitz homeomorphism.

Exercise 3. Let $\Phi : \omega \rightarrow \Omega$ be a C^1 -diffeomorphism. If $f \in W_{loc}^{1,1}(\Omega)$, prove that $f \circ \Phi \in W_{loc}^{1,1}(\omega)$ and that the chain rule holds, i.e.,

$$\partial_i (f \circ \Phi) = \sum_{j=1}^N [(\partial_j f) \circ \Phi] [\partial_i \Phi_j] \text{ in } \mathcal{D}'(\omega), \forall 1 \leq i \leq N.$$

Exercise 4. Let $\Phi : \omega \rightarrow \Omega$ be a C^1 -diffeomorphism. Let $u \in W_{loc}^{1,1}(\Omega)$ satisfy $-\Delta u = f \in L_{loc}^1(\Omega)$ in the distributions sense. Set $v := u \circ \Phi \in W_{loc}^{1,1}(\omega)$. Then, in the distributions sense, we have, in ω ,

$$-\operatorname{div}(A \nabla v) = g \in L_{loc}^1(\omega), \quad (2)$$

where

$$A = A(x) := |J \Phi| [(J \Phi)^{-1}] [{}^t[(J \Phi)^{-1}]], \quad g := |J \Phi| f \circ \Phi.$$

Exercise 5. Set $\mathbb{R}_+^N := \{x \in \mathbb{R}^N; x_N > 0\}$. Let $u \in W^{1,1}(\mathbb{R}_+^N)$. Let $h \in \mathbb{R}^{N-1} \times \{0\}$. Give a meaning to and prove the equality $\operatorname{tr} u(\cdot + h) = (\operatorname{tr} u)(\cdot + h)$.

Exercise 6. Let p, q be conjugated exponents, $g \in L^q(\mathbb{R}_+^N)$, $w \in W^{1,p}(\mathbb{R}_+^N)$, $h \in \mathbb{R}^{N-1} \times \{0\}$. Then

$$\left| \int_{\mathbb{R}_+^N} (g(x+h) - g(x)) w(x) dx \right| \leq |h| \|g\|_q \|\nabla w\|_p.$$

Exercise 7. Let $f \in L_{loc}^1(\mathbb{R}_+^N)$. Then

$$\lim_{t \rightarrow 0} \frac{f(\cdot + te_j) - f}{t} = \partial_j f \text{ in } \mathcal{D}'(\mathbb{R}_+^N), \forall 1 \leq j \leq N-1.$$

Exercise 8. Let $u \in W^{2,1}(\mathbb{R}_+^N)$.

1. Let $\Sigma := \mathbb{R}^{N-1} \times \{0\}$, that we identify with \mathbb{R}^{N-1} . When $\varphi \in C_c^2(\overline{\mathbb{R}_+^N})$, prove the generalized (second) Green formula

$$\int_{\mathbb{R}_+^N} (-\Delta u) \varphi = \int_{\mathbb{R}^{N-1}} [\operatorname{tr}_{|\Sigma} \partial_N u] \varphi - \int_{\mathbb{R}^{N-1}} [\operatorname{tr}_{|\Sigma} u] \partial_N \varphi + \int_{\mathbb{R}_+^N} u (-\Delta \varphi).$$

2. If $F : \mathbb{R}_+^N \rightarrow \mathbb{R}$, set

$$F^*(x) = F^*(x_1, \dots, x_N) := \begin{cases} F(x), & \text{if } x_N > 0 \\ -F(x_1, \dots, x_{N-1}, -x_N), & \text{if } x_N < 0 \end{cases}.$$

Let $u \in W^{2,1}(\mathbb{R}_+^N)$ satisfy $\operatorname{tr}_{|\Sigma} u = 0$. Prove that $-\Delta(u^*) = (-\Delta u)^*$.

