Introduction aux limites hydrodynamiques

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Chapter 1

Simple random walks in dimension 1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Unless stated otherwise, all the random variables will be defined on that space.

1.1 Definitions

1.1.1 On the lattice $\mathbb{Z}$

A *simple random walk on* $\mathbb{Z}$ is a sequence $(\mathrm{S}_t)_{t\in\mathbb{N}}$ of integers, constructed as follows: take

- $\mathrm{S}_0 \in \mathbb{Z}$ distributed according to some *initial probability* ν_0 on $\mathbb{Z}$,
- a sequence of i.i.d. random variables $(\varepsilon_k)_{k\in\mathbb{N}}$ which take values in $\{-1, 1\}$, *i.e.* there is $p \in [0, 1]$ such that

$$\mathbb{P}[\varepsilon_k = 1] = p, \qquad \mathbb{P}[\varepsilon_k = -1] = 1 - p, \qquad \text{for any } k \in \mathbb{N}.$$

We then define

$$\mathrm{S}_{t+1} = \mathrm{S}_t + \varepsilon_{t+1}, \qquad \text{therefore} \quad \mathrm{S}_t = \mathrm{S}_0 + \sum_{k=1}^{t} \varepsilon_k, \qquad t \in \mathbb{N}.$$

Note that $(\varepsilon_k)_{k\in\mathbb{N}}$ are the *independent steps* performed by a random walker (or random *particle*): S_{t+1} increases the value of S_t by 1 if $\varepsilon_{t+1} = 1$ (one step *to the right*) and decreases the value of S_t by 1 if $\varepsilon_{t+1} = -1$ (one step *to the left*).

We denote by $m := 2p - 1$ the *average displacement* $\mathbb{E}[\varepsilon_1]$. If $p = \frac{1}{2} \Leftrightarrow m = 0$, the random walk is *symmetric*, otherwise it is *asymmetric*.

We can easily represent the *graph* of the random walk (S_t), as the broken line joining the points of coordinates (i, S_i). If the random walk performs N steps, then there are 2^N possible *trajectories* starting from $(0, S_0)$ (at each step, there are two possibles choices).

Let us now state two important properties, which come easily from the definition. First, the law of large numbers and the central limit theorems give us information on the asymptotic behavior of the simple random walk:

PROPOSITION 1.1 (Asymptotic behavior). *Assume* $S_0 = 0$*:*

- *if* $p = \frac{1}{2}$*, then*
$$\lim_{t\to\infty} \frac{S_t}{t} = 0 \quad a.s. \qquad and \qquad \frac{S_t}{\sqrt{t}} \xrightarrow[t\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$
where $\mathcal{N}(0,1)$ *is the standard normal distribution. In particular, when* t *is large, with very high probability* $S_t \in [-2\sqrt{t}, 2\sqrt{t}]$
- *if* $p > \frac{1}{2}$ *then*
$$\lim_{t\to\infty} S_t = +\infty \quad a.s.$$
- *if* $p < \frac{1}{2}$ *then*
$$\lim_{t\to\infty} S_t = -\infty \quad a.s.$$

Second, the simple random walk is one of the most famous examples of the general *Markov chains*, which have the following property:

PROPOSITION 1.2 (Homogeneous Markov property). *For any* $t \geqslant 1$*, and any* $i_0, i_1, \dots, i_t \in \mathbb{Z}$*,*

$$\begin{aligned}\mathbb{P}\big(S_t = i_t \mid S_{t-1} = i_{t-1}, \dots, S_0 = i_0\big) &= \mathbb{P}\big(S_t = i_t \mid S_{t-1} = i_{t-1}\big) \\ &= \begin{cases} p & \text{if } i_t = i_{t-1} + 1, \\ 1-p & \text{if } i_t = i_{t-1} - 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

which only depends on the difference $i_t - i_{t-1}$*, but not on* t*.*

The function $P : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ *defined as* $P_{i,j} := \mathbb{P}(S_t = j | S_{t-1} = i)$ *is the* transition kernel *for the random walk, and it satisfies*

$$\sum_{j\in\mathbb{Z}} P_{i,j} = 1.$$

1.1.2 On the torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$

When the particle evolves on the periodic discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$, its position at time $t \in \mathbb{N}$ can easily be defined similarly: we start from S_0 distributed according to an initial probability measure ν_0 on $\mathbb{T}_N$, and then

$$S_t = \left(S_0 + \sum_{k=1}^{t} \varepsilon_k \right) \qquad \text{mod } N.$$

It has the same properties as the previous case, however, since the state space $\mathbb{T}_N$ is now *finite*, the transition probabilities are encoded by a $N \times N$–stochastic matrix:

Proposition 1.3 (Transition probabilities)**.** *For any $x, y \in \mathbb{T}_N$, let us introduce the* transition probabilities

$$\mathbf{Q}_{x,y} := \mathbb{P}\big(S_{t+1} = y \mid S_t = x\big) = \mathbb{P}\big(S_1 = y - x | S_0 = 0\big),$$

which do not depend on t, and only depend on $y - x$. Moreover, the transition probability matrix $\mathbf{Q} := (\mathbf{Q}_{x,y})_{x,y \in \mathbb{T}_N}$ *completely characterizes the distribution of S_t: for any $t, s \in \mathbb{N}$, for any $x, y \in \mathbb{T}_N$, we have*

$$\mathbb{P}\big(S_{t+s} = y \mid S_s = x\big) = (\mathbf{Q}^t)_{x,y} = (\mathbf{Q}^t)_{0,y-x} = \mathbb{P}\big(S_t = y - x | S_0 = 0\big). \tag{1.1}$$

Finally, for any $t \in \mathbb{N}$, the matrix $\mathbf{Q}^t$ is bi-stochastic*:*

$$\sum_{x \in \mathbb{T}_N} (\mathbf{Q}^t)_{x,y} = \sum_{y \in \mathbb{T}_N} (\mathbf{Q}^t)_{x,y} = 1.$$

Proof. By definition,

$$\mathbf{Q} = \begin{pmatrix} 0 & p & 0 & \cdots & 0 & 1-p \\ 1-p & 0 & p & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & 1-p & 0 & p \\ p & 0 & \cdots & 0 & 1-p & 0 \end{pmatrix}$$

First, for $t = 1$, by homogeneity we have: for any $s \in \mathbb{N}$,

$$\mathbb{P}\big(S_{s+1} = y \mid S_s = x\big) = \mathbf{Q}_{x,y}. \tag{1.2}$$

One proves (1.1) by induction. Let us prove it for $t = 2$:

$$\begin{aligned}
\mathbb{P}\big(S_{2+s} = y \mid S_s = x\big) &= \sum_{z\in\mathbb{T}_N} \mathbb{P}\big(S_{2+s} = y, S_{1+s} = z \mid S_s = x\big) \\
&= \sum_{z\in\mathbb{T}_N} \mathbb{P}\big(S_{2+s} = y \mid S_{1+s} = z \text{ and } S_s = x\big)\mathbb{P}\big(S_{1+s} = z \mid S_s = x\big) \\
&= \sum_{z\in\mathbb{T}_N} \mathbb{P}\big(S_{2+s} = y \mid S_{1+s} = z\big)\mathbf{Q}_{x,z} \qquad \text{(by Markov property and (1.2))} \\
&= \sum_{z\in\mathbb{T}_N} \mathbf{Q}_{z,y}\mathbf{Q}_{x,z} = (\mathbf{Q}^2)_{x,y} \qquad \text{(by (1.2))}.
\end{aligned}$$

□

1.1.3 Canonical measure

Let us denote $\mathcal{X} = \mathbb{Z}$ or $\mathbb{T}_N$ the space where the random walk evolves.

Note that for each $t \in \mathbb{N}$ fixed, the random variable S_t is defined on $(\Omega, \mathcal{F}, \mathbb{P})$. However it is often more convenient to consider the *canonical probability measure*, defined as follows:

let $\mathbb{P}_{\nu_0}$ be the probability law on the *path space* $\mathcal{X}^{\mathbb{N}}$ of the full trajectory $(S_t)_{t\in\mathbb{N}}$ starting from the initial probability measure ν_0 (on $\mathcal{X}$). More precisely, consider the measurable function[1]

$$\begin{aligned}
\varphi : (\Omega, \mathcal{F}, \mathbb{P}) &\longrightarrow \big(\mathcal{X}^{\mathbb{N}}, \mathcal{B}\big) \\
\omega &\longmapsto \big(S_t(\omega)\big)_{t\in\mathbb{N}}
\end{aligned}$$

and then, $\mathbb{P}_{\nu_0}$ is the *pushforward measure*

$$\mathbb{P}_{\nu_0} := \varphi_*(\mathbb{P}), \qquad \text{meaning, for any } B \in \mathcal{B},\ \varphi_*(\mathbb{P})(B) = \mathbb{P}\big(\varphi^{-1}(B)\big).$$

Its corresponding expectation on $\mathcal{X}^{\mathbb{N}}$ is denoted by $\mathbb{E}_{\nu_0}$.

1.2 Asymptotic behavior and universality

In order to illustrate the link between a *discrete* model and a *continuous* one, let us go back to the one-dimensional random walk on $\mathbb{Z}$.

[1]The σ–algebra $\mathcal{B}$ of $\mathcal{X}^{\mathbb{N}}$ is the smallest σ–algebra that makes the projection maps $x \in \mathcal{X}^{\mathbb{N}} \mapsto x_k \in \mathbb{T}_N$ measurable.

