HYDRODYNAMIC LIMIT FOR A CHAIN WITH THERMAL AND MECHANICAL BOUNDARY FORCES

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Abstract. We prove the hydrodynamic limit for a one dimensional harmonic chain with a random flip of the momentum sign. The system is open and subject to two thermostats at the boundaries and to an external tension at one of the endpoints. Under a diffusive scaling of space-time, we prove that the empirical profiles of the two locally conserved quantities, the volume stretch and the energy, converge to the solution of a non-linear diffusive system of conservative partial differential equations.

1. Introduction

The mathematical derivation of the macroscopic evolution of the conserved quantities of a physical system, from its microscopic dynamics, through a rescaling of space and time (so called hydrodynamic limit) has been the subject of much research in the last 40 years (cf. [9] and references within). Although heuristic assumptions like *local equilibrium* and *linear response* permit to formally derive the macroscopic equations [14], mathematical proofs are very difficult and most of the techniques used are based on relative entropy methods (cf. [9] and references within). Unfortunately, in the diffusive scaling when energy is one of the conserved quantities, relative entropy methods cannot be used. In some situations a different approach, based on Wigner distributions, is effective in controlling the macroscopic evolution of energy. This is the case for a chain of harmonic springs with a random flip of sign of the velocities, provided with periodic boundary conditions, for which the total energy and the total length of the system are the two conserved quantities, and where the hydrodynamic limit has been proven in [10].

The purpose of the present article is to deal with the case when microscopic mechanical forces and thermal heat baths acting on the boundaries, are present, and to determine macroscopic boundary conditions for the hydrodynamic diffusive equations. In the scaling limit the presence of boundary conditions is challenging, as the action of the forces and thermostats become singular. In [3] the authors prove the existence and uniqueness of the non-equilibrium stationary state, even in the anharmonic case. The existence of the Green-Kubo formula

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for the thermal conductivity is also proven in [3]. However, it turns out very difficult to control the limit properties as the size of the system becomes infinite (*i.e.* macroscopic). In particular the rigorous proof of the Fourier law, which states that the average energy current is inversely proportional to the size of the system and proportional to the negative temperature difference of the thermostats, is still an open problem. One of the main difficulties of this open dynamics is to prove that energy inside the system remains proportional to its size (a trivial fact for the periodic case where energy is globally conserved). In fact the random flip of the velocity signs does not move the energy in the system and the energy transport is entirely due to the hamiltonian part of the dynamics, that is very hard to control.

This difficulty forced us to consider a different energy conserving random dynamics, where kinetic energy is exchanged between nearest neighbor particles in a continuous random mechanism, see [11]. In this case the stochastic dynamics is also responsible for energy transport. The non-equilibrium stationary state (NESS) for this dynamics was already considered in [2], where the Fourier law was proven without external force, *i.e.* in mechanical (but not thermal) equilibrium. In [11] the validity of the Fourier law for the NESS is extended also to the situation when an external tension force is present (then the system is in both mechanical and thermal non-equilibrium). Furthermore, the existence of both stationary macroscopic profiles for the temperature and volume stretch, at least in some situations, are established in [11]. In particular, the temperature profile has the interesting feature that the stationary temperatures in the bulk can be higher than at the boundaries, a general behavior conjectured in the NESS for many systems [13]. Furthermore, because of the presence of other conservation laws, the stationary energy current can have the same sign as the gradient of the temperature – the phenomenon called *uphill diffusion phenomenon* in the literature.

Concerning the hydrodynamic limit, in the appendix section of [11] we have formulated a heuristic argument, based on entropy production estimates, that have not been proved there, and that substantiated the validity of the macroscopic equations governing the dynamics in the case of a random momentum exchange microscopic model, see Section 2.2 of [11]. Besides the aforementioned entropy production estimates, in order to obtain the hydrodynamic limit, one needs to establish also the property of *equipartition* of the random fluctuations of the mechanical and thermal components of the microscopic energy density, which was postulated in [11, identity (A.46)]. As we have pointed out in [11] this property seems to be out of reach of the relative entropy method and some other approach to resolve the difficulty is needed. In the present work we employ the Wigner distribution method to give a rigorous prove of the hydrodynamic limit, for an open system with a random flip of momenta, see Section 2 below for its precise formulation.

A crucial observation is the identity (8.9) that holds for the L^2 norm of the covariances of random fluctuations of momenta and stretches, which we obtain by careful analysis of time evolution of the Fourier-Wigner functions defined in Section 8.1. The last two terms in the right hand side of (8.9) correspond to the dissipation, due to the stochastic dynamics in the bulk. The remaining two terms describe the interaction between the fluctuation of the thermal and mechanical components of the kinetic energy at the boundary points and in the bulk of the system, respectively. In order to control these terms we need to control the rate of damping of the mechanical energy, which is done in Lemma 5.4. These controls allow us to prove that the L^2 norm of the covariances of random fluctuations of momenta and stretches, at the given time, grows with the logarithm of the size of the system: this is the content of Proposition 8.1. This in turn enables us to show, using again the properties of the Fourier-Wigner function dynamics, the already mentioned equipartition property, which is stated in Proposition 4.6 and proved in Section 8.3.

The next ingredient that is important in the hydrodynamic limit argument is the linear bound, in the system size, for the relative entropy of the chain, with respect to both the thermal equilibrium and local equilibrium probability measures. We establish this bound, together with some of its consequences, in Section 7 (see Proposition 7.1). A crucial property that allows us to control the entropy production, coming from the action of the external force, is the estimate of the damping rate of the time average of the momentum expectation at the respective endpoint of the system obtained in Proposition 4.2.

As we have already mentioned the model we consider in the present work, with the random flip of the sign of momenta, is more difficult to handle than the random momentum exchange one investigated in [11], due to the fact that the energy is not transported by the stochastic part of the dynamics. We believe that the method used in the present paper can be also applied to that model. In addition, the assumption that the forcing acting at the boundary is constant in time, is only made here to simplify the already complicated arguments for the entropy bound of Section 7.1 and the momentum damping estimates formulated in Proposition 4.2 and Lemma 5.4. At the expense of increasing the volume of the calculations, with some additional effort, one could extend the results of the present paper to the case when the tension is a C^1 smooth function of time.

A proof of the Fourier law in the stationary state remains an open problem for the random flip model. We hope that in the future we will also be able to extend the results of the present paper to the more challenging case of the chain of anharmonic springs.

Finally, concerning the organization of the paper. The description of the model and basic notation is presented in Section 2. The formulation of the main result, together with the auxiliary facts needed to carry out the proof are done in Section 3. For a reader convenience we sketch the structure of the main argument in Section 4. The proof of the hydrodynamic limit is carried out in Section 5.

It is contingent on a number of auxiliary results that are shown throughout the remainder of the paper. Namely, the estimates of the momentum and stretch averages are done in Section 6, the energy production bounds are obtained in Section 7, while Section 8 is devoted to showing the equipartition property. Finally, in the appendix sections we give the proofs of quite technical estimates used throughout Section 6.

2. Preliminaries

2.1. **Open chain of oscillators.** For $n \geqslant 1$ an integer we let $\mathbb{I}_n := \{0, 1, \dots, n\}$ and $\mathbb{I}_n^\circ := \{1, \dots, n-1\}$. The points 0 and n are the extremities of the chain. Let $\mathbb{I} := [0, 1]$ be the continuous counterpart. We suppose that the position and momentum of a harmonic oscillator at site $x \in \mathbb{I}_n$ are denoted by $(q_x, p_x) \in \mathbb{R}^2$. The interaction between two particles situated at $x-1, x \in \mathbb{I}_n^\circ$ is described by the quadratic potential energy

$$V(q_x - q_{x-1}) := \tfrac{1}{2}(q_x - q_{x-1})^2.$$

At the boundaries the system is connected to two Langevin heat baths at temperatures $T_0 := T_-$ and $T_n := T_+$. We also assume that a force (tension) of constant value $\overline{\tau}_+ \in \mathbb{R}$ is acting on the utmost right point $x = n$. Since the system is *unpinned*, the absolute positions q_x do not have precise meaning, and the dynamics depends only on the interparticle *stretch*

$$r_x := q_x - q_{x-1}, \qquad \text{for } x = 1, \dots, n,$$

and by convention throughout the paper we set $r_0 := 0$. The configurations are then described by

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_n, p_0, \dots, p_n) \in \Omega_n := \mathbb{R}^n \times \mathbb{R}^{n+1}. \tag{2.1}$$

The total energy of the chain is defined by the Hamiltonian:

$$\mathcal{H}_n(\mathbf{r}, \mathbf{p}) := \sum_{x \in \mathbb{I}_n} \mathcal{E}_x(\mathbf{r}, \mathbf{p}), \tag{2.2}$$

where the microscopic energy density is given by

$$\mathcal{E}_x(\mathbf{r}, \mathbf{p}) := \frac{p_x^2}{2} + V(r_x) = \frac{p_x^2}{2} + \frac{r_x^2}{2}, \quad x \in \mathbb{I}_n. \tag{2.3}$$

Finally, we assume that for each $x \in \mathbb{I}_n$ the momentum p_x can be flipped, at a random exponential time with intensity γn^2, to $-p_x$, with $\gamma > 0$.

Therefore, the microscopic dynamics of the process $\{(\mathbf{r}(t), \mathbf{p}(t))\}_{t \geqslant 0}$ describing the total chain is given in the bulk by

$$\begin{aligned} \mathrm{d}r_x(t) &= n^2 \left(p_x(t) - p_{x-1}(t)\right) \mathrm{d}t, && x \in \{1, \dots, n\}, \\ \mathrm{d}p_x(t) &= n^2 \left(r_{x+1}(t) - r_x(t)\right) \mathrm{d}t - 2p_x(t^-)\, \mathrm{d}\mathcal{N}_x(\gamma n^2 t), && x \in \mathbb{I}_n^\circ, \end{aligned} \tag{2.4}$$

and at the boundaries by

$$r_0(t) \equiv 0,$$

$$\mathrm{d}p_0(t) = n^2 r_1(t)\,\mathrm{d}t - 2p_0(t^-)\,\mathrm{d}\mathcal{N}_0(\gamma n^2 t) - \widetilde{\gamma} n^2 p_0\,\mathrm{d}t + n\sqrt{2\widetilde{\gamma} T_-}\mathrm{d}w_0(t), \tag{2.5}$$

$$\mathrm{d}p_n(t) = -n^2 r_n(t)\,\mathrm{d}t + n^2 \overline{\tau}_+\,\mathrm{d}t - 2p_n(t^-)\,\mathrm{d}\mathcal{N}_n(\gamma n^2 t) - \widetilde{\gamma} n^2 p_n\,\mathrm{d}t + n\sqrt{2\widetilde{\gamma} T_+}\mathrm{d}w_n(t),$$

where $w_0(t)$ and $w_n(t)$ are independent standard Wiener processes and $\mathcal{N}_x(t)$, $x \in \mathbb{I}_n$ are independent of them i.i.d. Poisson processes of intensity 1. Besides, $\widetilde{\gamma} > 0$ regulates the intensity of the Langevin thermostats. All processes are given over some probability space $(\Sigma, \mathcal{F}, \mathbb{P})$. The factor n^2 appearing in the temporal scaling comes from the fact that t, used in the equations above, is the *macroscopic* time, and the *microscopic* time scale is the diffusive one.

We assume that the initial data is random, distributed according to the probability distribution μ_n over Ω_n. We denote by $\mathbb{P}_n := \mu_n \otimes \mathbb{P}$ (resp. $\mathbb{E}_n$) the product probability distribution over $\Omega_n \times \Sigma$ (resp. its expectation).

Equivalently, the generator of this dynamics is given by

$$L := n^2 \left(A + \gamma S + \widetilde{\gamma} \widetilde{S} \right), \tag{2.6}$$

where

$$A := \sum_{x=1}^{n} (p_x - p_{x-1}) \partial_{r_x} + \sum_{x=1}^{n-1} (r_{x+1} - r_x) \partial_{p_x} + r_1 \partial_{p_0} + \left(\overline{\tau}_+(t) - r_n \right) \partial_{p_n} \tag{2.7}$$

and

$$SF(\mathbf{r}, \mathbf{p}) := \sum_{x=0}^{n} \left(F(\mathbf{r}, \mathbf{p}^x) - F(\mathbf{r}, \mathbf{p}) \right) \tag{2.8}$$

for any C^2–class smooth function F. Here $\mathbf{p}^x$ is the momentum configuration obtained from $\mathbf{p}$ with p_x replaced by $-p_x$. Finally, the generator of the Langevin heat bath at the boundary points equals:

$$\widetilde{S} = \sum_{x=0,n} \left(T_x \partial_{p_x}^2 - p_x \partial_{p_x} \right), \qquad \text{with} \quad T_0 := T_-, \quad T_n := T_+. \tag{2.9}$$

2.2. **Notations.** We collect here notations and conventions that we use throughout the paper.

- Given an integrable function $G : \mathbb{I} \to \mathbb{C}$, its Fourier transform is defined by

$$\mathcal{F}G(\eta) := \int_{\mathbb{I}} G(u) e^{-2i\pi u\eta}\, du, \qquad \eta \in \mathbb{Z}. \tag{2.10}$$

If $G \in L^2(\mathbb{I})$, then the inverse Fourier transform reads as

$$G(u) = \sum_{\eta \in \mathbb{Z}} e^{2i\pi u\eta} \mathcal{F}G(\eta), \qquad u \in \mathbb{I}, \tag{2.11}$$

where the sum converges in the L^2 sense.

- Given a sequence $\{f_x,\, x \in \mathbb{I}_n\}$, its Fourier transform is given by

$$\widehat{f}(k) = \sum_{x \in \mathbb{I}_n} f_x e^{-2i\pi xk}, \qquad k \in \widehat{\mathbb{I}}_n := \left\{0, \tfrac{1}{n+1}, \dots, \tfrac{n}{n+1}\right\}. \tag{2.12}$$

Reciprocally, for any $\widehat{f}:\widehat{\mathbb{I}}_n \to \mathbb{C}$, the inverse Fourier transform reads

$$f_x = \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \widehat{f}(k) e^{2i\pi xk}, \qquad x \in \mathbb{I}_n, \tag{2.13}$$

where we use the following short notation

$$\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} := \frac{1}{n+1} \sum_{k\in\widehat{\mathbb{I}}_n} \tag{2.14}$$

for the averaged summation over frequencies $k \in \widehat{\mathbb{I}}_n$. The Parseval identity can be then expressed as follows

$$\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \widehat{f}(k)\widehat{g}^\star(k) = \sum_{x\in\mathbb{I}_n} f_x g_x^\star, \qquad f, g : \mathbb{I}_n \to \mathbb{C}. \tag{2.15}$$

For a given function f we adopt the convention

$$f^+(k) := f(k) \quad \text{and} \quad f^-(k) := f^\star(-k), \qquad k \in \widehat{\mathbb{I}}_n. \tag{2.16}$$

According to our notation, given a configuration

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_n, p_0, \dots, p_n) \in \Omega_n := \mathbb{R}^n \times \mathbb{R}^{n+1}$$

we let, for any $k \in \widehat{\mathbb{I}}_n$,

$$\widehat{r}(k) := \sum_{x\in\mathbb{I}_n} r_x e^{-2\pi ixk}, \qquad \widehat{p}(k) := \sum_{x\in\mathbb{I}_n} p_x e^{-2\pi ixk},$$

recalling the convention $r_0 := 0$. Since the configuration components are real valued, the corresponding Fourier transforms have the property:

$$\widehat{p}^\star(k) = \widehat{p}(-k), \qquad \widehat{r}^\star(k) = \widehat{r}(-k). \tag{2.17}$$

• For a function $G : \mathbb{I} \to \mathbb{C}$, we define three discrete approximations: of the function itself, of its gradient and Laplacian, respectively by

$$\begin{aligned} G_x &:= G(\tfrac{x}{n}), && x \in \mathbb{I}_n, \\ (\nabla_n G)_x &:= n\big(G(\tfrac{x+1}{n}) - G(\tfrac{x}{n})\big), && x \in \{0, \dots, n-1\}, \\ (\Delta_n G)_x &:= n^2\big(G(\tfrac{x+1}{n}) + G(\tfrac{x-1}{n}) - 2G(\tfrac{x}{n})\big), && x \in \mathbb{I}_n^\circ. \end{aligned} \tag{2.18}$$

• Having two families of functions $f_i, g_i : A \to \mathbb{R}$, $i \in I$, where I, A are some sets we write $f_i \lesssim g_i$, $i \in I$ if there exists $C > 0$ such that

$$f_i(a) \leqslant C g_i(a), \qquad \text{for any } i \in I,\ a \in A.$$

If both $f_i \lesssim g_i, i \in I$ and $g_i \lesssim f_i, i \in I$, then we shall write $f_i \approx g_i, i \in I$.

3. Hydrodynamic limits: statements of the main results

In this section we state our main results, given below in Theorem 3.2, Theorem 3.3 and Theorem 3.5. Before that, let us give our first assumption on the initial probability distribution of the configurations.

Suppose that $T > 0$. Let us denote by $\nu_T(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p})$ the product Gaussian measure on Ω_n of null average and variance $T > 0$ given by

$$\nu_T(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \frac{e^{-\mathcal{E}_0/T}}{\sqrt{2\pi T}}\mathrm{d}p_0 \prod_{x=1}^{n} \frac{e^{-\mathcal{E}_x/T}}{\sqrt{2\pi T}}\mathrm{d}p_x \mathrm{d}r_x. \tag{3.1}$$

Let $\mu_n(t)$ be the probability law on Ω_n of the configurations $(\mathbf{r}(t), \mathbf{p}(t))$ and let $f_n(t, \mathbf{r}, \mathbf{p})$ be the density of the measure $\mu_n(t)$ with respect to ν_T.