(c) Warnings

A Exercise (Weierstrass' counterexample to Dirichlet's principle) Let $0 < \alpha < 1$ and set

$$v(x, y) := (x^2 - y^2) (-\ln(x^2 + y^2))^\alpha, \forall (x, y) \in \mathbb{D} := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}.$$

Prove that:

- (a) $v \notin C^2(\mathbb{D})$.
- (b) The distributional Laplacian $f := \Delta v$ is continuous on \mathbb{D} .
- (c) The equation $\Delta u = f$ has no classical (i.e., C^2) solution near the origin.

B Useful reference for items **B** and **C**: [IO]

Exercise Let $\alpha \in \mathbb{R} \setminus \{-1, 1 - N\}$ and set

$$u(x) := x_1 |x|^\alpha, \forall x \in \mathbb{R}^N \setminus \{0\}, \beta := -\frac{\alpha(\alpha + N)}{(\alpha + 1)(\alpha + N - 1)}.$$

Then

$$\sum_{1 \leq i \leq N} \partial_i \left(\sum_{1 \leq j \leq N} (\delta_{ij} + \beta x_i x_j |x|^{-2}) \partial_j u \right) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}.$$

C Theorem (Serrin) A homogeneous uniformly elliptic equation in divergence form may have locally unbounded $W_{loc}^{1,1}(\Omega)$ weak solutions.

More specifically, if $N \geq 2$ and $0 < \varepsilon < 1$, and we set

$$u(x) := \frac{x_1}{|x|^{N-1+\varepsilon}}, x \in \mathbb{R}^N,$$

and

$$A(x) := \text{Id}_N + \frac{b}{|x|^2} (x_i x_j)_{1 \leq i, j \leq N}, x \in \mathbb{R}^N, \text{ with } b := \frac{N-1}{\varepsilon(\varepsilon + N - 2)} - 1,$$

then $u \in W_{loc}^{1,1}(\mathbb{R}^N) \setminus L_{loc}^\infty(\mathbb{R}^N)$, A is uniformly elliptic in \mathbb{R}^N , and

$$\text{div}(A \nabla u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

(d) Singular integrals

General reference: [8, Section 3]

A Proposition. With the above notation, let $K := \partial_k g_j = \partial_k \partial_j E$ (in the distributions sense). Then:

- (a) $K \in \mathcal{D}'(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$, and in particular (I) holds.

(b) $K \in \mathcal{S}'$ and, in the distributions sense,

$$\widehat{K}(\xi) = \ell_{j,k}, \text{ where } \ell_{j,k}(\xi) := -\frac{\xi_j \xi_k}{|\xi|^2}.$$

(c) For some finite C , we have $|\nabla K(x)| \leq C/|x|^{N+1}, \forall x \in \mathbb{R}^N \setminus \{0\}$.

Useful reference: [6, Theorem 2.3.4]

B **Marcinkiewicz interpolation theorem** (special case) Let (X, \mathcal{F}, μ) be a measured space. Let $1 < r < \infty$ and let T be a linear operator on $L^1 \cap L^r(X)$ such that, for every $f \in L^1 \cap L^r(X)$, Tf is a measurable function on X and, for some $K_1, K_r < \infty$, we have

$$\mu(\{|Tf| > t\}) \leq K_1 \frac{\|f\|_1}{t}, \forall f \in L^1 \cap L^r(X), \forall t > 0,$$

$$\mu(\{|Tf| > t\}) \leq K_r \frac{\|f\|_r^r}{t^r}, \forall f \in L^1 \cap L^r(X), \forall t > 0.$$

Then, for every $1 < p < r$ and some $C_p < \infty$,

$$\|Tf\|_p \leq C_p \|f\|_p, \forall f \in L^1 \cap L^r(X),$$

and in particular T admits a unique linear continuous extension from $L^p(X)$ into $L^p(X)$.

In the special case where μ is a Radon measure in \mathbb{R}^N , the same holds if T is initially defined on $C_c(\mathbb{R}^N)$ or even $C_c^\infty(\mathbb{R}^N)$.