We first define a *continuous time process* in the following way: for any $t \geqslant 0$, let $\mathrm{S}(t)$ be the *linear interpolation* between $\mathrm{S}_{[t]}$ and $\mathrm{S}_{[t]+1}$, *i.e.*

$$\mathrm{S}(t) := \mathrm{S}_{[t]} + (t - [t])(\mathrm{S}_{[t]+1} - \mathrm{S}_{[t]}).$$

Renormalising as in Proposition 1.1, the Central Limit Theorem gives

$$\mathrm{S}_n^*(t) := \frac{\mathrm{S}(nt)}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{dist.}} \mathcal{N}(0,t).$$

Moreover, for any $0 = t_0 < t_1 < \cdots < t_p$, the random variables $\{\mathrm{S}_n^*(t_i) - \mathrm{S}_n^*(t_{i-1})\}$, $1 \leqslant i \leqslant d$, are independent, and each one converges in distribution to a centered normal r.v.

$$\mathcal{N}(0,(t_i - t_{i-1})).$$

Remarkably, the CLT shows that this limit does not depend much on the law of the ε_k's. In fact, we obtain the same convergence as soon as $(\varepsilon_k)_{k\in\mathbb{N}}$ are *centered* i.i.d. and with variance equal to 1. We obtain a *universal limit of random walks*, which has the following properties:

Definition 1.4. *We call* standard Brownian motion started from 0 *a continuous-time stochastic process*

$$\mathrm{B} := (\mathrm{B}_t, t \geqslant 0) \quad \textit{which takes values in } \mathbb{R}$$

such that $\mathrm{B}_0 = 0$ *a.s. and, for any* $0 = t_0 < t_1 < \cdots < t_p$*,*

- *the variables* $\{\mathrm{B}_{t_i} - \mathrm{B}_{t_{i-1}}\}_{1\leqslant i\leqslant p}$ *are independent (we say that it has* independent increments*)*
- *for any* i*,* $\mathrm{B}_{t_i} - \mathrm{B}_{t_{i-1}}$ *has the same law has* $\mathcal{N}(0,(t_i - t_{i-1}))$ *(we say that the* increments are stationnary gaussian*)*
- *almost surely, the function* $t \mapsto \mathrm{B}_t$ *is continuous.*

We give a few important results:

Theorem 1.5 (Lévy, 1933)**.** *The one-dimensional standard Brownian motion exists.*

In other words, there exists $(\Omega, \mathcal{F}, \mathbb{P}_x)$ *and a stochastic process* $(\mathrm{B}_t, t \geqslant 0)$ *which is defined on that space and satisfies Definition 1.4.*

Proposition 1.6 (Markov property)**.** *Let* B *be a standard Brownian motion on* $\mathbb{R}$.

1. *For any $s > 0$, the process $(\mathrm{B}_{t+s} - \mathrm{B}_s, t \geqslant 0)$ is a Brownian motion independent from $\sigma(\mathrm{B}_u \,:\, 0 \leqslant u \leqslant s) =: \mathcal{F}_s$.*

2. *Let $\mathrm{T} \in \mathbb{R}_+ \cup \{\infty\}$ be a random variables such that, for any $t \geqslant 0$, $\{\mathrm{T} \leqslant t\}$ is $\mathcal{F}_t$–mesurable[2]. We denote by $\mathcal{F}_\mathrm{T}$ the σ-algebra of the events "before " T, namely*
$$\mathcal{F}_\mathrm{T} = \big\{ \mathrm{A} \in \mathcal{F}_\infty \;;\; \forall\, t \geqslant 0,\ \mathrm{A} \cap \{\mathrm{T} \leqslant t\} \in \mathcal{F}_t \big\},$$
where $\mathcal{F}_\infty$ is the σ–algebra generated by B_t, $t \in \mathbb{R}_+$.

   Then for any $t \geqslant 0$, the process $(\mathbf{1}_{\mathrm{T}<\infty}(\mathrm{B}_{t+\mathrm{T}} - \mathrm{B}_t), t \geqslant 0)$ is a standard Brownian motion independent from $\mathcal{F}_\mathrm{T}$.

1.2.1 Universality and Donsker Theorem

The convergence of random walks is even stronger. We already know, from the CLT, that
$$(\mathrm{S}^*_n(t_1), \ldots, \mathrm{S}^*_n(t_p)) \xrightarrow[n\to\infty]{\text{dist.}} (\mathrm{B}_{t_1}, \ldots, \mathrm{B}_{t_p})$$
(as random variables taken values in $\mathbb{R}^p$). We have the following stronger result

Theorem 1.7 (Donsker, 1956)**.** *For any $\mathrm{T} > 0$, the random function $\mathrm{S}^*_n : [0, \mathrm{T}] \to \mathbb{R}$ converges in distribution towards the standard Brownian motion B in the metric space $(\mathcal{C}([0,\mathrm{T}]), \|\cdot\|_\infty)$.*

Some idea of the proof. We build (S_n) on the same probability space as B, in a way that their trajectories are close to each other. □

This theorem is very powerful and it has several applications:

- *discrete → continuous* :

  We use a (simple!) random walk to show that $(\overline{\mathrm{B}}_t - \mathrm{B}_t, t \geqslant 0)$ has the same distribution as $(|\mathrm{B}|_t, t \geqslant 0)$, where $\overline{\mathrm{B}}_t := \sup_{s \leqslant t} \mathrm{B}_s$.