We now define the linear interpolation between the inverse boundary temperatures T_-^{-1} and T_+^{-1} by

$$\beta(u) := \left(T_+^{-1} - T_-^{-1}\right) u + T_-^{-1}, \qquad u \in \mathbb{I}. \tag{3.2}$$

Recall the definition of its discrete approximation: $\beta_x := \beta(x/n)$, $x \in \mathbb{I}_n$. Let $\widetilde{\nu}$ be the corresponding inhomogeneous product measure with tension $\overline{\tau}_+$:

$$\widetilde{\nu}(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \frac{e^{-\beta_0 p_0^2/2}}{\sqrt{2\pi\beta_0^{-1}}}\mathrm{d}p_0 \prod_{x=1}^{n} \exp\left\{ - \beta_x \left(\mathcal{E}_x - \overline{\tau}_+ r_x\right) - \mathcal{G}(\beta_x, \overline{\tau}_+)\right\}\mathrm{d}r_x \mathrm{d}p_x, \tag{3.3}$$

where the Gibbs potential is

$$\mathcal{G}(\beta, \tau) := \log \int_{\mathbb{R}^2} e^{-\frac{\beta}{2}(r^2+p^2)+\beta\tau r}\mathrm{d}p\mathrm{d}r = \frac{1}{2}\beta\tau^2 + \frac{1}{2}\log\left(2\pi\beta^{-1}\right), \tag{3.4}$$

for $\beta > 0$, $\tau \in \mathbb{R}$. Consider then the density

$$\widetilde{f}_n(t) := f_n(t)\frac{\mathrm{d}\nu_T}{\mathrm{d}\widetilde{\nu}}. \tag{3.5}$$

and define the relative entropy

$$\widetilde{\mathbf{H}}_n(t) := \int_{\Omega_n} \widetilde{f}_n(t) \log \widetilde{f}_n(t) \mathrm{d}\widetilde{\nu}. \tag{3.6}$$

In the whole paper we assume

$$\widetilde{f}_n(0) \in C^2(\Omega_n) \qquad \text{and} \qquad \widetilde{\mathbf{H}}_n(0) \lesssim n, \qquad n \geqslant 1. \tag{3.7}$$

3.1. Empirical distributions of the averages.

We are interested in the evolution of the *microscopic profiles* of stretch, momentum, and energy, which we now define. For any $n \geqslant 1$, $t \geqslant 0$ and $x \in \mathbb{I}_n$, let

$$\overline{r}_x^{(n)}(t) := \mathbb{E}_n\big[r_x(t)\big], \qquad \overline{p}_x^{(n)}(t) := \mathbb{E}_n\big[p_x(t)\big], \qquad \overline{\mathcal{E}}_x^{(n)}(t) := \mathbb{E}_n\big[\mathcal{E}_x(t)\big]. \tag{3.8}$$

Moreover, we denote by $\widehat{\overline{r}}^{(n)}(t,k)$, $\widehat{\overline{p}}^{(n)}(t,k)$, with $k \in \widehat{\mathbb{I}}_n$, the Fourier transforms of the first two fields defined in (3.8). We shall make the following hypothesis:

Assumption 3.1. *We assume*

(1) an energy bound on the initial data:

$$\sup_{n\geqslant 1} \frac{1}{n+1} \sum_{x\in\mathbb{I}_n} \overline{\mathcal{E}}_x^{(n)}(0) < +\infty\ ; \tag{3.9}$$

(2) a uniform bound satisfied by the spectrum of the initial averages:

$$\sup_{n\geqslant 1} \Big(\sup_{k\in\widehat{\mathbb{I}}_n} |\widehat{\overline{r}}^{(n)}(0,k)| + \sup_{k\in\widehat{\mathbb{I}}_n} |\widehat{\overline{p}}^{(n)}(0,k)| \Big) < +\infty. \tag{3.10}$$

3.2. **Convergence of the average stretch and momentum.** In order to state the convergence results for the profiles, we extend the definition (3.8) to profiles on $\mathbb{I}$, as follows: for any $u\in\mathbb{I}$ and $x\in\mathbb{I}_n$ let

$$\begin{cases} \overline{r}^{(n)}(t,u) &= \mathbb{E}_n\big[r_x(t)\big],\\ \overline{p}^{(n)}(t,u) &= \mathbb{E}_n\big[p_x(t)\big],\\ \overline{\mathcal{E}}^{(n)}(t,u) &= \mathbb{E}_n\big[\mathcal{E}_x(t)\big], \end{cases} \qquad \text{if } u\in\Big[\tfrac{x}{n+1},\tfrac{x+1}{n+1}\Big). \tag{3.11}$$

Let $r(t,u)$ be the solution of the following partial differential equation

$$\partial_t r(t,u) = \frac{1}{2\gamma}\partial^2_{uu} r(t,u), \qquad (t,u)\in\mathbb{R}_+\times\mathbb{I}, \tag{3.12}$$

with the boundary and initial conditions:

$$\begin{aligned} &r(t,0)=0, \qquad r(t,1)=\overline{\tau}_+,\\ &r(0,u)=r_0(u), \end{aligned} \tag{3.13}$$

for any $(t,u)\in\mathbb{R}_+\times\mathbb{I}$. To guarantee the regularity of the solution of the above problem we assume that

$$r_0\in C^2(\mathbb{I}) \qquad \text{and} \qquad r_0(1)=\overline{\tau}_+. \tag{3.14}$$

Let $p_0\in C(\mathbb{I})$ be an initial momentum profile. Our first result can be formulated as follows.

Theorem 3.2 (Convergence of the stretch and momentum profiles)**.** *Assume that the initial distribution of the stretch and momentum weakly converges to $r_0(\cdot), p_0(\cdot)$ introduced above, i.e. for any test function $G\in C^\infty(\mathbb{I})$ we have*

$$\begin{aligned} \lim_{n\to+\infty} \frac{1}{n+1}\sum_{x\in\mathbb{I}_n} \overline{r}_x^{(n)}(0) G_x &= \int_{\mathbb{I}} r_0(u)G(u)\mathrm{d}u,\\ \lim_{n\to+\infty} \frac{1}{n+1}\sum_{x\in\mathbb{I}_n} \overline{p}_x^{(n)}(0) G_x &= \int_{\mathbb{I}} p_0(u)G(u)\mathrm{d}u. \end{aligned} \tag{3.15}$$

Then, under Assumption 3.1, for any $t>0$ the following holds:

$$\lim_{n\to+\infty} \overline{r}^{(n)}(t,\cdot) = r(t,\cdot) \tag{3.16}$$

weakly in $L^2(\mathbb{I})$, where $r(\cdot)$ is the solution of (3.12)–(3.13). In addition, we have

$$\lim_{n\to+\infty} \int_0^t \big\|\overline{p}^{(n)}(s,\cdot)\big\|^2_{L^2(\mathbb{I})}\mathrm{d}s = 0. \tag{3.17}$$

The proof of this theorem is given in Section 5.2.

It is not difficult to prove (see Section 4 below) that, under the same assumptions as in Theorem 3.2, for each $t > 0$ the sequence of the squares of the mean stretches $\{[\overline{r}^{(n)}]^2(\cdot)\}_{n\geqslant 1}$ – the *mechanical energy* density – is sequentially $\star$–weakly compact in $(L^1([0,t];C(\mathbb{I})))^\star$. However, in order to characterize its convergence one needs substantial extra work, and this is why we state it as an additional important result.

Let $C_0^2(\mathbb{I})$ be the class of C^2 functions on $\mathbb{I}$ such that $G(0) = G(1) = 0$.

Theorem 3.3 (Convergence of the mechanical energy profile). *Assume that Assumption 3.1 holds. Then, for any test function $G \in L^1([0,t];C_0^2(\mathbb{I}))$ we have*

$$\lim_{n\to+\infty}\int_0^t \mathrm{d}s\int_{\mathbb{I}}\left(\overline{r}^{(n)}(s,u)\right)^2 G(s,u)\mathrm{d}u = \int_0^t \mathrm{d}s\int_{\mathbb{I}} r^2(s,u)G(s,u)\mathrm{d}u, \tag{3.18}$$

where $r(s,u)$ is the solution of (3.12)–(3.13).

The proof of this theorem is contained in Section 5.3.

3.3. **Convergence of the energy density average.** Our last result concerns the microscopic energy profile. To obtain the convergence of $\overline{\mathcal{E}}_x^{(n)}(t)$ for $t > 0$, we add an assumption on the *fluctuating part* of the initial data distribution. For any $x \in \mathbb{I}_n$, let

$$\begin{aligned}\widetilde{r}_x^{(n)}(t) &:= r_x(t) - \overline{r}_x^{(n)}(t),\\ \widetilde{p}_x^{(n)}(t) &:= p_x(t) - \overline{p}_x^{(n)}(t).\end{aligned} \tag{3.19}$$

Similarly as before, let $\widehat{\widetilde{r}}^{(n)}(t,k)$, $\widehat{\widetilde{p}}^{(n)}(t,k)$ be the Fourier transforms of the fields defined in (3.19). We shall assume the following hypothesis on the covariance of the stretch and momentum fluctuations.

Assumption 3.4. *The following correlations sums are finite:*

$$\begin{aligned}&\sup_{n\geqslant 1}\frac{1}{n+1}\sum_{x,x'\in\mathbb{I}_n}\left(\mathbb{E}_n\left[\widetilde{p}_x^{(n)}(0)\widetilde{p}_{x'}^{(n)}(0)\right]\right)^2 < +\infty\\ &\sup_{n\geqslant 1}\frac{1}{n+1}\sum_{x,x'\in\mathbb{I}_n}\left(\mathbb{E}_n\left[\widetilde{r}_x^{(n)}(0)\widetilde{r}_{x'}^{(n)}(0)\right]\right)^2 < +\infty\\ &\sup_{n\geqslant 1}\frac{1}{n+1}\sum_{x,x'\in\mathbb{I}_n}\left(\mathbb{E}_n\left[\widetilde{p}_x^{(n)}(0)\widetilde{r}_{x'}^{(n)}(0)\right]\right)^2 < +\infty.\end{aligned} \tag{3.20}$$

Let $e(t,u)$ be the solution of the initial-boundary value problem for the inhomogeneus heat equation

$$\partial_t e(t,u) = \frac{1}{4\gamma}\partial_{uu}^2\left\{e(t,u) + \frac{1}{2}r^2(t,u)\right\}, \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{I}, \tag{3.21}$$

with the boundary and initial conditions

$$\begin{aligned}&e(t,0) = T_-, \qquad e(t,1) = T_+ + \frac{1}{2}\overline{\tau}_+^2,\\ &e(0,u) = e_0(u),\end{aligned} \tag{3.22}$$

for any $(t,u) \in \mathbb{R}_+ \times \mathbb{I}$. Here $r(t,u)$ is the solution of (3.12)–(3.13), and e_0 is non-negative. Our principal result concerning the convergence of the energy functional is contained in the following:

Theorem 3.5 (Convergence of the total energy profile)**.** *Similarly to* (3.15)*, assume that the initial distribution of the energy converges weakly to some* $e_0 \in C(\mathbb{I})$*,* i.e. *for any* $G \in C^\infty(\mathbb{I})$ *we have:*

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{x\in\mathbb{I}_n} \overline{\mathcal{E}}_x^{(n)}(0) G_x = \int_{\mathbb{I}} e_0(u) G(u) \mathrm{d}u. \tag{3.23}$$

Then, under Assumptions 3.1 and 3.4, for any $t > 0$ *and any test function* $G \in L^1([0,t]; C_0^2(\mathbb{I}))$ *we have*

$$\lim_{n\to+\infty} \int_0^t \mathrm{d}s \int_{\mathbb{I}} \overline{\mathcal{E}}^{(n)}(s,u) G(s,u) \mathrm{d}u = \int_0^t \mathrm{d}s \int_{\mathbb{I}} e(s,u) G(s,u) \mathrm{d}u, \tag{3.24}$$

where $e(\cdot)$ *is the solution of* (3.21)–(3.22)*.*

The proof of this theorem is presented in Section 5.4.

4. Sketches of proof and Equipartition of energy

In this section we present some essential intermediate results which will be used to prove the convergence theorems, and which are consequences of the various assumptions made. We have decided to expose them in an independent section in order to emphasize the main steps of the proofs, and to highlight the role of our hypotheses.

4.1. **Consequences of Assumption 3.1.**

4.1.1. *The boundary terms.* An important feature of our model is the presence of $\overline{\tau}_+ \neq 0$. A significant part of the work consists in estimating boundary terms. Let us first state in this section the crucial bounds that we are able to get, under Assumption 3.1, and which concern the extremity points $x = 0$ and $x = n$. One of the most important result is the following:

Proposition 4.1. *Under Assumption 3.1, for any* $t > 0$ *we have*

$$\int_0^t \left|\overline{p}_0(s) - \overline{p}_n(s)\right|^2 \mathrm{d}s \lesssim \frac{1}{n^2}, \qquad n \geqslant 1, \tag{4.1}$$

and

$$\int_0^t \left|\overline{p}_0(s) + \overline{p}_n(s)\right|^2 \mathrm{d}s \lesssim \frac{\log(n+1)}{n^2}, \qquad n \geqslant 1. \tag{4.2}$$

This result is proved in Section 6.1. Another consequence of Assumption 3.1 is the following one-point estimate, which uses the previous result (4.1), but allows us to get a sharper bound:

Proposition 4.2. *Under Assumption 3.1, for any* $t \geqslant 0$ *we have*

$$\left|\int_0^t \overline{p}_0^{(n)}(s) \mathrm{d}s\right| \lesssim \frac{1}{n} \qquad \textit{and} \qquad \left|\int_0^t \overline{p}_n^{(n)}(s) \mathrm{d}s\right| \lesssim \frac{1}{n}, \qquad n \geqslant 1. \tag{4.3}$$

This proposition is proved in Section 6.2.

Remark 4.3. *In fact, in the whole paper, only the second estimate in* (4.3) *will be used. However, in its proof, the first estimate comes freely.*

4.1.2. *Estimates in the bulk.* Provided with a good control on the boundaries, one can then obtain several estimates in the bulk of the chain. Two of them are used several times in the argument, and can be proved independently of each other. The first one is

Proposition 4.4 (L^2 bound on average momenta and stretches)**.** *Under Assumption 3.1, for any $t > 0$*

$$\frac{1}{n+1}\sup_{s\in[0,t]}\sum_{x\in\mathbb{I}_n}\left\{\left(\overline{r}_x^{(n)}(s)\right)^2+\left(\overline{p}_x^{(n)}(s)\right)^2\right\}\lesssim 1,\qquad n\geqslant 1. \tag{4.4}$$

In addition, for any $t > 0$ we have

$$n\sum_{x\in\mathbb{I}_n}\int_0^t\left(\overline{p}_x^{(n)}(s)\right)^2\mathrm{d}s\lesssim 1,\qquad n\geqslant 1. \tag{4.5}$$

The proof of Proposition 4.4 can be found in Section 6.3 below, and makes use of Proposition 4.2. Here we formulate some of its immediate consequences:

- thanks to (4.4) we conclude that that for each $t > 0$ the sequence of the averages $\{\overline{r}^{(n)}(t)\}_{n\geqslant 1}$ is bounded in $L^2(\mathbb{I})$, thus it is weakly compact. Therefore, to prove Theorem 3.2 one needs to identify the limit in (3.16), which is carried out in Section 5.2,
- the second equality (3.17) of Theorem 3.2 simply follows from (4.5).
- finally, the estimate (4.4) implies in particular that

$$\sup_{n\geqslant 1}\sup_{s\in[0,t]}\left\|[\overline{r}^{(n)}]^2(s,\cdot)\right\|_{L^1(\mathbb{I})}\lesssim 1. \tag{4.6}$$

  Therefore, we conclude that, for each $t > 0$ the sequence $\{[\overline{r}^{(n)}]^2(\cdot)\}_{n\geqslant 1}$ is sequentially $\star$−weakly compact in $(L^1([0,t];C(\mathbb{I})))^\star$, as claimed. This is the first step to prove Theorem 3.3.

The second important estimate focuses on the microscopic energy averages and is formulated as follows:

Proposition 4.5 (Energy bound)**.** *Under Assumption 3.1, for any $t \geqslant 0$ we have*

$$\sup_{s\in[0,t],\,n\geqslant 1}\left\{\frac{1}{n+1}\sum_{x\in\mathbb{I}_n}\overline{\mathcal{E}}_x^{(n)}(s)\right\}<+\infty. \tag{4.7}$$

This estimate is proved in Section 7.1, using a bound on the *entropy production*, given in Proposition 7.3 below. Thanks to Proposition 4.5 the sequence $\{\overline{\mathcal{E}}^{(n)}(\cdot)\}_{n\geqslant 1}$ is sequentially $\star$−weakly compact in $(L^1([0,t];C_0^2(\mathbb{I})))^\star$ for each $t > 0$. Therefore, to prove Theorem 3.5, one needs to identify the limit. This identification requires the extra Assumption 3.4.

4.2. **Consequence of Assumption 3.4.** The proof of Theorem 3.5 is based on a *mechanical and thermal energy equipartition* result given as follows:

Proposition 4.6 (Equipartition of energy)**.** *Under Assumptions 3.1 and 3.4, for any complex valued test function* $G \in C_0^\infty([0,+\infty) \times \mathbb{I} \times \mathbb{T})$ *we have*

$$\lim_{n\to+\infty} \int_0^t \frac{1}{n} \sum_{x\in\mathbb{I}_n} G_x(s)\mathbb{E}_n\Big[\big(\widetilde{r}_x^{(n)}(s)\big)^2 - \big(\widetilde{p}_x^{(n)}(s)\big)^2\Big]\mathrm{d}s = 0. \tag{4.8}$$

The proof of this result is presented in Section 8 (cf. conclusion in Section 8.3), and uses some of the results above, namely Proposition 4.1 and Proposition 4.5.

5. Proofs of the hydrodynamic limit theorems

In the present section we show Theorems 3.2, 3.3 and 3.5 announced in Section 3. The proof of the latter is contingent on several intermediate results:

- first of all, to prove the three results we need specific boundary estimates which will be all stated in Section 5.1 (see Lemma 5.1), and which are byproducts of Proposition 4.2, Proposition 4.4 and Proposition 4.5 ;
- the proof of Theorem 3.3 requires moreover Lemma 5.2, which is based on a detailed analysis of the average dynamics $(\overline{r}_x^{(n)}, \overline{p}_x^{(n)})_{x\in\mathbb{I}_n}$ (that will be carried out in Section 6) ;
- finally, to show Theorem 3.5 we need: first, a uniform L^2 bound on the averages of momentum, see Lemma 5.4 below. The latter will be proved in Section 6.4, as a consequence of Proposition 4.1 ; second, the equipartition result for the fluctuation of the potential and kinetic energy of the chain, which has already been stated in Section 4, see Proposition 4.6.

5.1. **Treatment of boundary terms.** First of all, the conservation of the energy gives the following microscopic identity:

$$n^{-2}L\mathcal{E}_x(t) = j_{x-1,x}(t) - j_{x,x+1}(t), \qquad x \in \mathbb{I}_n^o, \tag{5.1}$$

where

$$j_{x,x+1}(t) := j_{x,x+1}(\mathbf{r}(t),\mathbf{p}(t)), \qquad \text{with } j_{x,x+1}(\mathbf{r},\mathbf{p}) := -p_x r_{x+1}, \tag{5.2}$$

are the *microscopic currents*. At the boundaries we have

$$n^{-2}L\mathcal{E}_0(t) = -j_{0,1}(t) + \widetilde{\gamma}\left(T_- - p_0^2(t)\right) \tag{5.3}$$

$$n^{-2}L\mathcal{E}_n(t) = j_{n-1,n}(t) + \overline{\tau}_+ p_n(t) + \widetilde{\gamma}\left(T_+ - p_n^2(t)\right). \tag{5.4}$$

One can see that boundaries play an important role. Before proving the hydrodynamic limit results, one needs to understand very precisely how boundary variables behave. This is why we start with collecting here all the estimates that are essential in the following argument. Their proofs require quite some work, and for the sake of clarity this will be postponed to Section 7.3.