C **Calderón-Zygmund decomposition, second form** Let $f \in C_c(\mathbb{R}^N)$ and $t > 0$. Then, with finite constants independent of f and t there exist: a family of disjoint cubes $C_n \subset \mathbb{R}^N$ and functions $g, h_n \in L_c^\infty(\mathbb{R}^N)$ (depending on f and t) such that

(a) $g = f$ in $\mathbb{R}^N \setminus \cup_n C_n$.

(b) $|g| \leq Ct$.

(c) $\text{supp } h_n \subset C_n, \forall n$.

(d) $\int h_n = 0, \forall n$.

(e) $\int |h_n| \leq Ct, \forall n$.

(f) $f = g + \sum_n h_n$ (pointwise).

(g) $\sum_n |C_n| \leq C \frac{\|f\|_1}{t}$.

(h) $\|g\|_1 + \sum_n \|h_n\|_1 \leq C \|f\|_1$.

(i) $\left\| \sum_n h_n \right\|_2^2 = \sum_n \|h_n\|_2^2 \leq \|f\|_2^2$.

D Calderón-Zygmund theorem adapted to the Laplace equation Let $K \in \mathcal{S}'(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ satisfy

- (i) \widehat{K} is a bounded *real* function.
- (ii) $|\nabla K(x)| \leq C/|x|^{N+1}, \forall x \in \mathbb{R}^N \setminus \{0\}$, for some finite C .

Let $Tf := K * f, \forall f \in C_c^\infty(\mathbb{R}^N)$. Then

$$\|Tf\|_p \leq C_{p,N} \|f\|_p, \forall 1 < p < \infty, \forall f \in C_c^\infty(\mathbb{R}^N).$$

In particular, for $1 < p < \infty$, T admits a unique linear continuous extension from $L^p(\mathbb{R}^N)$ into itself.

Corollary. Let $1 < p < \infty$ and $f \in L_c^p(\mathbb{R}^N)$, and set $u := E * f$. Then

$$\|\partial_j \partial_k u\|_p \leq C_{p,N} \|f\|_p, \forall 1 \leq j, k \leq N.$$

E A standard “elliptic estimate” Let $1 < p < \infty$, $K \subset \Omega \subset \mathbb{R}^N$, with K compact and Ω open. If $-\Delta u = f \in L^p(\Omega)$, then $u \in W_{loc}^{2,p}(\Omega)$ and, for some finite $C = C_{p,N,\Omega,K}$,

$$\|u\|_{W^{2,p}(K)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}).$$

(e) L^p regularity theory

Useful references: [4, Chapter 9] for the regularity theory, [8, Section 1.5] for trace theory

A Main regularity theorem (Calderón, Zygmund, Koselev, Greco, Agmon, Douglis, Nirenberg, ...) Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ -domain. Let $1 < p < \infty$ and $f \in L^p(\Omega)$. Then the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

has a unique (generalized) solution $u \in W^{2,p}(\Omega)$. In addition, for some finite C independent of f , $\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_p$.

Exercise. The above u is not only a distributional solution, but also a *strong solution*, in the sense that for a.e. $x \in \Omega$ we have

$$-\sum_{j=1}^N \partial_{jj} u(x) = f(x).$$

B Toolbox In what follows, $\omega, \Omega \subset \mathbb{R}^N$ are bounded open sets.

For the record.

Rademacher’s theorem A Lipschitz function $f : \Omega \rightarrow \mathbb{R}$ is differentiable a.e., and its distributional gradient and point gradient coincide.

Useful reference: [3, Section 3.1.2]

Exercise. Let $\Phi : \bar{\omega} \rightarrow \bar{\Omega}$ be a $C^{1,1}$ -diffeomorphism and $1 \leq p \leq \infty$. Prove that $f \mapsto \|f \circ \Phi\|_{W^{2,p}(\omega)}$ is equivalent to the usual norm on $W^{2,p}(\Omega)$.