- *continuous → discrete* :

  We use direct computations on the Brownian motion to show that the last moment where the random walk changes sign is asymptotically distributed as the *Arcsin law*:
$$\sup\big\{1 \leqslant k \leqslant n-1 \;;\; \mathrm{S}_k \mathrm{S}_{k+1} \leqslant 0\big\} \xrightarrow[n\to\infty]{\text{dist.}} \mathrm{A} \sim \text{ Arcsin law.}$$

[2] In other words the "information" contained in the r.v. T only depends on the past and present, not on future.

Chapter 2

Independent random walks on $\mathbb{T}_N$

We investigate here a toy model of a *particle system*, where indistinguishable particles move as simple independent random walks on one-dimensional lattices, with probability p to jump the right, and probability $1-p$ to jump to the left.

Let us start by distinguishing all particles: let K denote the total number of particles at time 0 and take $x_1, \dots, x_K \in \mathbb{Z}$ which will correspond to their initial positions. Let $\{(\varepsilon_n^i)\}_{i=1,\dots,K}$ be K independent copies of sequences of i.i.d. random variables such that

$$\mathbb{P}(\varepsilon_n^i = 1) = p, \qquad \mathbb{P}(\varepsilon_n^i = -1) = 1-p.$$

We define, for any $t \in \mathbb{N}$ (the time variable), any $i \in \{1, \dots, K\}$,

$$X_t^i = \left(x_i + \sum_{n=1}^{t} \varepsilon_n^i \right) \mod N, \tag{2.1}$$

which therefore represents the position at time t of the i–th particle on the torus. For any i, the sequence $(X_t^i)_{t\in\mathbb{N}}$ is a *simple random walk on the torus* $\mathbb{T}_N$ starting at $(x_i \mod N)$ with *transition matrix* $\mathbf{Q}$.

Now, we are not interested in the individual position of *each* particle, but only in the total number of particles at each site. For a site $x \in \mathbb{T}_N$, we denote by $\eta(x) \in \mathbb{N}$ the number of particles at site x for the *configuration* $\eta \in \mathbb{N}^{\mathbb{T}_N}$. The state space of the system, also called *configuration space* is $\Omega_N = \mathbb{N}^{\mathbb{T}_N}$.

From the construction above, we are interested in the *configuration of particles at time* $t \in \mathbb{N}$ which is given by

$$\eta_t(x) = \sum_{i=1}^{K} \mathbf{1}\{X_t^i = x\}, \qquad x \in \mathbb{T}_N.$$

REMARK 2.1. Conversely, given an initial configuration $\eta \in \Omega_N$, to define the evolution of the system, we can first label all particles and then let them evolve according to the stochastic dynamics described above. In fact, in the following we will start from η_0 distributed according to a certain probability measure μ_0 on Ω_N, and thefefore K will become a random variable, given by

$$K = \sum_{x \in \mathbb{T}_N} \eta_0(x).$$

DEFINITION 2.2. *Let μ_0 be an* initial probability measure *on Ω_N. We denote by $\mathbb{P}_{\mu_0}$ the canonical probability measure on the path space $\mathcal{E}_N := (\Omega_N)^{\mathbb{N}}$ induced by the independent random walks dynamics described above and the initial measure μ_0. Expectation with respect to $\mathbb{P}_{\mu_0}$ is denoted by $\mathbb{E}_{\mu_0}$.*

We denote by η_0 the initial configuration *distributed according to μ_0. Then, for any $t \in \mathbb{N}$ and $x \in \mathbb{T}_N$ we have the representation*

$$\eta_t(x) = \sum_{y \in \mathbb{T}_N} \sum_{i=1}^{\eta_0(y)} \mathbf{1}\{X_t^{i,y} = x\},$$

where $\left\{(X_t^{i,y})_{t \in \mathbb{N}} \,;\; y \in \mathbb{T}_N \,, i \in \mathbb{N}\right\}$ are independent simple random walks on $\mathbb{T}_N$, with transition matrix **Q***, such that for any i, $X_0^{i,y} = y$ a.s.*

For any probability measure μ on Ω_N we denote by E_μ the expectation with respect to μ, namely $E_\mu[f] = \int_{\Omega_N} f(\eta)\mu(d\eta)$. Note the difference between E_μ and $\mathbb{E}_\mu$: if η_0 is distributed according to μ, then we have $\mathbb{E}_\mu[F(\eta_0)] = E_\mu[F(\eta)]$ for any bounded continuous function F on Ω_N.

2.1 Equilibrium states

One of the first questions to answer is the characterization of all *invariant probability distributions*, which have the following property: if the initial configuration $\eta_0 \in \mathbb{N}^{\mathbb{T}_N}$ is distributed according to an invariant probability measure μ on Ω_N, then, the configurations keep the same distribution forever, *i.e.* η_t is distributed according to μ for any $t \in \mathbb{N}$.