Lemma 5.1 (Boundary estimates)**.** *The following holds: for any $t \geqslant 0$*

(i) (Momentum correlations)

$$\lim_{n\to\infty} \mathbb{E}_n\left[\int_0^t p_0(s)p_1(s)\mathrm{d}s\right] = 0, \qquad \lim_{n\to\infty} \mathbb{E}_n\left[\int_0^t p_{n-1}(s)p_n(s)\mathrm{d}s\right] = 0. \tag{5.5}$$

(ii) (Boundary correlations)

$$\left|\mathbb{E}_n\left[\int_0^t p_0(s)r_1(s)\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad \left|\mathbb{E}_n\left[\int_0^t p_n(s)r_n(s)\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad n \geqslant 1. \tag{5.6}$$

(iii) (Boundary stretches)

$$\left|\mathbb{E}_n\left[\int_0^t r_1(s)\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad \left|\mathbb{E}_n\left[\int_0^t \big(r_n(s) - \overline{\tau}_+\big)\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad n \geqslant 1. \tag{5.7}$$

(iv) (Boundary temperatures, part I)

$$\left|\mathbb{E}_n\left[\int_0^t \big(T_- - p_0^2(s)\big)\,\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad \left|\mathbb{E}_n\left[\int_0^t \big(T_+ - p_n^2(s)\big)\,\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad n \geqslant 1. \tag{5.8}$$

(v) (Mechanical energy at the boundaries, part I)

$$\mathbb{E}_n\left[\int_0^t \big(r_1^2(s) + r_n^2(s)\big)\,\mathrm{d}s\right] \lesssim 1, \qquad n \geqslant 1. \tag{5.9}$$

(vi) (Boundary currents)

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^t j_{0,1}(s)\mathrm{d}s\right] = 0, \qquad \lim_{n\to\infty} \mathbb{E}\left[\int_0^t j_{n-1,n}(s)\mathrm{d}s\right] = 0. \tag{5.10}$$

(vii) (Mechanical energy at the boundaries, part II) : *at the left boundary point*

$$\left|\mathbb{E}_n\left[\int_0^t \Big(r_1^2(s) - T_-\Big)\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad n \geqslant 1 \tag{5.11}$$

and at the right boundary point

$$\left|\mathbb{E}_n\left[\int_0^t \Big(r_n^2(s) - \overline{\tau}_+^2 - T_+\Big)\mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \qquad n \geqslant 1. \tag{5.12}$$

(viii) (Boundary temperatures, part II)

$$\sum_{x=0,n} T_x \mathbb{E}_n\left[\int_0^t \big(T_x - p_x^2(s)\big)\,\mathrm{d}s\right] \lesssim \frac{1}{n}, \qquad n \geqslant 1. \tag{5.13}$$

Provided with all the previous results which have been stated (but not proved yet), we are ready to prove Theorem 3.2 and 3.5. Before that, in order to make the presentation unequivocal, let us draw in Figure 1 a diagram with the previous statements, and the sections where they will be proved into parentheses.

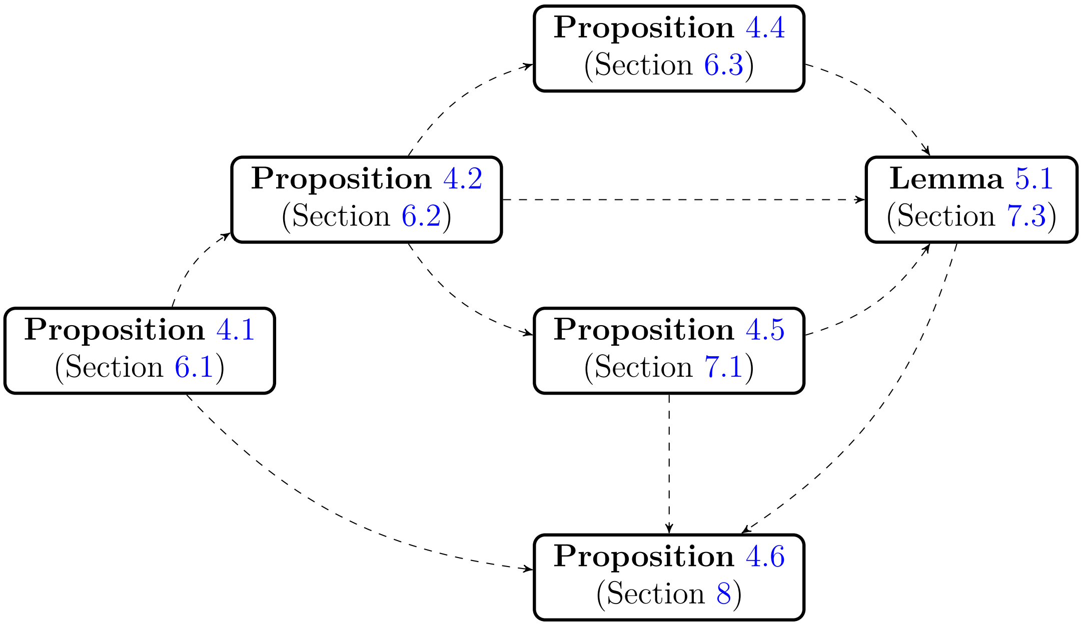

FIGURE 1. An arrow from A to B means that A is used to prove B, but is not necessarily a direct implication.

5.2. **Proof of Theorem 3.2.** Let us start with the diffusive equation (3.12), which can be formulated in a weak form as:

$$\int_0^1 G(u)\big(r(t,u)-r(0,u)\big)\mathrm{d}u = \frac{1}{2\gamma}\int_0^t \mathrm{d}s\int_0^1 G''(u)r(s,u)\mathrm{d}u - \frac{1}{2\gamma}G'(1)\overline{\tau}_+ t, \qquad t\geqslant 0, \quad (5.14)$$

for any test function $G\in C_0^2(\mathbb{I})$. Existence and uniqueness of such weak solutions in an appropriate space of integrable functions are standard.

By the microscopic evolution equations, see (2.4)–(2.5), we have (cf. (2.18))

$$\mathbb{E}_n\left[\frac{1}{n+1}\sum_{x\in\mathbb{I}_n} G_x\left(r_x(t)-r_x(0)\right)\right] = \frac{n^2}{n+1}\mathbb{E}_n\left[\int_0^t \mathrm{d}s\sum_{x=1}^n G_x\left(p_x(s)-p_{x-1}(s)\right)\right]$$
$$= \mathbb{E}_n\left[\int_0^t \mathrm{d}s\left\{-\sum_{x=1}^{n-1}(\nabla_n G)_x\, p_x(s) - (n+1)G_1 p_0(s)\right\}\right] + o_n(1). \quad (5.15)$$

As usual, the symbol $o_n(1)$ denotes an expression that vanishes with $n\to+\infty$. The dynamics of the averages $(\overline{\mathbf{r}}(t),\overline{\mathbf{p}}(t))$ is easy to deduce from the evolution equations (see also (6.2) where it is detailed). We can therefore rewrite the right hand side of (5.15) as

$$\mathbb{E}_n\left[-\int_0^t \mathrm{d}s\left\{\sum_{x=1}^{n-1}\frac{1}{2\gamma}(\nabla_n G)_x\left(r_{x+1}(s)-r_x(s)\right)+\frac{1}{2\gamma+\widetilde{\gamma}}(\nabla_n G)_0 r_1(s)\right\}\right]$$
$$+\mathbb{E}_n\left[\frac{1}{2\gamma n^2}\sum_{x=1}^{n-1}(\nabla_n G)_x\left(p_x(t)-p_x(0)\right)+\frac{1}{(2\gamma+\widetilde{\gamma})n^2}(\nabla_n G)_0\left(p_0(t)-p_0(0)\right)\right]+o_n(1). \tag{5.16}$$

Since G is smooth we have $\lim_{n\to+\infty}\sup_{x\in\mathbb{I}_n}\left|(\nabla_n G)_x - G'(x)\right| = 0$. Using this and Proposition 4.4 one can show that the second expression in (5.16) converges to 0, leaving as the only possible significant the first term. Summing by parts and recalling that $G(0)=0$, it can be rewritten as

$$\mathbb{E}_n\left[\int_0^t \frac{1}{2\gamma}\left\{\frac{1}{n+1}\sum_{x=2}^{n-1}(\Delta_n G)_x\, r_x(s)-(\nabla_n G)_{n-1}\, r_n(s)\right\}\mathrm{d}s\right]$$
$$-\left(\frac{1}{2\gamma+\widetilde{\gamma}}(\nabla_n G)_0-\frac{1}{2\gamma}(\nabla_n G)_1\right)\mathbb{E}_n\left[\int_0^t r_1(s)\mathrm{d}s\right]. \tag{5.17}$$

Therefore, we need to understand the macroscopic behavior of the boundary strech variables, which is done thanks to Lemma 5.1: from (5.7) we conclude that the second term vanishes, as $n\to+\infty$. Using again (5.7) but for the right boundary we infer that (5.17) can be written as

$$\int_0^t \mathrm{d}s\left\{\frac{1}{2\gamma}\int_0^1 G''(u)\overline{r}^{(n)}(s,u)\mathrm{d}u - G'(1)\overline{\tau}_+\right\}+o_n(t), \tag{5.18}$$

where $\lim_{n\to+\infty}\sup_{s\in[0,t]} o_n(s)=0$. Thanks to Proposition 4.4 we know that for a given $t_*>0$

$$\sup_{n\geqslant 1}\sup_{s\in[0,t_*]}\left\|\overline{r}^{(n)}(s,\cdot)\right\|_{L^2(\mathbb{I})}\lesssim 1. \tag{5.19}$$

The above means, in particular that the sequence $\{\overline{r}^{(n)}(\cdot)\}_{n\geqslant 1}$ is bounded in the space $L^\infty([0,t_*];L^2(\mathbb{I}))$. As this space is dual to the separable Banach space $L^1([0,t_*];L^2(\mathbb{I}))$, the sequence $\{\overline{r}^{(n)}(\cdot)\}_{n\geqslant 1}$ is $\star$-weakly sequentially compact. Suppose that $r\in L^\infty([0,t_*];L^2(\mathbb{I}))$ is its $\star$-weakly limiting point. Any limiting point of the sequence satisfies (5.14), which shows that has to be unique and as a result $\{\overline{r}^{(n)}(\cdot)\}_{n\geqslant 1}$ is $\star$-weakly convergent to $r\in L^\infty([0,t_*];L^2(\mathbb{I}))$, the solution to (3.12)–(3.13).

5.3. **Proof of Theorem 3.3.** The following estimate shall be crucial in our subsequent argument.

Lemma 5.2. *For any $t>0$ we have*

$$n\sum_{x=0}^{n-1}\int_0^t\left(\overline{r}_{x+1}^{(n)}(s)-\overline{r}_x^{(n)}(s)\right)^2\mathrm{d}s\lesssim 1,\qquad n\geqslant 1. \tag{5.20}$$

The proof of the lemma uses Proposition 4.2, and is postponed to Section 6.5.

Define $\overline{r}^{(n)}_{\rm int} : [0,+\infty)\times\mathbb{I}\to\mathbb{R}$ as the function obtained by the piecewise linear interpolation between the nodal points $(x/(n+1),\overline{r}_x)$, $x=0,\dots,n+1$. Here we let $\overline{r}_{n+1} := \overline{r}_n$. As a consequence of Lemma 5.2 above we obtain

Lemma 5.3. *For any $t\geqslant 0$ we have*

$$\sup_{n\geqslant 1}\int_0^t \left\|\overline{r}^{(n)}_{\rm int}(s,\cdot)\right\|^2_{H^1(\mathbb{I})}{\rm d}s = \mathfrak{h}(t) < +\infty, \tag{5.21}$$

where $H^1(\mathbb{I})$ is the H^1 Sobolev norm: $\|F\|^2_{H^1(\mathbb{I})} := \|F\|^2_{L^2(\mathbb{I})} + \|F'\|^2_{L^2(\mathbb{I})}$. Moreover,

$$\lim_{n\to+\infty}\sup_{u\in\mathbb{I}}\left|\int_0^t \overline{r}^{(n)}_{\rm int}(s,u){\rm d}s - \int_0^t r(s,u){\rm d}s\right| = 0. \tag{5.22}$$

Proof. It is easy to see that

$$\left\|\overline{r}^{(n)}_{\rm int}(t,\cdot)-\overline{r}^{(n)}(t,\cdot)\right\|^2_{L^2(\mathbb{I})} = \frac{1}{3(n+1)}\sum_{x=0}^{n-1}\Big(\overline{r}_{x+1}(t)-\overline{r}_x(t)\Big)^2, \qquad n\geqslant 1. \tag{5.23}$$

Thanks to (5.20) we obtain (5.21). Using (5.23) we also get

$$\lim_{n\to+\infty}\int_0^t \left\|\overline{r}^{(n)}_{\rm int}(s,\cdot)-\overline{r}^{(n)}(s,\cdot)\right\|^2_{L^2(\mathbb{I})}\,{\rm d}s = 0, \qquad t>0. \tag{5.24}$$

From the proof of Theorem 3.2 given in Section 5.2 we know that the sequence $\int_0^t \overline{r}^{(n)}_{\rm int}(s,u){\rm d}s$ weakly converges in $L^2(\mathbb{I})$ to $\int_0^t r(s,u){\rm d}s$. From (5.21) and the compactness of Sobolev embedding into $C(\mathbb{I})$ in dimension 1 we conclude (5.22). □

Thanks to (5.21) we know that for any $t_*>0$ we have

$$\sup_{s\in[0,t_*]}\left\|[\overline{r}^{(n)}_{\rm int}]^2(s,\cdot)\right\|_{L^1(\mathbb{I})}\lesssim 1, \qquad n\geqslant 1. \tag{5.25}$$

The above implies that the sequence $\{[\overline{r}^{(n)}_{\rm int}]^2(\cdot)\}_{n\geqslant 1}$ is sequentially $\star-$weakly compact in $(L^1([0,t_*];C(\mathbb{I})))^\star$. One can choose a subsequence, that for convenience sake we denote by the same symbol, which is $\star-$weakly convergent in any $(L^1([0,t_*];C(\mathbb{I})))^\star$, $t_*>0$. We prove now that for any $G\in L^1([0,t_*];C_0^2(\mathbb{I}))$ we have

$$\lim_{n\to+\infty}\int_0^{t_*}{\rm d}t\int_{\mathbb{I}}\Big(\overline{r}^{(n)}_{\rm int}(t,u)\Big)^2 G(t,u){\rm d}u = \int_0^{t_*}{\rm d}t\int_{\mathbb{I}} r^2(t,u)G(t,u){\rm d}u, \tag{5.26}$$

where $r(\cdot)$ is the solution of (3.12)–(3.13). By a density argument is suffices only to consider functions of the form $G(t,u)=\mathbf{1}_{[0,t_*)}(t)G(u)$, where $G\in C_0^2(\mathbb{I})$, $t_*>0$. To prove (5.26) it suffices therefore to show that

$$\lim_{n\to+\infty}\frac{1}{n+1}\int_0^{t_*}{\rm d}t\sum_{x\in\mathbb{I}_n}\left\{[\overline{r}^{(n)}_x(t)]^2 - r^2\left(t,\tfrac{x}{n+1}\right)\right\}G_x = 0. \tag{5.27}$$

Let $M \geqslant 1$ be an integer, that shall be specified later on, and $t_\ell := \ell t_*/M$, for $\ell = 0, \dots, M$. The expression under the limit in (5.27) can be rewritten as $B_n^1(M) + B_n^2(M) + o_n(1)$, where $o_n(1) \to 0$, as $n \to +\infty$, and

$$
\begin{aligned}
B_n^1(M) &:= \frac{1}{n+1} \sum_{\ell=0}^{M-1} \sum_{x \in \mathbb{I}_n} G_x \Big\{ \overline{r}_x^{(n)}(t_\ell) \int_{t_\ell}^{t_{\ell+1}} \overline{r}_x^{(n)}(t) \mathrm{d}t - r\left(t_\ell, \tfrac{x}{n+1}\right) \int_{t_\ell}^{t_{\ell+1}} r\left(t, \tfrac{x}{n+1}\right) \mathrm{d}t \Big\} \\
B_n^2(M) &:= \frac{1}{n+1} \sum_{\ell=0}^{M-1} \sum_{x \in \mathbb{I}_n} G_x \int_{t_\ell}^{t_{\ell+1}} \overline{r}_x^{(n)}(t) \Big\{ \int_{t_\ell}^{t} \frac{\mathrm{d}}{\mathrm{d}s} \overline{r}_x^{(n)}(s) \mathrm{d}s \Big\} \mathrm{d}t \\
&= \frac{n^2}{n+1} \sum_{\ell=0}^{M-1} \sum_{x \in \mathbb{I}_n} G_x \int_{t_\ell}^{t_{\ell+1}} \overline{r}_x^{(n)}(t) \int_{t_\ell}^{t} \Big(\overline{p}_x^{(n)}(s) - \overline{p}_{x-1}^{(n)}(s) \Big) \mathrm{d}s \mathrm{d}t. \qquad (5.28)
\end{aligned}
$$

The last equality follows from (2.4)–(2.5). In what follows we prove that

$$
\lim_{M \to +\infty} \limsup_{n \to +\infty} |B_n^j(M)| = 0, \qquad j = 1, 2. \qquad (5.29)
$$

Summing by parts in the utmost right hand side of (5.28) we obtain $B_n^2(M) = \sum_{j=1}^3 B_{n,j}^2(M)$, where

$$
\begin{aligned}
B_{n,1}^2 &:= \frac{n^2}{n+1} \sum_{\ell=0}^{M-1} \sum_{x=1}^{n-1} \int_{t_\ell}^{t_{\ell+1}} \left(G_x \overline{r}_x^{(n)}(t) - G_{x+1} \overline{r}_{x+1}^{(n)}(t) \right) \int_{t_\ell}^{t} \overline{p}_x^{(n)}(s) \mathrm{d}s \mathrm{d}t \\
B_{n,2}^2 &:= \frac{n^2}{n+1} \sum_{\ell=0}^{M-1} \int_{t_\ell}^{t_{\ell+1}} G_n\, \overline{r}_n^{(n)}(t) \left\{ \int_{t_\ell}^{t} \overline{p}_n^{(n)}(s) \mathrm{d}s \right\} \mathrm{d}t \\
B_{n,3}^2 &:= -\frac{n^2}{n+1} \sum_{\ell=0}^{M-1} \int_{t_\ell}^{t_{\ell+1}} G_1\, \overline{r}_1^{(n)}(t) \left\{ \int_{t_\ell}^{t} \overline{p}_0^{(n)}(s) \mathrm{d}s \right\} \mathrm{d}t.
\end{aligned}
$$