Lemma. Let $A \in \text{Lip}_{loc}(\omega)$, $v \in W_{loc}^{1,1}(\omega)$, $\zeta \in \text{Lip}_{loc}(\omega)$, and $g \in L_{loc}^1(\omega)$. If (2) holds, then

$$-\text{div}(A\nabla(\zeta v)) = \zeta g - v \text{div}(A\nabla\zeta) - (A\nabla v) \cdot \nabla\zeta - (A\nabla\zeta) \cdot \nabla v.$$

Adapted covering lemma. Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ -domain. Given a number $\varepsilon > 0$, there exist:

- (i) An integer m ,
- (ii) m balls $B_j = B_{\rho_j}(0)$, $1 \leq j \leq m$,
- (iii) $C^{1,1}$ diffeomorphisms $\Phi_j : B_j \rightarrow \mathbb{R}^N$, $1 \leq j \leq m$,
- (iv) $\psi \in C_c^\infty(\Omega)$ and $\zeta_j \in C_c^\infty(B_j)$, $1 \leq j \leq m$,

such that:

- (a) $\Omega \subset \cup_j \Phi_j(B_j)$,
- (b) $B_j \cap (\Phi_j)^{-1}(\Omega) = B_j^+ := \{x = (x_1, \dots, x_N) \in B_j; x_N > 0\}$,
- (c) $B_j \cap (\Phi_j)^{-1}(\partial\Omega) = B_j^0 := \{x = (x_1, \dots, x_N) \in B_j; x_N = 0\}$,
- (d) $B_j \cap (\Phi_j)^{-1}((\mathbb{R}^N \setminus \bar{\Omega})) = B_j^- := \{x = (x_1, \dots, x_N) \in B_j; x_N < 0\}$,
- (e) $\rho_j \leq 1$, $1 \leq j \leq m$,
- (f) Given $u : \Omega \rightarrow \mathbb{R}$, if we set $u_0 := \psi u : \Omega \rightarrow \mathbb{R}$, $v_j := u|_{\Phi_j(B_j^+)} \circ \Phi_j : B_j^+ \rightarrow \mathbb{R}$, and $w_j := \zeta_j v_j$, $1 \leq j \leq m$, then

$$\|u\|_{W^{2,p}(\Omega)} \sim \|u_0\|_{W^{2,p}(\Omega)} + \sum_{1 \leq j \leq m} \|w_j\|_{W^{2,p}(B_j^+)}, \quad \forall 1 \leq p \leq \infty,$$

- (g) If we set

$$A(x) = A_j(x) := |J\Phi_j(x)| [(J\Phi_j)^{-1}(x)] [{}^t[(J\Phi_j)^{-1}(x)]], \quad 1 \leq j \leq m, \quad x \in B_j,$$

then $x \mapsto A(x)$ is Lipschitz and $A(x) = \text{Id} + B(x)$, where $\|B(x)\|_\infty \leq \varepsilon$, $1 \leq j \leq m$, $x \in B_j$,

- (h) If $u \in W_{loc}^{1,1}(\Omega)$ satisfies, in $\mathcal{D}'(\Omega)$, $-\Delta u = f \in L_{loc}^1(\Omega)$, then u_0 and w_j satisfy

$$\begin{aligned} -\Delta u_0 &= \psi f - 2\nabla\psi \cdot \nabla u - u \Delta\psi \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ -\text{div}(A_j \nabla w_j) &= \zeta_j (f|_{\Phi_j(B_j^+)} \circ \Phi_j) |J\Phi_j| - v_j \text{div}(A_j \nabla \zeta_j) \\ &\quad - (A_j \nabla v_j) \cdot \nabla \zeta_j - (A_j \nabla \zeta_j) \cdot \nabla v_j \text{ in } \mathcal{D}'(B_j^+), \quad 1 \leq j \leq m. \end{aligned}$$

Exercise. Let $u \in W^{1,1}(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$. Prove that $\text{tr}(\varphi u) = \varphi|_{\partial\Omega} \text{tr } u$.