Let $\mathcal{P}_\alpha$ be the *Poisson distribution* of parameter $\alpha \geqslant 0$, namely the probability measure on $\mathbb{N}$ given by

$$\mathcal{P}_\alpha(k) = e^{-\alpha}\frac{\alpha^k}{k!}, \qquad k \in \mathbb{N}.$$

We introduce a probability measure on Ω_N as follows:

DEFINITION 2.3. *Let $\rho : \mathbb{T}_N \to \mathbb{R}_+$ be a fixed positive function. We call* Poisson measure on $\mathbb{T}_N$ associated to the function ρ *a probability measure on the configuration space $\Omega_N = \mathbb{N}^{\mathbb{T}_N}$, denoted by $\nu^N_{\rho(\cdot)}$, having the following two properties:*

1. *Under $\nu^N_{\rho(\cdot)}$, the random variables $\{\eta(x)\}_{x\in\mathbb{T}_N}$ are independent.*
2. *For every fixed $x \in \mathbb{T}_N$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho(x)$.*

In other words, $\nu^N_{\rho(\cdot)}$ is the product measure

$$\nu^N_{\rho(\cdot)} = \bigotimes_{x\in\mathbb{T}_N} \mathcal{P}_{\rho(x)}.$$

If the function ρ is constant equal to α, we denote this measure by ν^N_α.

In the following, expectation with respect to any measure ν is denoted by E_ν.

The measure $\nu^N_{\rho(\cdot)}$ is fully characterized by its *multidimensional Laplace transform*, given by: for any positive sequence $\{\lambda(x)\}_{x\in\mathbb{T}_N}$,

$$\begin{aligned}\mathrm{E}_{\nu^N_{\rho(\cdot)}}\Big[\exp\Big\{-\sum_{x\in\mathbb{T}_N}\lambda(x)\eta(x)\Big\}\Big] &= \prod_{x\in\mathbb{T}_N}\exp\Big\{\rho(x)\big(e^{-\lambda(x)}-1\big)\Big\}\\ &= \exp\Big\{\sum_{x\in\mathbb{T}_N}\rho(x)\big(e^{-\lambda(x)}-1\big)\Big\}.\end{aligned}$$

The first result consists in proving that the Poisson measures associated to constant functions are invariant:

PROPOSITION 2.4. *If η_0 is distributed according to ν^N_α, then η_t is distributed according to ν^N_α for any $t \in \mathbb{N}$.*

Proof. By assumption, we know that, for any $y \in \mathbb{T}_N$, $\eta_0(y)$ is distributed according to $\mathcal{P}_\alpha$. We want to get the distribution of η_t. Since a probability measure on Ω_N is characterized by its multidimensional Laplace transform, let us compute the following expectation, for any $t \in \mathbb{N}$:

$$\mathbb{E}_{\nu^N_\alpha}\Big[\exp\Big\{-\sum_{x\in\mathbb{T}_N}\lambda(x)\eta_t(x)\Big\}\Big].$$

For any i and any site $y \in \mathbb{T}_N$ we denote by $\mathrm{X}^{i,y}_t$ the position at time t of the i–th particle initially at y. In this way, the number of particles at site x at time

t is explicitely given by

$$\eta_t(x) = \sum_{y \in \mathbb{T}_N} \sum_{i=1}^{\eta_0(y)} \mathbf{1}\{X_t^{i,y} = x\},$$

with the convention $\sum_{\varnothing} = 0$. From this identity, and inverting the order of summations, we get

$$\sum_{x \in \mathbb{T}_N} \lambda(x)\eta_t(x) = \sum_{y \in \mathbb{T}_N} \sum_{i=1}^{\eta_0(y)} \lambda\big(X_t^{i,y}\big).$$

Recall that each particle evolves independently from each other. Let us denote by $\mathbf{S}_t$ the position at time t of a generic random walk on the torus $\mathbb{T}_N$ starting from 0 (with transition probability matrix $\mathbf{Q}$) and by $\mathbb{E}_{\text{rw}}$ the expectation with respect to its probability distribution. Then, for any i and any $y \in \mathbb{T}_N$, $X_t^{i,y}$ has the same distribution as $y + \mathbf{S}_t$. In particular, from Proposition 1.3 we have

$$\mathbb{E}_{\text{rw}}\big[F(\mathbf{S}_t)\big] = \sum_{x \in \mathbb{T}_N} (\mathbf{Q}^t)_{0,x}\, F(x),$$

for any bounded and continuous function F. Besides, from the independence of the particles trajectories, if we take K continuous and bounded functions $F_1, \dots, F_K$, we can write,

$$\mathbb{E}_{\nu_\alpha^N}\Big[\prod_{i=1}^{K} F_i(X_t^{i,y})\Big] = \prod_{i=1}^{k} \mathbb{E}_{rw}\big[F_i(y + \mathbf{S}_t)\big].$$