We have $B_{n,1}^2 = B_{n,1,1}^2 + B_{n,1,2}^2$, where

$$
\begin{aligned}
B_{n,1,1}^2 &:= -\frac{n}{n+1} \sum_{\ell=0}^{M-1} \sum_{x=1}^{n-1} \int_{t_\ell}^{t_{\ell+1}} \overline{r}_x^{(n)}(t) (\nabla_n G)_x \left\{ \int_{t_\ell}^{t} \overline{p}_x^{(n)}(s) \mathrm{d}s \right\} \mathrm{d}t, \\
B_{n,1,2}^2 &:= \frac{n^2}{n+1} \sum_{\ell=0}^{M-1} \sum_{x=1}^{n-1} \int_{t_\ell}^{t_{\ell+1}} G_{x+1} \big(\overline{r}_x^{(n)}(t) - \overline{r}_{x+1}^{(n)}(t) \big) \left\{ \int_{t_\ell}^{t} \overline{p}_x^{(n)}(s) \mathrm{d}s \right\} \mathrm{d}t.
\end{aligned}
$$

By the Cauchy-Schwarz inequality we can bound $|B_{n,1,2}^2|$ from above by

$$
\begin{aligned}
&\frac{n^2 \|G\|_\infty}{n+1} \left\{ \sum_{x=1}^{n-1} \int_0^{t_*} \left(\overline{r}_{x+1}^{(n)}(t) - \overline{r}_x^{(n)}(t) \right)^2 \mathrm{d}t \right\}^{1/2} \left\{ \sum_{\ell=0}^{M-1} \sum_{x=1}^{n-1} \int_{t_\ell}^{t_{\ell+1}} \left\{ \int_{t_\ell}^{t} \overline{p}_x^{(n)}(s) \mathrm{d}s \right\}^2 \mathrm{d}t \right\}^{1/2} \\
&\lesssim \left\{ (n+1) \sum_{x=1}^{n-1} \int_0^{t_*} \left(\overline{r}_{x+1}^{(n)}(t) - \overline{r}_x^{(n)}(t) \right)^2 \mathrm{d}t \right\}^{1/2} \left\{ n \sum_{\ell=0}^{M-1} \sum_{x=1}^{n-1} \int_{t_\ell}^{t_{\ell+1}} \left\{ \int_{t_\ell}^{t} \overline{p}_x^{(n)}(s) \mathrm{d}s \right\}^2 \mathrm{d}t \right\}^{1/2} \\
&\lesssim \left\{ n \sum_{\ell=0}^{M-1} \sum_{x=1}^{n-1} \int_{t_\ell}^{t_{\ell+1}} (t - t_\ell) \int_{t_\ell}^{t} \left(\overline{p}_x^{(n)}(s) \right)^2 \mathrm{d}s \mathrm{d}t \right\}^{1/2} \lesssim \left\{ \frac{n}{M^2} \sum_{x=1}^{n-1} \int_0^{t} \left(\overline{p}_x^{(n)}(s) \right)^2 \mathrm{d}s \right\}^{1/2}
\end{aligned}
$$

by virtue of Lemma 5.2. Using Proposition 4.4 (estimate (4.5)), we conclude $|B^2_{n,1,2}| \lesssim 1/M$ and $\lim_{M\to+\infty}\limsup_{n\to+\infty}|B^2_{n,1,2}| = 0$. The argument for $|B^2_{n,1,1}|$ is analogous. As a result we conclude $\lim_{M\to+\infty}\limsup_{n\to+\infty}|B^2_{n,1}| = 0$.

Concerning $B^2_{n,2}$ we can write

$$B^2_{n,2} = -\frac{n}{n+1}\sum_{\ell=0}^{M-1}\int_{t_\ell}^{t_{\ell+1}}(\nabla_n G)_n\,\overline{r}^{(n)}_n(t)\left\{\int_{t_\ell}^{t}\overline{p}^{(n)}_n(s)\mathrm{d}s\right\}\mathrm{d}t.$$

Therefore, we have

$$\begin{aligned}|B^2_{n,2}| &\lesssim \sum_{\ell=0}^{M-1}\int_{t_\ell}^{t_{\ell+1}}\left|\overline{r}^{(n)}_n(t)\right|\left\{\int_{t_\ell}^{t}\left|\overline{p}^{(n)}_n(s)\right|\mathrm{d}s\right\}\mathrm{d}t\\ &\lesssim\left\{\int_0^{t_*}|\overline{r}^{(n)}_n(t)|\mathrm{d}t\right\}\left\{t^\star\int_0^{t_*}\left(\overline{p}^{(n)}_n(s)\right)^2\mathrm{d}s\right\}^{1/2}\lesssim\frac{1}{\sqrt{n}},\end{aligned}$$

by virtue of Lemma 5.1–(5.9) and Proposition 4.4 (estimate (4.5)). We conclude therefore that $\lim_{M\to+\infty}\limsup_{n\to+\infty}|B^2_{n,2}| = 0$. An analogous argument shows that also $\lim_{M\to+\infty}\limsup_{n\to+\infty}|B^2_{n,3}| = 0$. Thus, (5.29) holds for $j = 2$.

We have $B^1_n(M) = B^1_{n,1}(M) + B^1_{n,2}(M)$, where

$$\begin{aligned}B^1_{n,1}(M) &:= \frac{1}{n+1}\sum_{\ell=0}^{M-1}\sum_{x\in\mathbb{I}_n}G_x\overline{r}^{(n)}_x(t_\ell)\int_{t_\ell}^{t_{\ell+1}}\left[r\left(t_\ell,\tfrac{x}{n+1}\right)-r\left(t,\tfrac{x}{n+1}\right)\right]\mathrm{d}t,\\ B^1_{n,2}(M) &:= \frac{1}{n+1}\sum_{\ell=0}^{M-1}\sum_{x\in\mathbb{I}_n}G_x\overline{r}^{(n)}_x(t_\ell)\int_{t_\ell}^{t_{\ell+1}}\left[\overline{r}^{(n)}_x(t)-r\left(t_\ell,\tfrac{x}{n+1}\right)\right]\mathrm{d}t.\end{aligned}$$

where r is the solution of (3.12)–(3.13). By the regularity of the $r(t,u)$, Lemma 5.3 and estimate (5.25) we can easily conclude that $\lim_{M\to+\infty}\limsup_{n\to+\infty}|B^1_{n,j}| = 0$, $j = 1,2$. Thus, (5.29) holds also for $j = 1$, which ends the proof of Theorem 3.3.

5.4. **Proof of Theorem 3.5.** Concerning equation (3.21)–(3.22), its weak formulation is as follows: for any test function $G \in L^1([0,+\infty); C^2_0(\mathbb{I}))$ which is compactly supported, we have

$$\begin{aligned}0 = &\int_{\mathbb{I}}G(0,u)e_0(u)\mathrm{d}u + \int_0^{+\infty}\int_{\mathbb{I}}\left(\partial_s G(s,u)+\frac{1}{4\gamma}\partial_u^2 G(s,u)\right)e(s,u)\mathrm{d}s\mathrm{d}u\\ &+\frac{1}{8\gamma}\int_0^{+\infty}\int_{\mathbb{I}}\partial_u^2 G(s,u)r^2(s,u)\mathrm{d}s\mathrm{d}u\\ &-\frac{1}{4\gamma}\int_0^{+\infty}\left(\partial_u G(s,1)\left(T_+ + \overline{\tau}_+^2\right) - T_-\,\partial_u G(s,0)\right)\mathrm{d}s. \qquad (5.30)\end{aligned}$$

Given a non-negative initial data $e_0 \in L^1(\mathbb{I})$ and the macroscopic stretch $r(\cdot,\cdot)$ (determined via (5.14)) one can easily show that the respective weak formulation of the boundary value problem for a linear heat equation, resulting from (5.30), admits a unique measure valued solution.

Recall that the averaged energy density function $\overline{\mathcal{E}}^{(n)}(t,u)$ has been defined in (3.11). It is easy to see, thanks to Proposition 4.5, that for any $t_* > 0$ we have

$$\mathbf{E}(t_*) := \sup_{n\geqslant 1}\sup_{t\in[0,t_*]}\left\|\overline{\mathcal{E}}^{(n)}(t)\right\|_{L^1(\mathbb{I})} < +\infty.$$

Thus the sequence

$$E_n(t) := \int_0^t \overline{\mathcal{E}}^{(n)}(s)\mathrm{d}s, \qquad n\geqslant 1,\ t\in[0,t_*] \tag{5.31}$$

lies in the space $C\big([0,t_*],\mathcal{M}(\mathbf{E}(t_*))$, where $\mathcal{M}(\mathbf{E}(t_*))$ is the space of all Borel measures on $\mathbb{I}$ with mass less than, or equal to, $\mathbf{E}(t_*)$, equipped with the topology of weak convergence of measures. Since $\mathbb{I}$ is compact, the space $\mathcal{M}(\mathbf{E}(t_*))$ is compact and metrizable. The sequence (5.31) is equicontinuous in the space $C\big([0,t_*];\mathcal{M}(\mathbf{E}(t_*))\big)$, therefore it is sequentially compact by virtue of the Ascoli-Arzelà Theorem, see e.g. [8, p. 234].

Suppose that $E(\cdot)\in C\big([0,t_*],\mathcal{M}(\mathbf{E}(t_*))\big)$ is the limiting point of $\{E_n\}_{n\geq 1}$, as $n\to+\infty$. We shall show that for any G as in (5.30) we have

$$\begin{aligned}\int_0^t \mathrm{d}s\int_{\mathbb{I}} G(s,u)E(s,\mathrm{d}u) &= \int_0^t \mathrm{d}s\int_{\mathbb{I}}\partial_s G(s,u)E(s,\mathrm{d}u) + \int_{\mathbb{I}} G(0,u)e_0(u)\mathrm{d}u\\ &+\frac{1}{4\gamma}\int_0^t \mathrm{d}s\int_{\mathbb{I}}\partial_u^2 G(s,u)E(s,\mathrm{d}u) + \frac{1}{8\gamma}\int_0^t \mathrm{d}s\int_{\mathbb{I}}\mathrm{d}u\ \partial_u^2 G(s,u)\Big(\int_0^s r^2(\sigma,u)\mathrm{d}\sigma\Big)\\ &-\frac{1}{4\gamma}\int_0^t s\Big(\partial_u G(s,1)\left(T_+ + \overline{\tau}_+^2\right) - T_-\partial_u G(s,0)\Big)\mathrm{d}s, \qquad t\in[0,t_*].\end{aligned} \tag{5.32}$$

This identifies the limit E of $\{E_n\}$ as a function $E:[0,+\infty)\times\mathbb{I}\to\mathbb{R}$ that is the unique solution of the problem

$$\partial_t E(t,u) = \frac{1}{4\gamma}\partial_{uu}^2\left\{E(t,u) + \frac{1}{2}\int_0^t r^2(s,u)\mathrm{d}s\right\} + e_0(u), \quad (t,u)\in\mathbb{R}_+\times\mathbb{I}, \tag{5.33}$$

with the boundary conditions

$$E(t,0) = T_- t, \qquad E(t,1) = \left(T_+ + \frac{1}{2}\overline{\tau}_+^2\right)t, \qquad t\geqslant 0 \tag{5.34}$$

and the initial condition $E(0,u)=0$. Here $r(t,u)$ is the solution of (3.12).

Concerning the limit identification for $\{\overline{\mathcal{E}}^{(n)}\}_{n\geqslant 1}$ we can write

$$\int_0^t\int_{\mathbb{I}}\overline{\mathcal{E}}^{(n)}(s,u)G(s,u)\mathrm{d}u\mathrm{d}s = \int_{\mathbb{I}} E_n(t,u)G(t,u)\mathrm{d}u - \int_0^t\int_{\mathbb{I}} E_n(s,u)\partial_s G(s,u)\mathrm{d}u\mathrm{d}s,$$

and, by passing to the limit $n\to\infty$, we get that the left hand side converges to

$$\int_{\mathbb{I}} E(t,u)G(t,u)\mathrm{d}u - \int_0^t\int_{\mathbb{I}} E(s,u)\partial_s G(s,u)\mathrm{d}u\mathrm{d}s = \int_0^t\int_{\mathbb{I}}\partial_s E(s,u)G(s,u)\mathrm{d}u\mathrm{d}s.$$

Hence, any $\star$−weak limiting point $e\in(L^1([0,t_*];C_0^2(\mathbb{I})))^\star$ of the sequence $\{\overline{\mathcal{E}}^{(n)}\}_{n\geqslant 1}$ is given by $e(t,u)=\partial_t E(t,u)$, which in turn satisfies (5.30) and Theorem 3.5 would then follow. Therefore one is left with proving (5.32).

Consider now a smooth test function $G \in C^\infty([0,+\infty)\times\mathbb{I})$ such that $G(s,0) = G(s,1) \equiv 0$, $s \geqslant 0$. Then, from (5.1), we get

$$\int_{\mathbb{I}} G(t,u)\mathrm{d}u\overline{\mathcal{E}}^{(n)}(t,u) - \int_{\mathbb{I}} G(0,u)\overline{\mathcal{E}}^{(n)}(0,u)\mathrm{d}u$$
$$= \frac{1}{n}\sum_{x\in\mathbb{I}_n} \mathbb{E}_n\left[G_x(t)\mathcal{E}_x(t)\right] - \frac{1}{n}\sum_{x\in\mathbb{I}_n} \mathbb{E}_n\left[G_x(0)\mathcal{E}_x(0)\right] + o_n(1)$$
$$= \frac{1}{n}\sum_{x=1}^{n-1}\int_0^t \mathbb{E}_n\left[\partial_s\big(G_x(s)\mathcal{E}_x(s)\big)\right]\mathrm{d}s + o_n(1) =: \mathrm{I}_n + \mathrm{II}_n + o_n(1),$$

where $o_n(1) \to 0$, as $n \to +\infty$ and

$$\mathrm{I}_n := \frac{1}{n}\int_0^t \sum_{x=0}^{n-1} \mathbb{E}_n\left[\partial_s G_x(s)\mathcal{E}_x^{(n)}(s)\right]\mathrm{d}s = \int_0^t \int_{\mathbb{I}} \partial_s G(s,u)\overline{\mathcal{E}}^{(n)}(s,u)\mathrm{d}u\mathrm{d}s + o_n(1),$$
$$\mathrm{II}_n := \frac{1}{n}\int_0^t \sum_{x=0}^{n-1} \mathbb{E}_n\left[G_x(s)\partial_s\mathcal{E}_x^{(n)}(s)\right]\mathrm{d}s.$$

Thanks to (5.1), and after an integration by parts, we can write

$$\mathrm{II}_n = \frac{1}{n}\int_0^t \sum_{x=1}^{n-1} \mathbb{E}_n\left[G_x(s)\big(j_{x-1,x} - j_{x,x+1}\big)(s)\right]\mathrm{d}s = \mathrm{II}_{n,1} + \mathrm{II}_{n,2}, \tag{5.35}$$

with

$$\mathrm{II}_{n,1} := \int_0^t \sum_{x=1}^{n-2} \mathbb{E}\big[(\nabla_n G)_x(s)\; j_{x,x+1}(s)\big]\mathrm{d}s,$$
$$\mathrm{II}_{n,2} := \int_0^t \mathbb{E}\big[-nG_{n-1}(s)j_{n-1,n}(s) + nG_1(s)j_{0,1}(s)\big]\mathrm{d}s.$$

By Lemma 5.1–(5.10), we conclude that $\mathrm{II}_{n,2} = o_n(1)$.

By a direct calculation we conclude the following *fluctuation-dissipation relation* for the microscopic currents:

$$j_{x,x+1} = n^{-2}Lg_x - (V_{x+1} - V_x), \qquad x \in \mathbb{I}_n^{\circ}, \tag{5.36}$$

with

$$g_x := -\frac{1}{4}p_x^2 + \frac{1}{4\gamma}p_x(r_x + r_{x+1}), \qquad V_x := \frac{1}{4\gamma}\big(r_x^2 + p_x p_{x-1}\big). \tag{5.37}$$

Using the notation $g_x(t) := g_x(\mathbf{r}(t), \mathbf{p}(t))$ (and similarly for other local functions), this allows us to write $\mathrm{II}_{n,1} = \sum_{j=1}^4 \mathrm{II}_{n,1,j}$, where

$$\begin{aligned}
\mathrm{II}_{n,1,1} &:= \int_0^t \frac{1}{n}\sum_{x=2}^{n-2} \mathbb{E}_n\big[(\Delta_n G)_x(s) V_x(s)\big]\mathrm{d}s\ ,\\
\mathrm{II}_{n,1,2} &:= \int_0^t \frac{1}{n^2}\sum_{x=1}^{n-2} \mathbb{E}_n\big[(\nabla_n G)_x(s) L g_x(s)\big]\mathrm{d}s,\\
\mathrm{II}_{n,1,3} &:= -\int_0^t \mathrm{d}s\ \mathbb{E}_n\big[(\nabla_n G)_{n-2}(s) V_{n-1}(s)\big],\\
\mathrm{II}_{n,1,4} &:= \int_0^t \mathrm{d}s\ \mathbb{E}_n\big[(\nabla_n G)_1(s) V_1(s)\big].
\end{aligned}$$

We have

$$\begin{aligned}
\mathrm{II}_{n,1,2} = &\frac{1}{n^2}\sum_{x=1}^{n-2}(\nabla_n G)_x(t)\mathbb{E}_n\left[g_x(t)\right] - \frac{1}{n^2}\sum_{x=1}^{n-2}(\nabla_n G)_x(0)\mathbb{E}_n\left[g_x(0)\right]\\
&- \int_0^t \frac{1}{n^2}\sum_{x=1}^{n-2}(\nabla_n \partial_s G)_x(s)\mathbb{E}_n\left[g_x(s)\right]\mathrm{d}s,
\end{aligned}$$

which vanishes, thanks to Proposition 4.5. By Lemma 5.1–(5.5) and (5.11)– we have:

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{II}_{n,1,4} &= \lim_{n\to\infty}\int_0^t \mathbb{E}_n\big[(\nabla_n G)_1(s)V_1(s)\big]\mathrm{d}s = \frac{1}{4\gamma}\lim_{n\to\infty}\int_0^t \partial_u G(s,0)\mathbb{E}_n\big[r_1^2(s)\big]\mathrm{d}s\\
&= \frac{T_-}{4\gamma}\int_0^t \partial_u G(s,0)\mathrm{d}s, \qquad (5.38)
\end{aligned}$$

which takes care of the left boundary condition. Concerning the right one:

$$\mathrm{II}_{n,1,3} = -\int_0^t \mathbb{E}_n\big[(\nabla_n G)_{n-2}(s)V_{n-1}(s)\big]\mathrm{d}s\ = J_{n,1} + J_{n,2}, \qquad (5.39)$$

where

$$\begin{aligned}
J_{n,1} &:= \int_0^t \mathbb{E}_n\big[(\nabla_n G)_{n-2}(s)(V_n - V_{n-1})(s)\big]\mathrm{d}s,\\
J_{n,2} &:= -\int_0^t \mathbb{E}_n\big[(\nabla_n G)_{n-2}(s)V_n(s)\big]\mathrm{d}s.
\end{aligned}$$

By virtue of Lemma 5.1–(5.12) we have

$$\lim_{n\to+\infty} J_{n,2} = -\lim_{n\to+\infty}\mathbb{E}_n\left[\int_0^t (\nabla_n G)_{n-2}(s)V_n(s)\mathrm{d}s\right] = -\frac{\left(T_+ + \overline{\tau}_+^2\right)}{4\gamma}\int_0^t \partial_u G(s,1)\mathrm{d}s.$$