Exercise. Let Ω be a C^1 -domain, and let $\Psi : U \rightarrow \mathbb{R}^N$ be a C^1 -diffeomorphism from an open set $U \subset \mathbb{R}^N$ into its image. Set $\Xi := \Psi|_U : U \rightarrow \Psi(U)$ and $\Phi := \Xi^{-1}$. Set also $\Sigma := \partial\Omega \cap U$ and $\Lambda := \Psi(\Sigma)$. Let $u \in W^{1,1}(U)$ and set $v := u \circ \Phi$. Give a meaning to and prove the equality $\text{tr}_{\Psi(\Sigma)} v = (\text{tr}_{\Sigma}(u)) \circ [(\Psi|_{\Lambda})^{-1}]$.

C Theorem (Higher order regularity) Let $k \geq 0$, $\Omega \in C^{k+1,1}$, and $1 < p < \infty$. If $f \in W^{k,p}(\Omega)$, then the solution u of (3) satisfies $u \in W^{k+2,p}(\Omega)$ and, for some finite C independent of f , $\|u\|_{W^{k+2,p}(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}$.

D For the record, we mention some results in lower order regularity theory.

Theorem. Let $\Omega \in C^{1,1}$ and $1 < p < \infty$. For $F \in L^p(\Omega; \mathbb{R}^N)$, the equation

$$-\Delta u = \text{div } F \text{ in } \mathcal{D}'(\Omega)$$

has a unique solution $u \in W_0^{1,p}(\Omega)$. In addition, with some finite constant C independent of F , we have the estimate $\|\nabla u\|_p \leq C\|F\|_p$.

Theorem (Stampacchia) Let $\Omega \in C^{1,1}$. For $f \in L^1(\Omega)$, the equation

$$-\Delta u = f \text{ in } \mathcal{D}'(\Omega)$$

has a unique solution $u \in W_0^{1,1}(\Omega)$. Moreover, this u satisfies $u \in \cap_{1 \leq p < N/(N-1)} W_0^{1,p}(\Omega)$ and, with finite constants C_p independent of f ,

$$\|\nabla u\|_p \leq C_p \|f\|_1, \quad \forall 1 \leq p < \frac{N}{N-1}.$$

Useful reference: [9, Section 4.1]

Big picture of the proofs of Theorems **A** and **C**

I. Preliminaries

- (a) Localization of the problem.
- (b) The effects of the change of coordinates: equation, traces.
- (c) Equivalence of norms. Reduction to local estimates.
- (d) Choice of adapted neighborhoods of points of $\partial\Omega$.

II. L^2 theory

- (a) Interior estimates.
- (b) Boundary estimates *via* Nirenberg's quotient method.

III. $L^p, p > 2$, theory

- (a) Interior estimates *via* Calderón-Zygmund theory. Induction principle.
- (b) Boundary estimates. Reflection and contraction principles.

IV. $L^p, p < 2$, theory

- (a) Uniqueness *via* duality.
- (b) *A priori* estimates *via* uniqueness.
- (c) Existence *via a priori* estimates.

V. Higher order regularity by induction

(f) A glimpse of the C^α regularity theory

Useful reference: [4, Lemma 4.4, Theorem 6.14, Theorem 6.19]. For the record:

Theorem (C^α regularity) (Kellogg) Let $0 < \alpha < 1, k \geq 0, \Omega \in C^{k+2, \alpha}$. If $f \in C^{k, \alpha}(\overline{\Omega})$, then the solution of (3) satisfies $u \in C^{k+2, \alpha}(\overline{\Omega})$. In addition, for some finite C independent of f , $\|u\|_{C^{k+2, \alpha}(\overline{\Omega})} \leq C\|f\|_{C^{k, \alpha}(\overline{\Omega})}$.