We now use these relations in order to compute[1]:

$$
\begin{aligned}
\mathbb{E}_{\nu_\alpha^{\mathrm{N}}}\Big[\exp\Big\{-\sum_{x\in\mathbb{T}_{\mathrm{N}}}\lambda(x)\eta_t(x)\Big\}\Big] &= \prod_{y\in\mathbb{T}_{\mathrm{N}}}\mathbb{E}_{\nu_\alpha^{\mathrm{N}}}\Big[\exp\Big\{-\sum_{i=1}^{\eta_0(y)}\lambda\big(\mathrm{X}_t^{i,y}\big)\Big\}\Big] \\
&= \prod_{y\in\mathbb{T}_{\mathrm{N}}}\int_{\Omega_{\mathrm{N}}}\nu_\alpha^{\mathrm{N}}(d\eta)\mathbb{E}_{\nu_\alpha^{\mathrm{N}}}\Big[\exp\Big\{-\sum_{i=1}^{\eta(y)}\lambda\big(\mathrm{X}_t^{i,y}\big)\Big\}\Big] \\
&= \prod_{y\in\mathbb{T}_{\mathrm{N}}}\int_{\Omega_{\mathrm{N}}}\nu_\alpha^{\mathrm{N}}(d\eta)\Big(\mathbb{E}_{\mathrm{rw}}\Big[\exp\{-\lambda(y+\mathbf{S}_t)\}\Big]\Big)^{\eta(y)} \\
&= \prod_{y\in\mathbb{T}_{\mathrm{N}}}\exp\Big\{\alpha\Big(\mathbb{E}_{\mathrm{rw}}[\exp\{-\lambda(y+\mathbf{S}_t)\}]-1\Big)\Big\} \\
&= \exp\Big\{\sum_{y\in\mathbb{T}_{\mathrm{N}}}\alpha\Big(\sum_{z\in\mathbb{T}_{\mathrm{N}}}(\mathbf{Q}^t)_{0,z}\,e^{-\lambda(y+z)}-1\Big)\Big\} \\
&= \exp\Big\{\sum_{y\in\mathbb{T}_{\mathrm{N}}}\alpha\Big(\sum_{z\in\mathbb{T}_{\mathrm{N}}}(\mathbf{Q}^t)_{0,z-y}\,e^{-\lambda(z)}-1\Big)\Big\} \\
&= \exp\Big\{\alpha\sum_{y\in\mathbb{T}_{\mathrm{N}}}\sum_{z\in\mathbb{T}_{\mathrm{N}}}(\mathbf{Q}^t)_{y,z}\big(e^{-\lambda(z)}-1\big)\Big\} \\
&= \exp\Big\{\alpha\sum_{z\in\mathbb{T}_{\mathrm{N}}}\big(e^{-\lambda(z)}-1\big)\Big\},
\end{aligned}
$$

since the matrix **Q** is bi-stochastic. □

REMARK 2.5. The parameter α is related to the conservation of the total number of particles. In fact, note that

$$\int_{\Omega_{\mathrm{N}}}\nu_\alpha^{\mathrm{N}}(d\eta)\Big(\frac{1}{\mathrm{N}}\sum_{x\in\mathbb{T}_{\mathrm{N}}}\eta(x)\Big)=\sum_{k\geqslant 0}e^{-\alpha}\frac{\alpha^k}{k!}k=\alpha.$$

Furthermore, from the law of large numbers,

$$\lim_{\mathrm{N}\to\infty}\frac{1}{\mathrm{N}}\sum_{x\in\mathbb{T}_{\mathrm{N}}}\eta(x)=\alpha,\qquad \text{a.s.}$$

[1]We also use the fact that

$$\sum_{k\in\mathbb{N}}\lambda^k\mathcal{P}_\alpha(k)=\exp(\alpha(\lambda-1))$$

2.2 Local equilibrium

We want to rigorously perform a limit in which the distance between particles converges to 0, so as to pass from a *microscopic* description to a *macroscopic* one. This point does not present any difficulty in formalization. We will consider the torus $\mathbb{T}_N$ as embedded in the continuous torus $\mathbb{T} = [0, 1) = \mathbb{R}/\mathbb{Z}$. In this way, each macroscopic point $u \in \mathbb{T}$ us associated with a microscopic site $x = [uN] \in \mathbb{T}_N$, and reciprocally, each site $x \in \mathbb{T}_N$ is associated with a macroscopic point $u = \frac{x}{N} \in \mathbb{T}$.