On the other hand, using (5.36) for $x = n-1$, the term $J_{n,1}$ equals

$$J_{n,1} = -\frac{1}{n^2}\mathbb{E}_n\left[\int_0^t (\nabla_n G)_{n-2}(s)Lg_{n-1}(s)\mathrm{d}s\right] + \int_0^t (\nabla_n G)_{n-2}(s)\mathbb{E}_n\big[j_{n-1,n}(s)\big]\mathrm{d}s. \qquad (5.40)$$

From Lemma 5.1–(5.10) we conclude that the second term vanishes, with $n \to +\infty$. By integration by parts the first term equals

$$\frac{1}{n^2}\mathbb{E}_n\Big[(\nabla_n G)_{n-2}(0)g_{n-1}(0) - (\nabla_n G)_{n-2}(t)g_{n-1}(t)\Big] + \frac{1}{n^2}\mathbb{E}_n\left[\int_0^t (\nabla_n \partial_s G)_{n-2}(s)g_{n-1}(s)\mathrm{d}s\right], \quad (5.41)$$

which vanishes, thanks to Proposition 4.5. Summarizing, we have shown that

$$\lim_{n\to+\infty} \mathrm{II}_{n,1,3} = -\lim_{n\to+\infty} \mathbb{E}_n\left[\int_0^t (\nabla_n G)_{n-2}(s)V_{n-1}(s)\mathrm{d}s\right] = -\frac{\left(T_+ + \overline{\tau}_+^2\right)}{4\gamma}\int_0^t \partial_u G(s,1)\mathrm{d}s. \quad (5.42)$$

Now, for the bulk expression $\mathrm{II}_{n,1,1}$, we can write $\mathrm{II}_{n,1,1} = \mathcal{J}_{n,1} + \mathcal{J}_{n,2}$, where

$$\mathcal{J}_{n,1} := \frac{1}{4\gamma}\int_0^t \frac{1}{n}\sum_{x=2}^{n-2}\mathbb{E}_n\big[(\Delta_n G)_x(s)r_x^2(s)\big]\mathrm{d}s\,,$$

$$\mathcal{J}_{n,2} := \frac{1}{4\gamma}\int_0^t \frac{1}{n}\sum_{x=2}^{n-2}\mathbb{E}_n\big[(\Delta_n G)_x(s)p_x(s)p_{x-1}(s)\big]\mathrm{d}s\,.$$

5.4.1. *Estimates of* $\mathcal{J}_{n,2}$. After a direct calculation, it follows from (2.6) that

$$n^{-2}Lh_x = (W_{x+1} - W_x) - p_x p_{x-1}, \qquad x = 2,\dots,n-2, \quad (5.43)$$

with

$$h_x := \frac{1}{2\gamma}\left(\frac{1}{2}(r_x + r_{x-1})^2 + p_{x-1}p_x - r_x^2\right),$$

$$W_x := \frac{1}{2\gamma}p_{x-2}(r_{x-1} + r_x).$$

Substituting into the expression for $\mathcal{J}_{n,2}$ we conclude that $\mathcal{J}_{n,2} = K_{n,1} + K_{n,2}$, where $K_{n,1}$ and $K_{n,2}$ correspond to $(W_{x+1} - W_x)$ and $n^{-2}Lh_x$, respectively. Using the summation by parts to deal with $K_{n,1}$, performing time integration in the case of $K_{n,2}$ and subsequently invoking the energy bound from Proposition 4.5, we conclude that

$$\lim_{n\to+\infty} \mathcal{J}_{n,2} = 0. \quad (5.44)$$

5.4.2. *Limit of* $\mathcal{J}_{n,1}$. We write $\mathcal{J}_{n,1} = \mathcal{J}_{n,1,1} + \mathcal{J}_{n,1,2} + \mathcal{J}_{n,1,3}$, where

$$\mathcal{J}_{n,1,1} := \frac{1}{4\gamma}\int_0^t \frac{1}{n}\sum_{x=2}^{n-2}\mathbb{E}_n\big[(\Delta_n G)_x(s)\mathcal{E}_x(s)\big]\mathrm{d}s\,,$$

$$\mathcal{J}_{n,1,2} := \frac{1}{8\gamma}\int_0^t \frac{1}{n}\sum_{x=2}^{n-2}(\Delta_n G)_x(s)\Big([\overline{r}_x^{(n)}(s)]^2 - [\overline{p}_x^{(n)}(s)]^2\Big)\mathrm{d}s,$$

$$\mathcal{J}_{n,1,3} := \frac{1}{8\gamma}\int_0^t \frac{1}{n}\sum_{x=2}^{n-2}\mathbb{E}_n\Big[(\Delta_n G)_x(s)\Big([\widetilde{r}_x^{(n)}(s)]^2 - [\widetilde{p}_x^{(n)}(s)]^2\Big)\Big]\mathrm{d}s.$$

To deal with the term $\mathcal{J}_{n,1,3}$ we use Proposition 4.6, which allows us to conclude that $\lim_{n\to+\infty}\mathcal{J}_{n,1,3}=0$.

Concerning the term $\mathcal{J}_{n,1,2}$ we first note that $\lim_{n\to+\infty}(\mathcal{J}_{n,1,2}-\widetilde{\mathcal{J}}_{n,1,2})=0$, where

$$\widetilde{\mathcal{J}}_{n,1,2}:=\frac{1}{8\gamma}\int_0^t\frac{1}{n}\sum_{x=2}^{n-2}(\Delta_n G)_x(s)\big(\overline{r}_x^{(n)}(s)\big)^2\mathrm{d}s.$$

This is a consequence of the following result, proved in Section 6.4 (and which uses Proposition 4.1).

Lemma 5.4. *Under Assumption 3.1, for any $t>0$ we have*

$$\int_0^t\sup_{x\in\mathbb{I}_n}\big|\overline{p}_x^{(n)}(s)\big|^2\mathrm{d}s\lesssim\frac{\log^2(n+1)}{n^2},\qquad n\geqslant 1. \tag{5.45}$$

Next, using Theorem 3.3, we conclude that

$$\lim_{n\to+\infty}\widetilde{\mathcal{J}}_{n,1,2}=\frac{1}{8\gamma}\int_0^t\mathrm{d}s\int_{\mathbb{I}}\mathrm{d}u\,(\partial_u^2G)(s,u)r^2(s,u), \tag{5.46}$$

where $r(\cdot)$ is the solution of (3.12).

Summarizing, the results announced above, allow us to conclude that

$$\begin{aligned}&\int_{\mathbb{I}}G(t,u)\overline{\mathcal{E}}^{(n)}(t,u)\mathrm{d}u-\int_{\mathbb{I}}G(0,u)\overline{\mathcal{E}}^{(n)}(0,u)\mathrm{d}u\\&-\frac{1}{4\gamma}\int_0^t\int_{\mathbb{I}}(\partial_u^2G)(s,u)\overline{\mathcal{E}}^{(n)}(s,u)\mathrm{d}u\mathrm{d}s=\frac{1}{8\gamma}\int_0^t\int_{\mathbb{I}}(\partial_u^2G)(s,u)r^2(s,u)\mathrm{d}u\mathrm{d}s\\&+\frac{T_-}{4\gamma}\int_0^t\partial_uG(s,0)\mathrm{d}s-\frac{\big(T_++\overline{\tau}_+^2\big)}{4\gamma}\int_0^t\partial_uG(s,1)\mathrm{d}s+o_n(t),\end{aligned}$$

where $\lim_{n\to\infty}\sup_{s\in[0,t_*]}|o_n(s)|=0$ for a fixed $t_*>0$. Given $t>0$ we can take, as test function, $G(s,u):=H(t,u)$, for any $s\in[0,t]$, with an arbitrary compactly supported $H\in C([0,+\infty);C_0^2(\mathbb{I}))$. Integrating over $t\in[0,t_*]$ we obtain that $E_n(t)$, cf. (5.31), satisfies

$$\begin{aligned}&\int_0^{t_*}\int_{\mathbb{I}}H(s,u)E_n(s,u)\mathrm{d}s\mathrm{d}u\\&=\int_0^{t_*}\mathrm{d}s\int_{\mathbb{I}}\partial_sH(s,u)E_n(s,u)\mathrm{d}u+\int_{\mathbb{I}}H(0,u)\overline{\mathcal{E}}^{(n)}(0,u)\mathrm{d}u\\&+\frac{1}{4\gamma}\int_0^{t_*}\mathrm{d}s\int_{\mathbb{I}}\partial_u^2H(s,u)E_n(s,u)\mathrm{d}u+\frac{1}{8\gamma}\int_0^{t_*}\mathrm{d}s\int_{\mathbb{I}}\partial_u^2H(s,u)\Big(\int_0^s r^2(\sigma,u)\mathrm{d}\sigma\Big)\mathrm{d}u\\&-\frac{1}{4\gamma}\int_0^{t_*}s\Big(\partial_uH(s,1)\left(T_++\overline{\tau}_+^2\right)-T_-\partial_uH(s,0)\Big)\mathrm{d}s+o_n(t_*),\end{aligned}$$

with $o_n(t_*)\to 0$, as $n\to+\infty$. This obviously implies (5.32) and ends the proof of Theorem 3.5.

6. Dynamics of the averages

This part aims at proving previous results which have been left aside:

- (Sections 6.1 and 6.2) Proposition 4.1 and Proposition 4.2, which only deal with the extremity points $x = 0$ and $x = n$;
- (Section 6.3) Proposition 4.4 which gives an L^2 bound on all the averages *(its proof uses Proposition 4.2)* ;
- (Section 6.4) Lemma 5.4, which controls $\sup_{x\in\mathbb{I}_n} |\overline{p}_x^{(n)}(s)|^2$ *(its proof uses Proposition 4.1)* ;
- (Section 6.5) finally, Lemma 5.2, which gives a bound on the H^1–norm of the stretch averages *(its proof uses Proposition 4.2, Proposition 4.4 and Proposition 4.5)*.

All their proofs are based on a refined analysis of the system of equations satisfied by the averages of momenta and stretches.

To simplify the notation, in the present section we omit writing the superscript n by the averages $\overline{p}_x^{(n)}(t)$, $\overline{r}_x^{(n)}(t)$ defined in (3.11). Their dynamics is given by the following system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{r}_x(t) = n^2\left(\overline{p}_x(t) - \overline{p}_{x-1}(t)\right), \qquad x = 1,\dots,n \tag{6.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p}_x(t) = n^2\left(\overline{r}_{x+1}(t) - \overline{r}_x(t)\right) - 2\gamma n^2 \overline{p}_x(t), \qquad x \in \mathbb{I}_n^{\circ} \tag{6.2}$$

and at the boundaries: $\overline{r}_0(t) \equiv 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p}_0(t) = n^2\,\overline{r}_1(t) - n^2(2\gamma + \widetilde{\gamma})\overline{p}_0(t), \tag{6.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p}_n(t) = -n^2\,\overline{r}_n(t) + n^2\,\overline{\tau}_+(t) - n^2(2\gamma + \widetilde{\gamma})\overline{p}_n(t). \tag{6.4}$$

We have allowed above the forcing $\tau_+(t)$ to depend on t. Although in most cases we shall consider $\tau_+(t) \equiv \tau_+$ constant, yet in some instances we also admit to be in the form $\tau_+(t) = \mathbf{1}_{[0,t_*)}(t)\tau_+$ for some $t_* > 0$, $\tau_+ \in \mathbb{R}$.

The resolution of these equations will allow us to get several crucial estimates. For that purpose, we first rewrite the system in terms of Fourier transforms, and we will then take its Laplace transform. Let us define

$$\widehat{\overline{r}}(t,k) = \mathbb{E}_n\big[\widehat{r}(t,k)\big], \qquad \widehat{\overline{p}}(t,k) = \mathbb{E}_n\big[\widehat{p}(t,k)\big].$$

From (6.1) and (6.2) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\widehat{\overline{r}}(t,k)\\ \widehat{\overline{p}}(t,k)\end{pmatrix} = n^2 A\begin{pmatrix}\widehat{\overline{r}}(t,k)\\ \widehat{\overline{p}}(t,k)\end{pmatrix} + n^2\overline{\tau}_+(t)\begin{pmatrix}0\\ e^{2\pi i k}\end{pmatrix} - n^2\overline{p}_0(t)\begin{pmatrix}1\\ \widetilde{\gamma}\end{pmatrix} + n^2\overline{p}_n(t)\begin{pmatrix}1\\ -\widetilde{\gamma}e^{2\pi i k}\end{pmatrix}, \tag{6.5}$$

where

$$A = \begin{pmatrix}0 & 1 - e^{-2i\pi k}\\ e^{2i\pi k} - 1 & -2\gamma\end{pmatrix}.$$

Assuming that $\overline{\tau}_+(t) \equiv \overline{\tau}_+$, $t \geqslant 0$ we can rewriting equation (6.5) in the mild formulation and obtain

$$\begin{aligned}\begin{pmatrix}\widehat{\overline{r}}(t,k)\\ \widehat{\overline{p}}(t,k)\end{pmatrix} &= \exp\left\{n^2 A t\right\}\begin{pmatrix}\widehat{\overline{r}}(0,k)\\ \widehat{\overline{p}}(0,k)\end{pmatrix} + n^2\overline{\tau}_+\int_0^t \exp\left\{n^2A(t-s)\right\}\begin{pmatrix}0\\ e^{2\pi i k}\end{pmatrix}\mathrm{d}s\\ &\quad - n^2\int_0^t \exp\left\{n^2A(t-s)\right\}p_0(s)\begin{pmatrix}1\\ \widetilde{\gamma}\end{pmatrix}\mathrm{d}s\\ &\quad + n^2\int_0^t \exp\left\{n^2A(t-s)\right\}p_n(s)\begin{pmatrix}1\\ -\widetilde{\gamma}e^{2\pi i k}\end{pmatrix}\mathrm{d}s. \end{aligned}\tag{6.6}$$

Denoting by $\lambda_\pm(k) = -\left(\gamma \pm \sqrt{\gamma^2 - 4\sin^2(\pi k)}\right)$ the eigenvalues of A we obtain the following autonomous integral equation for $\{\overline{p}_x(t)\}_{x\in\mathbb{I}_n}$

$$\overline{p}_x(t) = \widehat{\mathcal{T}}_x^{(n)}(t) + \int_0^t q_{1,x}^{(n)}(t-s)\overline{p}_{0,n}^{\mathrm{diff}}(s)\mathrm{d}s + \int_0^t q_{2,x}^{(n)}(t-s)\overline{p}_{0,n}^{\mathrm{sum}}(s)\mathrm{d}s, \tag{6.7}$$

where

$$\begin{aligned} q_{1,x}^{(n)}(t) &:= \sum_{k\in\widehat{\mathbb{I}}_n}\frac{n^2}{\lambda_-(k)-\lambda_+(k)}e^{2\pi i k x}(e^{2\pi i k}-1)\left(e^{n^2t\lambda_+(k)} - e^{n^2t\lambda_-(k)}\right)\\ q_{2,x}^{(n)}(t) &:= \widetilde{\gamma}\sum_{k\in\widehat{\mathbb{I}}_n}\frac{n^2}{\lambda_-(k)-\lambda_+(k)}e^{2\pi i k x}\left(\lambda_-(k)e^{n^2t\lambda_-(k)} - \lambda_+(k)e^{n^2t\lambda_+(k)}\right),\\ \overline{p}_{0,n}^{\mathrm{diff}}(s) &:= \overline{p}_0(s) - \overline{p}_n(s),\\ \overline{p}_{0,n}^{\mathrm{sum}}(s) &:= \overline{p}_0(s) + e^{2\pi i k}\overline{p}_n(s),\\ \widehat{\mathcal{T}}_x^{(n)}(t) &:= \sum_{k\in\widehat{\mathbb{I}}_n}e^{2\pi i k x}\frac{\left(-(1-e^{2\pi i k})\widehat{\overline{r}}(0,k) + \overline{\tau}_+e^{2\pi i k}\right)\left(e^{n^2t\lambda_-(k)} - e^{n^2t\lambda_+(k)}\right)}{\lambda_-(k)-\lambda_+(k)},\\ &\quad + \sum_{k\in\widehat{\mathbb{I}}_n}e^{2\pi i k x}\frac{\widehat{\overline{p}}(0,k)\left(\lambda_-(k)e^{t\lambda_-(k)} - \lambda_+(k)e^{t\lambda_+(k)}\right)}{\lambda_-(k)-\lambda_+(k)}. \end{aligned}$$

6.1. **Proof of Proposition 4.1.** We define, for any $x \in \mathbb{I}_n$, and any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > 0$, the Laplace transforms:

$$\widetilde{\overline{p}}_x(\lambda) := \int_0^{+\infty} e^{-\lambda t}\overline{p}_x(t)\mathrm{d}t, \qquad \widetilde{\overline{\tau}}_+(\lambda) := \int_0^{+\infty} e^{-\lambda t}\overline{\tau}_+(t)\mathrm{d}t$$

and

$$\widetilde{\overline{r}}(\lambda,k) := \int_0^{+\infty} e^{-\lambda t}\widehat{\overline{r}}(t,k)\mathrm{d}t, \qquad \widetilde{\overline{p}}(\lambda,k) := \int_0^{+\infty} e^{-\lambda t}\widehat{\overline{p}}(t,k)\mathrm{d}t.$$

Performing the Laplace transform on both sides of (6.5) we obtain the following system

$$\begin{pmatrix}\widetilde{\widehat{r}}(\lambda,k)\\ \widetilde{\widehat{p}}(\lambda,k)\end{pmatrix} = (\lambda - n^2A)^{-1}\begin{pmatrix}\widehat{r}(0,k)\\ \widehat{p}(0,k)\end{pmatrix} + n^2\widetilde{\widetilde{\tau}}_+(\lambda)(\lambda - n^2A)^{-1}\begin{pmatrix}0\\ e^{2\pi ik}\end{pmatrix} \tag{6.8}$$
$$- n^2\widetilde{\widetilde{p}}_0(\lambda)(\lambda - n^2A)^{-1}\begin{pmatrix}1\\ \widetilde{\gamma}\end{pmatrix} + n^2\widetilde{\widetilde{p}}_n(\lambda)(\lambda - n^2A)^{-1}\begin{pmatrix}1\\ -\widetilde{\gamma}e^{2\pi ik}\end{pmatrix}.$$

Here

$$(\lambda - n^2A)^{-1} = \frac{1}{n^2\Delta(\lambda/n^2,k)}\begin{pmatrix}\frac{\lambda}{n^2} + 2\gamma & 1 - e^{-2i\pi k}\\ e^{2i\pi k} - 1 & \frac{\lambda}{n^2}\end{pmatrix}$$

and $\Delta(\lambda,k) := \lambda^2 + 2\gamma\lambda + 4\sin^2(\pi k)$. Let

$$\widetilde{\widetilde{p}}^{\text{diff}}_{0,n}(\lambda) := \widetilde{\widetilde{p}}_0(\lambda) - \widetilde{\widetilde{p}}_n(\lambda) \quad \text{and} \quad \widetilde{\widetilde{p}}^{(+)}_{0,n}(\lambda) := \widetilde{\widetilde{p}}_0(\lambda) + \widetilde{\widetilde{p}}_n(\lambda).$$