Lemma (Hölder estimates for the Newtonian potential) (Korn) Let $0 < \alpha < 1$. If $f \in C_c^\alpha(\mathbb{R}^N)$ and $u := E * f$, then, for some finite C independent of f ,

$$|D^2 u|_{C^\alpha(\mathbb{R}^N)} \leq C|f|_{C^\alpha(\mathbb{R}^N)}.$$

(g) Power growth nonlinearities. Bootstrap

Useful reference: [8, Section 3.3.2]. In what follows, we assume that $N \geq 3$.

Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function satisfying

$$|f(x, t)| \leq C(1 + |t|^p), \quad \forall x \in \Omega, \forall t \in \mathbb{R}.$$

Let u satisfy

$$\begin{aligned} u &\in H_{loc}^1(\Omega), \quad x \mapsto f(x, u(x)) \in L_{loc}^1(\Omega) \\ -\Delta u &= f(x, u(x)) \text{ in } \mathcal{D}'(\Omega). \end{aligned}$$

A Exercise. Assume that $p < \frac{N+2}{N-2}$. Then $u \in W_{loc}^{2, r}(\Omega), \forall r < \infty$.

B Proposition. The same holds when $p = \frac{N+2}{N-2}$.

Moreover, if $u \in H_{loc}^1(\Omega)$ satisfies

$$-\Delta u = a(x)u + b(x), \quad \text{with } a \in L_{loc}^{N/2}(\Omega), \quad b \in L_{loc}^\infty(\Omega),$$

then $u \in L_{loc}^r(\Omega), \forall r < \infty$.

C Exercise. Let $p > \frac{N+2}{N-2}$. Prove that the equation $-\Delta u = |u|^p$ has a locally unbounded solution $u \in H^1 \cap L^p(B_1(0))$, of the form $u(x) = \lambda|x|^{-\alpha}$, for appropriate constants $\lambda, \alpha > 0$.

(h) A glimpse of the De Giorgi regularity theory

Useful references: [4, Sections 8.5–8.9], [5, Chapter 4]. For the record:

A **Theorem** (local boundedness; **Stampacchia**, Ladyzhenskaya, Uraltseva, Trudinger,...) Let $A = A(x)$ be uniformly elliptic in $\Omega := B_1(0)$. Let $u \in H^1(\Omega)$ satisfy $-\operatorname{div}(A\nabla u) = f \in L^p(\Omega)$, where $p > \frac{N}{2}$. Then $u \in L^\infty_{loc}(\Omega)$ and, with a finite constant depending only on $0 < R < 1$ and p ,

$$\|u\|_{L^\infty(B_R(0))} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}).$$

B **Theorem** (local C^α regularity; **De Giorgi**, Nash, Ladyzhenskaya, Uraltseva, **Moser**,...) There exists some $0 < \alpha < 1$ depending only on p and the ellipticity constants of A such that the above u belongs to $C^\alpha_{loc}(\Omega)$ and satisfies, with a finite constant C depending only on R and p

$$|u(x) - u(y)| \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}), \quad \forall x, y \in B_R(0).$$

(i) Wente estimates. Compensation phenomena

Useful references: [1], [2], [7, Section 10.3]

A **Theorem** (Wente) Let $\Omega \in C^{1,1}$ be a bounded domain in \mathbb{R}^2 , and let $F \in H^1(\Omega; \mathbb{R}^2)$. Then the problem

$$\begin{cases} -\Delta u = \det(JF) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a (unique) weak solution $u \in H^1_0(\Omega)$. In addition, we have $u \in C(\overline{\Omega})$ and, for some finite constant independent of F , we have the *Wente estimates*

$$\|u\|_\infty + \|\nabla u\|_2 \leq C\|\nabla F\|_2.$$

B For the record:

Theorem (Fefferman, Stein, Coifman, Lions, Meyer, Semmes) If $F \in W^{1,N}_c(\mathbb{R}^N; \mathbb{R}^N)$, and we set $u := E * [\det(JF)]$, then $D^2u \in L^1(\mathbb{R}^N)$ and, with a finite constant independent of F and of its support,

$$\|D^2u\|_1 \leq C\|\nabla F\|_N.$$

References

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