In order to see a non-trivial evolution of the system, we initially distribute particles according to a Poisson measure with *slowly varying parameter*. The resulting measure on Ω_N is defined similarly as in Definition 2.3:

Definition 2.6. *Let $\rho_0 : \mathbb{T} \to \mathbb{R}_+$ be a smooth initial* density profile. *We call* local equilibrium associated to the function ρ_0 *a probability measure on $\Omega_N = \mathbb{N}^{\mathbb{T}_N}$, denoted by $\nu^N_{\rho_0(\cdot)}$, having the following two properties:*

1. *Under $\nu^N_{\rho_0(\cdot)}$, the random variables $\{\eta(x)\}_{x\in\mathbb{T}_N}$ are independent.*
2. *For any $x \in \mathbb{T}_N$, $\eta(x)$ is distributed according to a Poisson law of parameter $\rho_0(\frac{x}{N})$.*

In other words, $\nu^N_{\rho_0(\cdot)}$ is the product measure

$$\nu^N_{\rho(\cdot)} = \bigotimes_{x\in\mathbb{T}_N} \mathcal{P}_{\rho_0(\frac{x}{N})}.$$

As the parameter N increases to infinity, the discrete torus $\mathbb{T}_N$ tends to the full lattice $\mathbb{Z}$. We can also define a Poisson measure on the space $\mathbb{N}^{\mathbb{Z}}$, with constant parameter $\alpha > 0$, in a very similar way as in Definition 2.3.

The term *local equilibrium* comes from the following observation: as $N \uparrow \infty$, when we look "close" to a point $u \in \mathbb{T}$ (i.e. around $x = [uN]$), we observe a Poisson measure of parameter (almost) constant equal to $\rho_0(u)$.

To perform the limit $N \uparrow \infty$, we embed the space $\mathbb{N}^{\mathbb{T}_N}$ into $\mathbb{N}^{\mathbb{Z}}$ be identifying a configuration on the torus to a periodic configuration on the full lattice, as done for instance in (2.1). We can also see ρ_0 as a function on $\mathbb{R}$ by periodization. The configuration space $\mathbb{N}^{\mathbb{Z}}$ is endowed with the product topology. We denote by $\mathcal{M}_1$ the space of probability measures on $\mathbb{N}^{\mathbb{Z}}$ endowed with the weak topology.

The notion of *local equilibrium* is a bit more general:

DEFINITION 2.7. *We say that a probability measure μ^N on Ω_N is a* local equilibrium *associated to the continuous function $\rho_0 : \mathbb{T} \to [0,1]$ if, for any continuous function $\varphi : [0,1] \to \mathbb{R}$, the following convergence holds, in probability (with η distributed according to μ^N):*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{T}_N} \eta(x)\varphi\left(\tfrac{x}{N}\right) = \int_{\mathbb{T}} \rho_0(u)\varphi(u)du. \tag{2.2}$$

In particular, under $\nu^N_{\rho_0(\cdot)}$, we have the following convergence[2]: locally around any point $u \in \mathbb{T}$,

$$\lim_{\varepsilon\to 0} \lim_{N\to+\infty} \frac{1}{2\varepsilon} \sum_{|y-[uN]|\leqslant\varepsilon} \eta(y) = \rho_0(u),$$

which means that the average density of particles in the box of size 2ε around the site $[uN]$ is asymptotically equal to $\rho_0(u)$.

REMARK 2.8. The fact that $\nu^N_{\rho_0(\cdot)}$ indeed satisfies (2.2) is a consequence of Chebyshev's inequality (Exercise).

2.3 Hydrodynamic equation

We now turn to the study of the distribution of particles at later times, starting from a product measure with slowly varying parameter. We would like to obtain the same kind of convergence as in (2.2).

We can repeat the computations we did to prove Proposition 2.4, and, if η_0 is distributed according to $\nu^N_{\rho_0(\cdot)}$, then

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\left[\exp\left\{-\sum_{x\in\mathbb{T}_N}\lambda(x)\eta_0(x)\right\}\right] = \exp\left\{\sum_{z\in\mathbb{T}_N}\left(e^{-\lambda(z)}-1\right)\rho_0\left(\tfrac{z}{N}\right)\right\}$$

and we now obtain that

$$\begin{aligned}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\left[\exp\left\{-\sum_{x\in\mathbb{T}_N}\lambda(x)\eta_t(x)\right\}\right] &= \exp\left\{\sum_{y\in\mathbb{T}_N}\rho_0\left(\tfrac{y}{N}\right)\sum_{z\in\mathbb{T}_N}(\mathbf{Q}^t)_{y,z}\left(e^{-\lambda(z)}-1\right)\right\}\\ &= \exp\left\{\sum_{z\in\mathbb{T}_N}\left(e^{-\lambda(z)}-1\right)\left(\sum_{y\in\mathbb{T}_N}(\mathbf{Q}^t)_{y,z}\,\rho_0\left(\tfrac{y}{N}\right)\right)\right\}.\end{aligned}$$

[2]Take the approximation of the Dirac function δ_u given by $\psi_\varepsilon(v) = \frac{1}{2\varepsilon}$ if $|v-u| < \varepsilon$ and 0 otherwise.