Since $\widehat{\sum}_{k\in\widehat{\mathbb{I}}_n}\widehat{r}(0,k) = r_0 = 0$ we have

$$\begin{aligned}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{-4\sin^2(\pi k)\widehat{r}(0,k)}{\Delta(\lambda/n^2,k)} &= \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\big(-4\sin^2(\pi k) + \Delta(\lambda/n^2,k)\big)\,\widehat{r}(0,k)}{\Delta(\lambda/n^2,k)}\\ &= \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\lambda/n^2(\lambda/n^2 + 2\gamma)\,\widehat{r}(0,k)}{\Delta(\lambda/n^2,k)}.\end{aligned} \tag{6.9}$$

Using (6.8) and (6.9) we conclude the expressions for $\widetilde{\widetilde{p}}^{\text{diff}}_{0,n}(\lambda)$ and $\widetilde{\widetilde{p}}^{(+)}_{0,n}(\lambda)$

$$\widetilde{\widetilde{p}}^{\text{diff}}_{0,n}(\lambda) = \frac{1}{n^2\mathbf{e}_{\text{d,n}}(\lambda/n^2)}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{(\lambda/n^2 + 2\gamma)\widehat{r}(0,k)}{\Delta(\lambda/n^2,k)} \tag{6.10}$$
$$+ \frac{1}{n^2\mathbf{e}_{\text{d},n}(\lambda/n^2)}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{(1 - e^{-2\pi ik})\widehat{p}(0,k)}{\Delta(\lambda/n^2,k)} - \frac{2\widetilde{\widetilde{\tau}}_+(\lambda)}{\mathbf{e}_{\text{d},n}(\lambda/n^2)}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\sin^2(\pi k)}{\Delta(\lambda/n^2,k)}$$

and

$$\widetilde{\widetilde{p}}^{(+)}_{0,n}(\lambda) = \frac{2i}{n^2\mathbf{e}_{\text{s},n}(\lambda/n^2)}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\sin(2\pi k)\widehat{r}(0,k)}{\Delta(\lambda/n^2,k)} \tag{6.11}$$
$$+ \frac{\lambda}{n^4\mathbf{e}_{\text{s},n}(\lambda/n^2)}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{(1 + e^{-2\pi ik})\widehat{p}(0,k)}{\Delta(\lambda/n^2,k)} + \frac{2\lambda\widetilde{\widetilde{\tau}}_+(\lambda)}{n^2\mathbf{e}_{\text{s},n}(\lambda/n^2)}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\cos^2(\pi k)}{\Delta(\lambda/n^2,k)},$$

where

$$\begin{aligned}\mathbf{e}_{\text{d},n}(\lambda) &:= \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\lambda + 2\gamma + 2\widetilde{\gamma}\sin^2(\pi k)}{\lambda^2 + 2\gamma\lambda + 4\sin^2(\pi k)},\\ \mathbf{e}_{\text{s},n}(\lambda) &:= \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\lambda^2 + 2\gamma\lambda + 2\widetilde{\gamma}\lambda\cos^2(\pi k) + 4\sin^2(\pi k)}{\lambda^2 + 2\gamma\lambda + 4\sin^2(\pi k)}.\end{aligned}$$

6.1.1. *First part of Proposition 4.1: estimates of $\|\overline{p}_0 - \overline{p}_n\|_{L^2(\mathbb{R}_+)}$.* .
Let us fix $t_* > 0$ and consider $\overline{\tau}_+(t) := \overline{\tau}_+ \mathbf{1}_{[0,t_*]}(t)$. Then, for $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0$,

$$\widetilde{\overline{\tau}}_+(\lambda) = \overline{\tau}_+ \int_0^{t_*} e^{-\lambda t} \mathrm{d}t = \frac{\overline{\tau}_+}{\lambda}\left(1 - e^{-\lambda t_*}\right).$$

By the Plancherel Theorem we have

$$\|\overline{p}_0 - \overline{p}_n\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{2\pi} \int_{\mathbb{R}} \left|\widetilde{\overline{p}}^{\mathrm{diff}}_{0,n}(i\eta)\right|^2 \mathrm{d}\eta, \tag{6.12}$$

therefore, from (6.10), we can estimate

$$\|\overline{p}_0 - \overline{p}_n\|^2_{L^2(\mathbb{R}_+)} \lesssim P^{\mathrm{d}}_{n,1} + P^{\mathrm{d}}_{n,2} + P^{\mathrm{d}}_{n,3}, \tag{6.13}$$

where

$$P^{\mathrm{d}}_{n,1} := \frac{1}{n^2} \int_{\mathbb{R}} \left|\rho_{\mathrm{d},n}(\eta)\right|^2 \left|e_{\mathrm{d},n}(\eta)\right|^{-2} \mathrm{d}\eta, \tag{6.14}$$

$$P^{\mathrm{d}}_{n,2} := \frac{1}{n^2} \int_{\mathbb{R}} \left|\pi_{\mathrm{d},n}(\eta)\right|^2 \left|e_{\mathrm{d},n}(\eta)\right|^{-2} \mathrm{d}\eta, \tag{6.15}$$

$$P^{\mathrm{d}}_{n,3} := \frac{1}{n^2} \int_{\mathbb{R}} \left|\frac{\sin(n^2 \eta t_*/2)}{\eta}\right|^2 \left|a_n(\eta)\right|^2 \left|e_{\mathrm{d},n}(\eta)\right|^{-2} \mathrm{d}\eta, \tag{6.16}$$

and

$$e_{\mathrm{d},n}(\eta) := \mathbf{e}_{\mathrm{d},n}(i\eta) \tag{6.17}$$

$$\rho_{\mathrm{d},n}(\eta) := \widehat{\sum_{k \in \widehat{\mathbb{I}}_n}} \frac{(i\eta + 2\gamma)\widehat{\overline{r}}(0,k)}{4\sin^2(\pi k) - \eta^2 + 2i\gamma\eta}, \tag{6.18}$$

$$\pi_{\mathrm{d},n}(\eta) := \widehat{\sum_{k \in \widehat{\mathbb{I}}_n}} \frac{(1 - e^{-2\pi i k})\widehat{\overline{p}}(0,k)}{4\sin^2(\pi k) - \eta^2 + 2i\gamma\eta}, \tag{6.19}$$

$$a_n(\eta) := \widehat{\sum_{k \in \widehat{\mathbb{I}}_n}} \frac{2\sin^2(\pi k)}{4\sin^2(\pi k) - \eta^2 + 2i\gamma\eta} \tag{6.20}$$

After elementary, but somewhat tedious calculations, see the Appendix sections A, B, D and E for details, we conclude that: for any $\eta \in \mathbb{R}$,

$$\frac{1}{|\eta|} \lesssim |e_{\mathrm{d},n}(\eta)|, \tag{6.21}$$

$$|a_n(\eta)| \lesssim \frac{1}{1+\eta^2}, \tag{6.22}$$

$$|\pi_{\mathrm{d},n}(\eta)| \lesssim \frac{1}{\eta^2+1} \log\left(1 + |\eta|^{-1}\right), \tag{6.23}$$

$$\left|\frac{\rho_{\mathrm{d},n}(\eta)}{e_{\mathrm{d},n}(\eta)}\right| \lesssim \frac{1}{\eta^2+1}, \qquad n \geqslant 1. \tag{6.24}$$

Let us emphasize here that these bounds are obtained thanks to the assumption that we made on the spectrum of the averages at initial time, recall Assumption

3.1–(3.10). From (6.16), (6.21) and (6.22) we get

$$P^{\mathrm{d}}_{n,3} \lesssim \frac{1}{n^2}\int_0^1 \sin^2(n^2\eta t_*/2)\mathrm{d}\eta + \frac{1}{n^2}\int_1^{+\infty}\frac{\mathrm{d}\eta}{\eta^2} \lesssim \frac{1}{n^2}.$$

A similar argument, using (6.23) and (6.24), shows that $P^{\mathrm{d}}_{n,j} \lesssim n^{-2}$, for $j = 1,2$. As a result we conclude (4.1).

6.1.2. *Second part of Proposition 4.1: estimates of* $\|\overline{p}_0 + \overline{p}_n\|_{L^2(\mathbb{R}_+)}$. .

Recall that $\overline{\tau}_+(t) = \overline{\tau}_+\mathbf{1}_{[0,t_*)}(t)$. The strategy to estimate $\|\overline{p}_0 + \overline{p}_n\|_{L^2(\mathbb{R}_+)}$ is completely similar. First, we write

$$\|\overline{p}_0 + \overline{p}_n\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\widetilde{\widehat{\overline{p}}}{}^{(+)}_{0,n}(i\eta)\right|^2\mathrm{d}\eta, \tag{6.25}$$

with $\widetilde{\widehat{\overline{p}}}{}^{(+)}_{0,n}$ given by (6.11). Substituting from (6.11) we get

$$\|\overline{p}_0 + \overline{p}_n\|^2_{L^2(\mathbb{R}_+)} \lesssim P^{\mathrm{s}}_{n,1} + P^{\mathrm{s}}_{n,2} + P^{\mathrm{s}}_{n,3}, \tag{6.26}$$

where

$$P^{\mathrm{s}}_{n,1} := \frac{1}{n^2}\int_{\mathbb{R}}\left|\rho_{\mathrm{s},n}(\eta)\right|^2\left|e_{\mathrm{s},n}(\eta)\right|^{-2}\mathrm{d}\eta, \tag{6.27}$$

$$P^{\mathrm{s}}_{n,2} := \frac{1}{n^2}\int_{\mathbb{R}}\eta^2\left|\pi_{\mathrm{s},n}(\eta)\right|^2\left|e_{\mathrm{s},n}(\eta)\right|^{-2}\mathrm{d}\eta, \tag{6.28}$$

$$P^{\mathrm{s}}_{n,3} := \frac{1}{n^2}\int_{\mathbb{R}}\sin^2(n^2\eta t_*/2)\left|c_n(\eta)\right|^2\left|e_{\mathrm{s},n}(\eta)\right|^{-2}\mathrm{d}\eta \tag{6.29}$$

and

$$e_{\mathrm{s},n}(\eta) := \mathbf{e}_{\mathrm{s},n}(i\eta) \tag{6.30}$$

$$\rho_{\mathrm{s},n}(\eta) := \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{\sin(2\pi k)\widehat{\overline{r}}(0,k)}{-\eta^2 + 2i\gamma\eta + 4\sin^2(\pi k)}, \tag{6.31}$$

$$\pi_{\mathrm{s},n}(\eta) := \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{(1+e^{-2\pi ik})\widehat{\overline{p}}(0,k)}{-\eta^2 + 2i\gamma\eta + 4\sin^2(\pi k)}, \tag{6.32}$$

$$c_n(\eta) := \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{1+\cos(2\pi k)}{-\eta^2 + 4\sin^2(\pi k) + 2i\gamma\eta}. \tag{6.33}$$

We have, see the Appendix sections C, D, E, F, for any $\eta \in \mathbb{R}$

$$1 \lesssim |e_{\mathrm{s},n}(\eta)|, \tag{6.34}$$

$$|c_n(\eta)| \lesssim \frac{1}{\sqrt{|\eta|}(1+|\eta|^{3/2})}, \tag{6.35}$$

$$|\rho_{\mathrm{s},n}(\eta)| \lesssim \frac{1}{\eta^2+1}\log\left(1+|\eta|^{-1}\right), \tag{6.36}$$

$$|\pi_{\mathrm{s},n}(\eta)| \lesssim \frac{1}{|\eta|+\eta^2}, \qquad n \geqslant 1. \tag{6.37}$$

Using (6.34) and (6.35) we conclude that

$$P^{\rm s}_{n,3} \lesssim \frac{1}{n^2}\int_0^{+\infty}\frac{\sin^2(n^2\eta t_*/2)}{\eta}\frac{{\rm d}\eta}{1+\eta^3} \lesssim \frac{1}{n^2}\int_0^1 \frac{\sin^2(n^2\eta t_*/2)}{\eta}{\rm d}\eta + \frac{1}{n^2} \tag{6.38}$$
$$\lesssim \frac{1}{n^2}\int_0^{n^2t_*}\frac{\sin^2(\eta)}{\eta}{\rm d}\eta + \frac{1}{n^2} \lesssim \frac{\log(n^2t_*+1)}{n^2}, \qquad n\geqslant 1.$$

From (6.34), (6.36) and (6.37) we can easily obtain $P^{\rm s}_{n,j} \lesssim n^{-2}$, $n \geqslant 1$, for $j = 1, 2$. By virtue of (6.26) we conclude (4.2). Finally, Proposition 4.1 is proved. □

6.2. **Proof of Proposition 4.2.** To prove Proposition 4.2, we use once again the autonomous system of equations for the averages of the momenta (6.1)–(6.4), and we will prove two estimates, on

$$\int_0^t \big(\overline{p}_0(s)+\overline{p}_n(s)\big){\rm d}s \qquad \text{and} \qquad \int_0^t \big(\overline{p}_0(s)-\overline{p}_n(s)\big){\rm d}s.$$

Recall that $\overline{\tau}_+(t) = \overline{\tau}_+ \mathbf{1}_{[0,t_*)}(t)$. Note that for any $0 \leqslant a < b \leqslant t_*$ we can write

$$\int_a^b \overline{p}_x(s){\rm d}s = \int_{\mathbb{R}} \frac{e^{i\eta(b-a)}-1}{2\pi i\eta}e^{i\eta a}\widetilde{\overline{p}}_x(i\eta){\rm d}\eta. \tag{6.39}$$

Using (6.10) and (6.11) we conclude that

$$\frac{e^{i\eta(b-a)}-1}{2\pi i\eta}e^{i\eta a}\,\widetilde{\overline{p}}^{\rm diff}_{0,n}(i\eta) = \widetilde{P}^{\rm d}_{n,1}+\widetilde{P}^{\rm d}_{n,2}+\widetilde{P}^{\rm d}_{n,3}$$
$$\frac{e^{i\eta(b-a)}-1}{2\pi i\eta}e^{i\eta a}\,\widetilde{\overline{p}}^{(+)}_{0,n}(i\eta) = \widetilde{P}^{\rm s}_{n,1}+\widetilde{P}^{\rm s}_{n,2}+\widetilde{P}^{\rm d}_{n,3}$$

where, for $\iota \in \{{\rm d},{\rm s}\}$, we denote

$$\widetilde{P}^{\iota}_{n,1} := \frac{1-e^{i\eta(b-a)}}{2\pi i n^2\eta}e^{i\eta a}\rho_{\iota,n}\left(\frac{\eta}{n^2}\right)e^{-1}_{\iota,n}\left(\frac{\eta}{n^2}\right),$$
$$\widetilde{P}^{\iota}_{n,2} := \frac{e^{i\eta(b-a)}-1}{2\pi i n^2\eta}e^{i\eta a}\pi_{\iota,n}\left(\frac{\eta}{n^2}\right)e^{-1}_{\iota,n}\left(\frac{\eta}{n^2}\right),$$
$$\widetilde{P}^{\rm d}_{n,3} := -\frac{\widetilde{\overline{\tau}}_+}{2\pi\eta^2}\big(e^{i\eta(b-a)}-1\big)\big(e^{-i\eta t_*}-1\big)e^{i\eta a}a_n\left(\frac{\eta}{n^2}\right)e^{-1}_{{\rm d},n}\left(\frac{\eta}{n^2}\right),$$
$$\widetilde{P}^{\rm s}_{n,3} := \frac{\widetilde{\overline{\tau}}_+}{2\pi i n^2\eta}\big(e^{i\eta(b-a)}-1\big)\big(e^{-i\eta t_*}-1\big)e^{i\eta a}c_n\left(\frac{\eta}{n^2}\right)e^{-1}_{{\rm s},n}\left(\frac{\eta}{n^2}\right).$$

Hence,

$$\left|\int_a^b \big(\overline{p}_0(s)-\overline{p}_n(s)\big){\rm d}s\right| \leqslant \left\{\int_{\mathbb{R}}\frac{\sin^2(\eta(b-a)/2)}{4\pi^2\eta^2}{\rm d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}}\left|\widetilde{\overline{p}}^{\rm diff}_{0,n}(i\eta)\right|^2{\rm d}\eta\right\}^{1/2}$$
$$\lesssim \frac{(b-a)^{1/2}}{n}, \tag{6.40}$$

where we have used Proposition 4.1–(4.1). On the other hand

$$\left|\int_0^t \big(\overline{p}_0(s)+\overline{p}_n(s)\big)\mathrm{d}s\right| \lesssim \mathrm{I}_{n,1}+\mathrm{I}_{n,2}+\mathrm{I}_{n,3}, \tag{6.41}$$

where

$$\begin{aligned}
\mathrm{I}_{n,1} &:= \frac{1}{n^2}\int_{\mathbb{R}} \left|\sin(n^2\eta(b-a)/2)\right| \left|\rho_{\mathrm{s},n}(\eta)\right| \, |e_{\mathrm{s},n}(\eta)|^{-1}\frac{\mathrm{d}\eta}{|\eta|},\\
\mathrm{I}_{n,2} &:= \frac{1}{n^2}\int_{\mathbb{R}} \left|\sin(n^2\eta(b-a)/2)\right| \left|\pi_{\mathrm{s},n}(\eta)\right| \, |e_{\mathrm{s},n}(\eta)|^{-1}\mathrm{d}\eta,\\
\mathrm{I}_{n,3} &:= \frac{1}{n^2}\int_{\mathbb{R}} \left|\sin(n^2\eta(b-a)/2)\sin(n^2\eta t_*/2)\right| \left|c_n(\eta)\right| |e_{\mathrm{s},n}(\eta)|^{-1}\frac{\mathrm{d}\eta}{|\eta|}.
\end{aligned}$$

Suppose that $p \in (0, \frac{1}{2}]$ and $0 < b-a < 1$. Thanks to the estimates (6.34)–(6.37) we conclude that

$$\begin{aligned}
\mathrm{I}_{n,3} &\lesssim \frac{1}{n^2}\int_{(b-a)^{-p}}^{+\infty}\frac{\mathrm{d}\eta}{\eta^2} + \frac{1}{n^2}\int_0^{(b-a)^{-p}} \frac{|\sin(n^2\eta(b-a)/2)\sin(n^2\eta t_*/2)|\mathrm{d}\eta}{\sqrt{\eta}(1+\eta^{3/2})}\\
&\lesssim \frac{(b-a)^p}{n^2} + \mathrm{I}^1_{n,3} + \mathrm{I}^2_{n,3},
\end{aligned} \tag{6.42}$$

where $\mathrm{I}^j_{n,3}$, $j=1,2$ correspond to splitting the domain of integration in the last integral into $[0,1]$ and $[1,(b-a)^{-p}]$, respectively. We have

$$|\sin x| \lesssim x^p, \quad x>0.$$

Using this estimate to bound $|\sin(n^2\eta(b-a)/2)|$, since $|\sin(n^2\eta t_*/2)| \leqslant 1$, we can bound (recall that $p \in (0, \frac{1}{2}]$):

$$\mathrm{I}^1_{n,3} \lesssim \frac{(b-a)^p}{n^{2(1-p)}}\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1/2-p}} \lesssim \frac{(b-a)^p}{n}. \tag{6.43}$$

In addition,

$$\mathrm{I}^2_{n,3} \lesssim \frac{(b-a)^p}{n^{2(1-p)}}\int_1^{(b-a)^{-p}} \frac{\mathrm{d}\eta}{\eta^{2-p}} \lesssim \frac{(b-a)^p}{n}. \tag{6.44}$$

As a result, we obtain $\mathrm{I}_{n,3} \lesssim (b-a)^p/n$. Estimates for $\mathrm{I}_{n,j}$, $j=1,2$ are similar. Hence, for any $t>0$ we can find $p>0$ such that

$$\left|\int_a^b \big(\overline{p}_0(s)+\overline{p}_n(s)\big)\mathrm{d}s\right| \lesssim \frac{(b-a)^p}{n}, \qquad n \geqslant 1, \tag{6.45}$$

and this together with (6.40) implies (4.3). □

6.3. **Proof of Proposition 4.4.** Let

$$\Xi_n(t) := \sum_{x\in\mathbb{I}_n}\overline{p}_x^2(t) + \sum_{x=1}^n \overline{r}_x^2(t), \qquad \mathfrak{P}_n(t) := \sum_{x\in\mathbb{I}_n}\overline{p}_x^2(t).$$

Multiplying the equations (6.1), (6.2) by $\overline{r}_x(t)$ and $\overline{p}_x(t)$, respectively and (6.3), (6.4) by $\overline{p}_0(t)$ and $\overline{p}_n(t)$ and summing up we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Xi_n(t) = n^2\left\{-\gamma\mathfrak{P}_n(t) - \frac{1}{2}\widetilde{\gamma}\overline{p}_0^2(t) - \frac{1}{2}\widetilde{\gamma}\overline{p}_n^2(t) + \overline{p}_n(t)\overline{\tau}_+\right\}. \tag{6.46}$$

Hence, by virtue of (4.3), we conclude that for any $t_* > 0$

$$\frac{1}{2}\Big(\overline{\Xi}_n(t) - \overline{\Xi}_n(0)\Big) \leqslant n^2|\overline{\tau}_+|\left|\int_0^t \overline{p}_n(s)\mathrm{d}s\right| \lesssim n, \qquad t\in[0,t_*]. \tag{6.47}$$

and (4.4) follows. From (6.46) we obtain further

$$\frac{1}{2}\mathfrak{P}_n(t) \leqslant \frac{1}{2}\Xi_n(0) - n^2\int_0^t \mathfrak{P}_n(s)\mathrm{d}s + n^2|\overline{\tau}_+|\left|\int_0^t \overline{p}_n(s)\mathrm{d}s\right| \tag{6.48}$$

Using Assumption 3.1–(3.9) and Proposition 4.2 we conclude that given $t_* > 0$ we have

$$\mathfrak{P}_n(t) \lesssim n - n^2\int_0^t \overline{\mathfrak{P}}_n(s)\mathrm{d}s, \quad t\in[0,t_*],\ n\geqslant 1. \tag{6.49}$$

Therefore (4.5) follows upon an application of the Gronwall inequality. □

6.4. **Proof of Lemma 5.4.** To prove Lemma 5.4, we need an estimate for

$$\sup_{x\in\mathbb{I}_n}\left|\overline{p}_x^{(n)}(s)\right|^2,$$

which can be done using the explicit formula (6.7). Note first that we have the following inequalities for $\lambda_\pm(k)$:

$$-2\gamma \leqslant \lambda_+(k) \leqslant -\gamma \qquad \text{and} \qquad \frac{-4\sin^2(\pi k)}{\gamma} \leqslant \lambda_-(k) \leqslant \frac{-2\sin^2(\pi k)}{\gamma}.$$

Therefore, in order to estimate the members which appear in the right hand side of (6.7), let us introduce, for $\ell = 0,1,2$,

$$\overline{Q}_\ell^{(n)}(t) := \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\frac{|\sin(\pi k)|^\ell}{|\lambda_-(k)-\lambda_+(k)|}\Big(\exp\Big\{-\frac{2t\sin^2(\pi k)}{\gamma}\Big\} + \exp\{-\gamma t\}\Big).$$

Lemma 6.1. *We have*

$$\overline{Q}_\ell^{(n)}(t) \lesssim \frac{1}{(1+t)^{(1+\ell)/2}}, \qquad t\geqslant 0,\ n\geqslant 1,\ \ell=0,1,2. \tag{6.50}$$

Proof. We prove the result for $\ell = 2$. The argument in the remaining cases is similar. It suffices only to show that

$$\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\sin^2(\pi k)\exp\Big\{-\frac{2t\sin^2(\pi k)}{\gamma}\Big\} \lesssim \frac{1}{(1+t)^{3/2}}, \qquad t\geqslant 0,\ n\geqslant 1. \tag{6.51}$$

Since the left hand side of (6.51) is obviously bounded it suffices to show that

$$\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\sin^2(\pi k)\exp\Big\{-\frac{2t\sin^2(\pi k)}{\gamma}\Big\} \lesssim \frac{1}{t^{3/2}}, \qquad t\geqslant 0,\ n\geqslant 1, \tag{6.52}$$

which follows easily from the fact that $\sin^2(\pi k) \sim k^2$, as $|k| \ll 1$. $\square$

From (6.7) we get that for any $t_* > 0$

$$\sup_{x\in\mathbb{I}_n} |\overline{p}_x(t)|\mathbf{1}_{[0,t_*]}(t) \leqslant \mathrm{I}_n(t) + \mathrm{II}_n(t) + \mathrm{III}_n(t), \tag{6.53}$$

where

$$\mathrm{I}_n(t) := \overline{Q}_0^{(n)}(n^2t)\mathbf{1}_{[0,t_*]}(t),$$

$$\mathrm{II}_n(t) := \int_0^t q_1^{(n)}(t-s)\big|\overline{p}_{0,n}^{\mathrm{diff}}(s)\big|\mathbf{1}_{[0,t_*]}(s)\mathrm{d}s,$$

$$\mathrm{III}_n(t) := \int_0^t q_2^{(n)}(t-s)\big|\overline{p}_{0,n}^{\mathrm{sum}}(s)\big|\mathbf{1}_{[0,t_*]}(s)\mathrm{d}s$$

and $q_\ell^{(n)}(t) := n^2\overline{Q}_\ell^{(n)}(n^2t)\mathbf{1}_{[0,t_*]}(t)$, for $\ell = 1,2$. Therefore, from Lemma 6.1 we get

$$\|\mathrm{I}_n\|^2_{L^2[0,+\infty)} = \int_0^{t_*} \big|\overline{Q}_0^{(n)}(n^2t)\big|^2\mathrm{d}t \lesssim \frac{1}{n^2}\int_0^{n^2t_*}\frac{\mathrm{d}t}{1+t} \lesssim \frac{\log(n+1)}{n^2}.$$

We also have

$$\|q_1^{(n)}\|_{L^1[0,+\infty)} = n^2\int_0^{t_*}\overline{Q}_1^{(n)}(n^2t)\mathrm{d}t \lesssim \int_0^{n^2t_*}\frac{\mathrm{d}t}{1+t} \lesssim \log(n+1),$$

$$\|q_2^{(n)}\|_{L^1[0,+\infty)} = n^2\int_0^{t_*}\overline{Q}_2^{(n)}(n^2t)\mathrm{d}t \lesssim \int_0^{n^2t_*}\frac{\mathrm{d}t}{(1+t)^{3/2}} \lesssim 1, \qquad n \geqslant 1.$$

In order to estimate $|\overline{p}_{0,n}^{\mathrm{diff}}|$, recall Proposition 4.1: using (4.1) and the Young inequality for convolution we obtain

$$\|\mathrm{II}_n\|^2_{L^2[0,+\infty)} \leqslant \big\|q_1^{(n)}\big\|^2_{L^1[0,+\infty)}\big\|\overline{p}_{0,n}^{\mathrm{diff}}\mathbf{1}_{[0,t_*]}\big\|^2_{L^2[0,+\infty)} \lesssim \frac{\log^2(n+1)}{n^2},$$

and

$$\|\mathrm{III}_n\|^2_{L^2[0,+\infty)} \leqslant \big\|q_2^{(n)}\big\|^2_{L^1[0,+\infty)}\big\|\overline{p}_{0,n}^{\mathrm{sum}}\mathbf{1}_{[0,t_*]}\big\|^2_{L^2[0,+\infty)} \lesssim \frac{\log(n+1)}{n^2}.$$

Thus, the conclusion of Lemma 5.4 follows. $\square$

6.5. **Proof of Lemma 5.2.** To prove Lemma 5.2 we need to get an expression for

$$\sum_{x=0}^{n-1}\big(\overline{r}_{x+1}^{(n)}(t) - \overline{r}_x^{(n)}(t)\big)^2.$$

Using (6.1) and then summing by parts we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\frac{1}{n+1}\sum_{x\in\mathbb{I}_n}\overline{r}_x^2(t)\Bigg) = \frac{n^2}{n+1}\sum_{x=1}^{n}\overline{r}_x(t)\big(\overline{p}_x(t) - \overline{p}_{x-1}(t)\big)$$

$$= \frac{n^2}{n+1}\sum_{x=1}^{n-1}\overline{p}_x(t)\big(\overline{r}_x(t) - \overline{r}_{x+1}(t)\big) + \frac{n^2}{n+1}\overline{p}_n(t)\overline{r}_n(t) - \frac{n^2}{n+1}\overline{p}_0(t)\overline{r}_1(t).$$

Computing $\overline{p}_x(t)$ from (6.2) we can rewrite the utmost right hand side as

$$-\frac{n^2}{2\gamma(n+1)}\sum_{x=1}^{n-1}\big(\overline{r}_x(t)-\overline{r}_{x+1}(t)\big)^2-\frac{1}{2\gamma(n+1)}\sum_{x=1}^{n-1}\frac{\mathrm{d}\overline{p}_x(t)}{\mathrm{d}t}\big(\overline{r}_x(t)-\overline{r}_{x+1}(t)\big)$$
$$+\frac{n^2}{n+1}\overline{p}_n(t)\overline{r}_n(t)-\frac{n^2}{n+1}\overline{p}_0(t)\overline{r}_1(t).$$

Therefore, after integration by parts in the temporal variable, we get

$$\frac{n^2}{2\gamma(n+1)}\int_0^t\sum_{x=1}^{n-1}\big(\overline{r}_x(s)-\overline{r}_{x+1}(s)\big)^2\mathrm{d}s=\frac{1}{n+1}\sum_{x\in\mathbb{I}_n}\big(\overline{r}_x^2(t)-\overline{r}_x^2(0)\big)$$
$$-\frac{n^2}{2\gamma(n+1)}\int_0^t\sum_{x=1}^{n-1}\overline{p}_x(s)\big(\overline{p}_{x+1}(s)+\overline{p}_{x-1}(s)-2\overline{p}_x(s)\big)\mathrm{d}s$$
$$-\frac{1}{2\gamma(n+1)}\left(\sum_{x=1}^{n-1}\overline{p}_x(t)\big(\overline{r}_x(t)-\overline{r}_{x+1}(t)\big)-\sum_{x=1}^{n-1}\overline{p}_x(0)\big(\overline{r}_x(0)-\overline{r}_{x+1}(0)\big)\right)$$
$$+\frac{n^2}{n+1}\int_0^t\overline{p}_n(s)\overline{r}_n(s)\mathrm{d}s-\frac{n^2}{n+1}\int_0^t\overline{p}_0(s)\overline{r}_1(s)\mathrm{d}s.$$

Using (4.4) from Proposition 4.4 we conclude that the first and third expressions in the right hand side stay bounded, as $n\geqslant 1$. Summing by parts we conclude that the second expression equals

$$\frac{n^2}{2\gamma(n+1)}\int_0^t\sum_{x=1}^{n-2}\big(\overline{p}_{x+1}(s)-\overline{p}_x(s)\big)^2\mathrm{d}s$$
$$+\frac{n^2}{2\gamma(n+1)}\int_0^t\Big(\overline{p}_1(s)\big(\overline{p}_1(s)-\overline{p}_0(s)\big)-\overline{p}_{n-1}(s)\big(\overline{p}_n(s)-\overline{p}_{n-1}(s)\big)\Big)\mathrm{d}s.$$

The expression stays bounded, due to Proposition 4.4–(4.5).

Multiplying (6.3) by $\overline{p}_0(t)$ and integrating we get

$$\frac{1}{2n}\big(\overline{p}_0^2(t)-\overline{p}_0^2(0)\big)=n\int_0^t\overline{r}_1(s)\overline{p}_0(s)\mathrm{d}s-n(2\gamma+\widetilde{\gamma})\int_0^t\overline{p}_0^2(s)\mathrm{d}s. \tag{6.54}$$

From here, thanks to Proposition 4.4–(4.5) (which controls $\int\overline{p}_0^2$) and thanks to Proposition 4.5 (which controls the left hand side of (6.54) since $\overline{p}_0^2(t)\leqslant 2\overline{\mathcal{E}}_0(t)$ by convexity), we conclude that

$$n\left|\int_0^t\overline{r}_1(s)\overline{p}_0(s)\mathrm{d}s\right|\lesssim 1.$$

Multiplying (6.4) by $\overline{p}_n(t)$ and integrating we get

$$\frac{1}{2n}\big(\overline{p}_n^2(t)-\overline{p}_n^2(0)\big)=n\overline{\tau}_+\int_0^t\overline{p}_n(s)\mathrm{d}s-\ n\int_0^t\overline{r}_n(s)\overline{p}_n(s)\mathrm{d}s-n(2\gamma+\widetilde{\gamma})\int_0^t\overline{p}_n^2(s)\mathrm{d}s.$$

From here, thanks to Proposition 4.4–(4.5), Proposition 4.2 and Proposition 4.5 we conclude that

$$n\left|\int_0^t\overline{r}_n(s)\overline{p}_n(s)\mathrm{d}s\right|\lesssim 1.$$

This ends the proof of (5.20). □

7. Entropy production

This section is mainly devoted to proving Proposition 4.5, which will be concluded in Section 7.2. For that purpose, we will obtain an *entropy production bound* stated in Proposition 7.1 below, in Section 7.1. Its proof uses Proposition 4.2. This new result, together with the estimates given in Proposition 4.2 and Proposition 4.4, will also allow us to conclude the proof of Lemma 5.1 (which give all boundary estimates), in Section 7.3.

7.1. Entropy production bound.

To prove Proposition 4.5, we need to show that

$$\sup_{s\in[0,t]} \sum_{x\in\mathbb{I}_n} \mathbb{E}_n\big[\mathcal{E}_x(s)\big]$$

grows at most linearly in n. We first relate this quantity to the *entropy production*, as follows: recall that $f_n(t)$ is the density of the distribution $\mu_n(t)$ of $(\mathbf{r}(t),\mathbf{p}(t))$ with respect to ν_T, see (3.1). We denote the expectation with respect to ν_T by $\langle\!\langle\cdot\rangle\!\rangle_T$. Given a density $F\in L^2(\nu_T)$ we define the relative entropy $H_{n,T}[F]$ of the measure $\mathrm{d}\mu := F\mathrm{d}\nu_T$, with respect to ν_T by

$$H_{n,T}[F] := \langle\!\langle F\log F\rangle\!\rangle_T = \int_{\Omega_n} F\log F \mathrm{d}\nu_T. \tag{7.1}$$

We interpret $F\log F = 0$, whenever $F = 0$. Finally, we denote

$$\mathbf{H}_{n,T}(t) := H_{n,T}[f_n(t)]. \tag{7.2}$$

Then, by virtue of the entropy inequality, see e.g. [9, p. 338] (and also (7.18) below), and from our assumption (3.7) on the initial condition, we conclude that: for any $\alpha>0$ we can find $C_\alpha>0$ such that

$$\mathbb{E}_n\Bigg[\sum_{x\in\mathbb{I}_n}\mathcal{E}_x(t)\Bigg] \leqslant \frac{1}{\alpha}\big(C_\alpha n + \mathbf{H}_{n,T}(t)\big), \qquad t\geqslant 0. \tag{7.3}$$

This reduces the problem to showing a linear bound on $\mathbf{H}_{n,T}(t)$, which is the main result of this section.

Proposition 7.1. *Under Assumptions 3.1, for any $t, T>0$ we have*

$$\sup_{s\in[0,t]} \mathbf{H}_{n,T}(s) \lesssim n, \qquad n\geqslant 1. \tag{7.4}$$

In order to prove Proposition 7.1 (which will be achieved in Section 7.1.3), we first introduce another relative entropy which takes into account the boundary temperatures fixed at T_- and T_+ and explains how relate them to each other.

7.1.1. *Relative entropy of an inhomogeneous product measure.*

Recall the definition of the non-homogeneous product measure $\widetilde{\nu}$ given in (3.3) and of the density $\widetilde{f}_n(t)$ given in (3.5). The relative entropies $\widetilde{\mathbf{H}}_n(t)$ (defined in (3.6)) and $\mathbf{H}_{n,T}(t)$ (defined in (7.1)) are related by the following formula.