Therefore, at time t, η_t is still distributed according to a product Poisson measure with slowly varying parameter, which is now at site $z \in \mathbb{T}_N$,

$$\psi_{N,t}(z) := \sum_{y \in \mathbb{T}_N} (\mathbf{Q}^t)_{y,z}\, \rho_0\left(\tfrac{y}{N}\right) \tag{2.3}$$

instead of $\rho_0(\frac{z}{N})$. Namely, we have obtain the following:

LEMMA 2.9. *For any $t \in \mathbb{N}$,*

$$\eta_t \sim \bigotimes_{z \in \mathbb{T}_N} \mathcal{P}_{\psi_{N,t}(z)}.$$

where

$$\psi_{N,t}(z) = \sum_{x \in \mathbb{T}_N} (\mathbf{Q}^t)_{0,x}\, \rho_0\left(\tfrac{z-x}{N}\right) = \mathbb{E}_{\mathrm{rw}}\left[\rho_0\left(\tfrac{z-\mathbf{Y}_t}{N}\right)\right].$$

We now want to look close to a macroscopic point $u \in \mathbb{T}$, and therefore we need to understand the behavior of $\psi_{N,t}([uN])$ as $N \to \infty$, in other words we want to compute

$$\lim_{N\to\infty} \psi_{N,t}([uN]) = \lim_{N\to\infty} \mathbb{E}_{\mathrm{rw}}\left[\rho_0\left(u - \tfrac{1}{N}\mathbf{Y}_t\right)\right] \overset{??}{=} \rho(t,u).$$

We state the results in distinct propositions. First, let us assume that t is fixed and does not depend on N.

PROPOSITION 2.10 (No evolution). *Let $t \in \mathbb{N}$ be fixed. Then, for any $u \in \mathbb{T}$,*

$$\lim_{N\to\infty} \psi_{N,t}([uN]) = \rho_0(u).$$

This means that the density profile remains unchanged. The system did not have time to evolve and this reflects the fact that, at the macroscopic scale, particles did not move. We solve this problem by distinguishing two time scales: a microscopic time t and a macroscopic time t_N which is infinitely large with respect to t.

PROPOSITION 2.11 (Hyperbolic time scale). *Recall that $m = 2p - 1$, and let $t \in \mathbb{R}_+$ be fixed.*

Then, in the hyperbolic time scale $[tN]$, for any $u \in \mathbb{T}$,

$$\lim_{N\to\infty} \psi_{N,[tN]}([uN]) = \rho_0(u - mt) =: \rho(t,u).$$

In this new time scale, we observe a new density profile, which is the original one translated by mt. This new profile satisfies the partial differential equation:

$$\partial_t \rho + m\partial_u \rho = 0,$$

which corresponds to the macroscopic deterministic evolution for the unique conserved quantity (the density). An interacting particle system for which there exists a time and space macroscopic scales in which the conserved quantities evolve according to some partial differential equation is said to have a hydrodynamic description. This equation is called *hydrodynamic equation* associated to the system.

When m vanishes, the profile did not change in the hyperbolic time change. In this case, to observe an interesting evolution, we need to consider a larger time scale:

PROPOSITION 2.12. *Assume $p = \frac{1}{2}$, namely $m = 0$ and denote $\sigma = 1$ the variance of the elementary displacement. Let $t \in \mathbb{R}_+$ be fixed.*

Then, in the diffusive time scale $[t\mathrm{N}^2]$, for any $u \in \mathbb{T}$,

$$\lim_{\mathrm{N}\to\infty} \psi_{\mathrm{N},[t\mathrm{N}^2]}([u\mathrm{N}]) = \int_{\mathbb{R}} \rho_0(\theta)\mathrm{G}_t(u-\theta)d\theta,$$

where G_t is the density of the centered Gaussian distribution with variance $t\sigma = t$.

Since the Gaussian distribution is the fundamental solution of the heat equation, we obtain that the hydrodynamic equation in the diffusive time scale is

$$\partial_t \rho = \sigma \partial_u^2 \rho.$$

We now state the final result, which follows from the last three propositions:

THEOREM 2.13 (Hydrodynamic behaviors)**.** *Assume that η_0 is distributed according to $\nu^{\mathrm{N}}_{\rho_0(\cdot)}$ with $\rho_0 : \mathbb{T} \to \mathbb{R}_+$ a smooth initial density profile. Let $\varphi : \mathbb{T} \to \mathbb{R}$ be a smooth test function.*

Then, we have the following convergence in probability

$$\frac{1}{\mathrm{N}} \sum_{x\in\mathbb{T}_\mathrm{N}} \varphi\left(\tfrac{x}{\mathrm{N}}\right)\eta_{t_\mathrm{N}}(x) \xrightarrow[\mathrm{N}\to\infty]{} \int_{\mathbb{T}} \varphi(u)\rho(t,u)du,$$

where $\rho(t,u)$ is

- *constant equal to $\rho_0(u)$ if $t_\mathrm{N} \equiv t$ does not depend on* N,

- *solution to the transport equation*

$$\partial_t \rho + m \partial_u \rho = 0, \qquad \rho(0, \cdot) = \rho_0(\cdot)$$

  if $t_N = [tN]$ *and* $m \neq 0$,

- *solution to the heat equation*

$$\partial_t \rho = \partial_u^2 \rho, \qquad \rho(0, \cdot) = \rho_0(\cdot)$$

  if $t_N = [tN^2]$ *and* $m = 0$.