Proposition 7.2. *For any $T > 0$ and $n \geqslant 1$ we have*

$$\mathbf{H}_{n,T}(t) = \widetilde{\mathbf{H}}_n(t) - \int_{\Omega_n} \sum_{x\in\mathbb{I}_n} \left(\left(\beta_x - T^{-1}\right) \mathcal{E}_x - \beta_x \overline{\tau}_+ r_x \right) \widetilde{f}_n(t) \mathrm{d}\widetilde{\nu} \\ - \sum_{x=1}^{n} \left(\mathcal{G}(\beta_x, \overline{\tau}_+) - \mathcal{G}(T^{-1}, 0) \right) - \frac{1}{2} \log\left(T_- T^{-1} \right), \quad t \geqslant 0. \tag{7.5}$$

In addition, for any $t_* > 0$

$$\mathbf{H}_{n,T}(t) \lesssim \widetilde{\mathbf{H}}_n(t) + n, \qquad n \geqslant 1,\, t \in [0, t_*]. \tag{7.6}$$

Proof. Formula (7.5) can be obtained by a direct calculation. To prove the bound (7.6) note first that one can choose a sufficiently small $\alpha > 0$ so that

$$\sup_{n\geqslant 1} \frac{1}{n} \log\left\{ \int_{\Omega_n} \exp\left\{ -\alpha \sum_{x\in\mathbb{I}_n} \left(\left(\beta_x - T^{-1}\right) \mathcal{E}_x - \beta_x \overline{\tau}_+ r_x \right) \right\} \widetilde{\nu}(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) \right\} =: C_\alpha < +\infty. \tag{7.7}$$

From the entropy inequality we can write

$$-\sum_{x\in\mathbb{I}_n} \int_{\Omega_n} \left(\left(\beta_x - T^{-1}\right) \mathcal{E}_x - \beta_x \overline{\tau}_+ r_x \right) \widetilde{f}_n(t) \mathrm{d}\widetilde{\nu} \leqslant \frac{1}{\alpha} \left(C_\alpha n + \widetilde{\mathbf{H}}_n(t) \right).$$

Thus (7.6) follows from (7.5). □

7.1.2. *Estimate of $\widetilde{\mathbf{H}}_n(t)$.* Next step consists in estimating $\widetilde{\mathbf{H}}_n(t)$ by computing its derivative. Using the regularity theory for solutions of stochastic differential equations and Duhamel formula, see e.g. Section 8 of [3], we can argue that $\widetilde{f}_n(t, \mathbf{r}, \mathbf{p})$ is twice continuously differentiable in $(\mathbf{r}, \mathbf{p})$ and once in t, provided that $\widetilde{f}_n(0) \in C^2(\Omega_n)$, which is the case, due to (3.7). Using the dynamics (2.4)–(2.5) we therefore obtain:

$$\widetilde{\mathbf{H}}_n'(t) = -\left(T_+^{-1} - T_-^{-1} \right) n \sum_{x=0}^{n-1} \mathbb{E}_n\Big[j_{x,x+1}(t) \Big] + n^2 T_+^{-1} \overline{\tau}_+ \overline{p}_n^{(n)}(t) - n^2 \mathbf{D}(\widetilde{f}_n(t)), \tag{7.8}$$

where $j_{x,x+1}(t) := j_{x,x+1}(\mathbf{r}(t), \mathbf{p}(t))$, with $j_{x,x+1}$ given in (5.2), and the operator $\mathbf{D}$ is defined for any $F \geqslant 0$ such that $F \log F \in L^1(\widetilde{\nu})$ and $(\nabla_{\mathbf{p}} F)^{1/2} \in L^2(\widetilde{\nu})$, by

$$\mathbf{D}(F) := \gamma \sum_{x\in\mathbb{I}_n} \mathcal{D}_{x,\beta}(F) + \widetilde{\gamma} \sum_{x=0,n} T_x \int_{\Omega_n} \frac{\left(\partial_{p_x} F\right)^2}{F} \mathrm{d}\widetilde{\nu}, \tag{7.9}$$

with

$$\mathcal{D}_{x,\beta}(F) := -\int_{\Omega_n} F(\mathbf{r}, \mathbf{p}) \log\left(\frac{F(\mathbf{r}, \mathbf{p}^x)}{F(\mathbf{r}, \mathbf{p})} \right) \mathrm{d}\widetilde{\nu}. \tag{7.10}$$

It is standard to show, using the inequality $a \log(b/a) \leqslant 2\sqrt{a}(\sqrt{b} - \sqrt{a})$ for any $a, b > 0$, that: for any positive, measurable function F on Ω_n, and any $x \in \mathbb{I}_n$,

$$\mathcal{D}_{x,\beta}(F) \geqslant \int_{\Omega_n} \left(F(\mathbf{r}, \mathbf{p}^x) - F(\mathbf{r}, \mathbf{p}) \right)^2 \mathrm{d}\widetilde{\nu} \geqslant 0. \tag{7.11}$$

The main result of this section is the following:

Proposition 7.3 (Entropy production)**.** *For any* $t > 0$ *we have*

$$\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s \lesssim \frac{1}{n^2}\left(\widetilde{\mathbf{H}}_n(0) + n\right) \tag{7.12}$$

and

$$\sup_{s\in[0,t]} \widetilde{\mathbf{H}}_n(s) \lesssim \widetilde{\mathbf{H}}_n(0) + n, \qquad n \geqslant 1. \tag{7.13}$$

Proof. From (7.8) we get

$$\widetilde{\mathbf{H}}_n(t) = \widetilde{\mathbf{H}}_n(0) + \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n, \tag{7.14}$$

where

$$\begin{aligned}
\mathrm{I}_n &:= -\left(T_+^{-1} - T_-^{-1}\right) n \sum_{x=0}^{n-1} \int_0^t \mathbb{E}_n\left[j_{x,x+1}(s)\right]\mathrm{d}s,\\
\mathrm{II}_n &:= n^2 T_+^{-1}\overline{\tau}_+ \int_0^t \overline{p}_n^{(n)}(s)\mathrm{d}s,\\
\mathrm{III}_n &:= -n^2 \int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s \leqslant 0,
\end{aligned}$$

where the last inequality follows from (7.9) and (7.11). We now estimate I_n, II_n.

(i) Estimates of I_n*.* Recall the fluctuation-dissipation relation (5.36) and recall also the notation $g_x(t) := g_x(\mathbf{r}(t), \mathbf{p}(t))$ (and similarly for other local functions). We can write

$$|\mathrm{I}_n| \leqslant \mathrm{I}_{n,1} + \mathrm{I}_{n,2} + \mathrm{I}_{n,3},$$

where

$$\mathrm{I}_{n,1} := \frac{1}{n}\left|T_+^{-1} - T_-^{-1}\right| \left|\sum_{x=1}^{n-1} \mathbb{E}_n\left[g_x(t) - g_x(0)\right]\right|, \tag{7.15}$$

$$\mathrm{I}_{n,2} := \left|T_+^{-1} - T_-^{-1}\right| n \left|\int_0^t \mathbb{E}_n\left[V_n(s) - V_1(s)\right]\mathrm{d}s\right|, \tag{7.16}$$

$$\mathrm{I}_{n,3} := \left|T_+^{-1} - T_-^{-1}\right| n \left|\int_0^t \mathbb{E}_n\left[j_{0,1}(s)\right]\mathrm{d}s\right|. \tag{7.17}$$

To deal with $\mathrm{I}_{n,1}$ we invoke the entropy inequality: for any $\alpha > 0$ we have

$$\int_{\Omega_n} \Big(\sum_{x\in\mathbb{I}_n} \mathcal{E}_x\Big)\widetilde{f}_n(t)\mathrm{d}\widetilde{\nu} \leqslant \frac{1}{\alpha}\left\{\log\left(\int \exp\left\{\frac{\alpha}{2}\sum_{x\in\mathbb{I}_n}(p_x^2 + r_x^2)\right\}\mathrm{d}\widetilde{\nu}\right) + \widetilde{\mathbf{H}}_n(t)\right\} \tag{7.18}$$

for any $t \geqslant 0$. Recalling the definition of g_x given in (5.37) and choosing $\alpha > 0$ sufficiently small it allows us to estimate

$$\mathrm{I}_{n,1} \lesssim \frac{1}{n}\sum_{x\in\mathbb{I}_n} \overline{\mathcal{E}}_x(0) + \frac{1}{n}\sum_{x\in\mathbb{I}_n} \overline{\mathcal{E}}_x(t) \lesssim \frac{1}{n}\sum_{x\in\mathbb{I}_n} \overline{\mathcal{E}}_x(0) + 1 + \frac{1}{n}\widetilde{\mathbf{H}}_n(t), \qquad n \geqslant 1,\ t \geqslant 0.$$

To deal with $\mathrm{I}_{n,2}$, which involves boundary terms, we shall need some auxiliary estimates.

Lemma 7.4. *For any $t_* \geqslant 0$ we have: for $n \geqslant 1$, $t \in [0, t_*]$,*

$$\mathbb{E}_n\Big[\int_0^t \big(p_1^2(s) + p_n^2(s)\big)\mathrm{d}s\Big] \lesssim 1 + \int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s \tag{7.19}$$

and

$$\mathbb{E}_n\Big[\int_0^t \big(r_1^2(s) + r_n^2(s)\big)\mathrm{d}s\Big] \lesssim 1 + \frac{1}{n^2}\widetilde{\mathbf{H}}_n(t) + \int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s + \Big\{1 + \int_0^t \widetilde{\mathbf{H}}_n(s)\mathrm{d}s\Big\}^{1/2}\Big\{\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s\Big\}^{1/2}. \tag{7.20}$$

Proof. Note that

$$\mathbb{E}_n\big[p_n^2(s)\big] = \frac{1}{T_+}\int_{\Omega_n} \widetilde{f}_n(s)\mathrm{d}\widetilde{\nu} + \frac{1}{T_+}\int_{\Omega_n} p_n\partial_{p_n}\widetilde{f}_n(s)\mathrm{d}\widetilde{\nu}$$

Therefore, by the Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^t \mathbb{E}_n\big[p_n^2(s)\big]\mathrm{d}s &\lesssim 1 + \left|\int_0^t \int_{\Omega_n} p_n\,(\widetilde{f}_n(s))^{1/2}\frac{\partial_{p_n}\widetilde{f}_n(s)}{(\widetilde{f}_n(s))^{1/2}}\,\mathrm{d}\widetilde{\nu}\mathrm{d}s\right| \\ &\leqslant 1 + \Big\{\int_0^t \int_{\Omega_n} p_n^2\widetilde{f}_n(s)\mathrm{d}\widetilde{\nu}\mathrm{d}s\Big\}^{1/2}\Big\{\int_0^t \int_{\Omega_n} \frac{(\partial_{p_n}\widetilde{f}_n(s))^2}{\widetilde{f}_n(s)}\mathrm{d}\widetilde{\nu}\mathrm{d}s\Big\}^{1/2} \\ &\leqslant 1 + \Big\{\int_0^t \mathbb{E}_n\big[p_n^2(s)\big]\mathrm{d}s\Big\}^{1/2}\Big\{\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s\Big\}^{1/2}.\end{aligned}$$

A similar calculation can be made at the left boundary point. This yields (7.19).

To prove (7.20) note that, by (2.4)–(2.5),

$$\begin{aligned}&n^{-2}\mathbb{E}_n\big[p_n(t)r_n(t) - p_n(0)r_n(0)\big] \\ &= \int_0^t \mathbb{E}_n\big[p_n(s)(p_n(s) - p_{n-1}(s)) + (\overline{\tau}_+ - r_n(s))r_n(s) - (\widetilde{\gamma} + 2\gamma)p_n(s)r_n(s)\big]\mathrm{d}s.\end{aligned} \tag{7.21}$$

Thanks to the assumption on the initial energy bound (3.9) we have

$$\begin{aligned}\mathbb{E}_n\Big[\int_0^t r_n^2(s)\mathrm{d}s\Big] &\lesssim \int_0^t \mathbb{E}_n\big[p_n^2(s)\big]\mathrm{d}s + \left|\int_0^t \mathbb{E}_n\big[p_n(s)p_{n-1}(s)\big]\mathrm{d}s\right| \\ &+ \Big\{\int_0^t \mathbb{E}\big[r_n^2(s)\big]\mathrm{d}s\Big\}^{1/2} + (\widetilde{\gamma} + 2\gamma)\left|\int_0^t \mathbb{E}_n\big[p_n(s)r_n(s)\big]\mathrm{d}s\right| \\ &+ n^{-2}\mathbb{E}_n\big[p_n^2(t) + r_n^2(t)\big] + \frac{1}{n}, \qquad n \geqslant 1, t \geqslant 0.\end{aligned} \tag{7.22}$$

Similarly we get

$$\left|\int_0^t \mathbb{E}_n\big[p_n(s)r_n(s)\big]\mathrm{d}s\right| \lesssim \Big\{\int_0^t \mathbb{E}_n\big[r_n^2(s)\big]\mathrm{d}s\Big\}^{1/2}\Big\{\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s\Big\}^{1/2} \tag{7.23}$$

and

$$\left|\int_0^t \mathbb{E}_n\big[p_n(s)p_{n-1}(s)\big]\mathrm{d}s\right| \lesssim \left\{\int_0^t \mathbb{E}_n\big[p_{n-1}^2(s)\big]\mathrm{d}s\right\}^{1/2}\left\{\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s\right\}^{1/2}$$

for $n \geqslant 1$, $t \geqslant 0$. Using the entropy inequality the last bound leads to

$$\left|\int_0^t \mathbb{E}_n\big[p_n(s)p_{n-1}(s)\big]\mathrm{d}s\right| \lesssim \left\{1+\int_0^t \widetilde{\mathbf{H}}_n(s)\mathrm{d}s\right\}^{1/2}\left\{\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s\right\}^{1/2}, \quad (7.24)$$

for $n \geqslant 1$, $t \geqslant 0$. By the entropy inequality we also get

$$n^{-2}\mathbb{E}_n\Big[p_n^2(t)+r_n^2(t)\Big] \lesssim \frac{1}{n^2}\Big(1+\widetilde{\mathbf{H}}_n(t)\Big) \quad (7.25)$$

Substituting these bounds into (7.22) we conclude (7.20). The proofs for the case of the left boundary point are analogous. □

Returning to the proof of Proposition 7.3, with the help of Lemma 7.4, we get

$$\begin{aligned} \mathrm{I}_{n,2} \lesssim\, & n + n\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s + \frac{1}{n}\widetilde{\mathbf{H}}_n(t) \\ & + n\left\{1+\int_0^t \widetilde{\mathbf{H}}_n(s)\mathrm{d}s\right\}^{1/2}\left\{\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s\right\}^{1/2}, \qquad n \geqslant 1,\ t \geqslant 0. \end{aligned}$$

By an application of the Young inequality we conclude that for any $\alpha > 0$ we can choose $C > 0$ so that

$$\mathrm{I}_{n,2} \lesssim Cn+\alpha n^2\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s+\frac{C}{n}\widetilde{\mathbf{H}}_n(t)+C\int_0^t \widetilde{\mathbf{H}}_n(s)\mathrm{d}s, \qquad n \geqslant 1,\ t \geqslant 0. \quad (7.26)$$

An analogous bound holds for $\mathrm{I}_{n,3}$. Therefore we conclude that for any $\alpha > 0$,

$$\mathrm{I}_n \lesssim n + \alpha n^2\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s + \frac{1}{n}\widetilde{\mathbf{H}}_n(t) + \int_0^t \widetilde{\mathbf{H}}_n(s)\mathrm{d}s, \qquad n \geqslant 1,\ t \geqslant 0. \quad (7.27)$$

(ii) Estimates of II_n*.* To estimate II_n we need Proposition 4.2. Thanks to (4.3) we conclude that

$$|\mathrm{II}_n| = n^2T_+^{-1}|\overline{\tau}_+|\left|\int_0^t \overline{p}_n^{(n)}(s)\mathrm{d}s\right| \lesssim n, \quad n \geqslant 1. \quad (7.28)$$

Choosing α sufficiently small in (7.27) and substituting from (7.27) and (7.28) into (7.8) we conclude that there exists $c > 0$, for which

$$\widetilde{\mathbf{H}}_n(t) \lesssim \widetilde{\mathbf{H}}_n(0) + n + \int_0^t \widetilde{\mathbf{H}}_n(s)\mathrm{d}s - cn^2\int_0^t \mathbf{D}(\widetilde{f}_n(s))\mathrm{d}s. \quad (7.29)$$

This, by an application of the Gronwall inequality, in particular implies that

$$\widetilde{\mathbf{H}}_n(t) \lesssim \widetilde{\mathbf{H}}_n(0) + n. \quad (7.30)$$

Thus (7.13) follows. Estimate (7.12) is an easy consequence of (7.29) and (7.13). □

7.1.3. *Proof of Proposition 7.1.* From the assumption (3.7), one has

$$\widetilde{\mathbf{H}}_n(0) \lesssim n, \qquad n \geqslant 1.$$

Therefore, Proposition 7.1 directly follows from (7.6) and (7.13).

7.2. **The end of the proof of Proposition 4.5.** Proposition 4.5 directly follows from the entropy inequality (7.18) and bound (7.6).

7.3. **Boundary estimates: proof of Lemma 5.1.** The entropy production bound from Proposition 7.3 is also crucial in order to get information on the behavior of boundary quantities. We prove here all the estimates of Lemma 5.1.

Proof of Lemma 5.1, estimate (5.5). We start with the right boundary point $x = n$. The proof for $x = 1$ is similar. Using the definition (3.3) we can write

$$\begin{aligned}\left|\int_0^t \mathbb{E}_n\big[p_n(s)p_{n-1}(s)\big]\mathrm{d}s\right| &= \left|\int_0^t \mathrm{d}s \int_{\Omega_n} p_n p_{n-1} \widetilde{f}_n(s)\mathrm{d}\widetilde{\nu}\right| \\ &= T_+^{-1}\left|\int_0^t \mathrm{d}s \int_{\Omega_n} p_{n-1}(\widetilde{f}_n(s))^{1/2} \frac{\partial_{p_n}\widetilde{f}_n(s)}{(\widetilde{f}_n(s))^{1/2}}\mathrm{d}\widetilde{\nu}\right| \\ &\leqslant T_+^{-1}\left\{\int_0^t \mathrm{d}s \int_{\Omega_n} p_{n-1}^2 \widetilde{f}_n(s)\mathrm{d}\widetilde{\nu}\right\}^{1/2}\left\{\int_0^t \mathrm{d}s \int_{\Omega_n} \frac{(\partial_{p_n}\widetilde{f}_n(s))^2}{\widetilde{f}_n(s)}\mathrm{d}\widetilde{\nu}\right\}^{1/2}. \end{aligned} \tag{7.31}$$

Invoking the entropy production bound (7.12) we conclude that

$$\left|\int_0^t \mathbb{E}_n\big[p_n(s)p_{n-1}(s)\big]\mathrm{d}s\right| \lesssim \frac{1}{n}\left\{\mathbb{E}_n\left[\int_0^t p_{n-1}^2(s)\mathrm{d}s\right]\right\}^{1/2} \lesssim \frac{1}{\sqrt{n}}, \qquad n \geqslant 1, \tag{7.32}$$

in light of the energy estimate (4.7) of Proposition 4.5, which is now proved. □

Proof of Lemma 5.1, estimate (5.6). To show (5.6) note that

$$\begin{aligned}\left|\mathbb{E}_n\left[\int_0^t r_n(s)p_n(s)\mathrm{d}s\right]\right| &= \left|\int_0^t \mathrm{d}s \int r_n \partial_{p_n}\widetilde{f}_n(s)\mathrm{d}\widetilde{\nu}\right| \qquad (7.33)\\ &\leqslant \left\{\int_0^t \mathrm{d}s \int r_n^2 \widetilde{f}_n(s)\mathrm{d}\widetilde{\nu}\right\}^{1/2}\left\{\int_0^t \mathrm{d}s \int \frac{\big(\partial_{p_n}\widetilde{f}_n(s)\big)^2}{\widetilde{f}_n(s)}\mathrm{d}\widetilde{\nu}\right\}^{1/2} \\ &\lesssim \frac{1}{\sqrt{n}}\left\{\mathbb{E}_n\left[\int_0^t r_n^2(s)\mathrm{d}s\right]\right\}^{1/2} \lesssim \frac{1}{\sqrt{n}}, \end{aligned}$$

in light of Proposition 4.5–(4.7). The proof for the left boundary is similar. □

Proof of Lemma 5.1, estimate (5.7). From the time evolution of the dynamics (2.5) (see also (6.4)) we obtain for $x = n$

$$\frac{1}{n^2}\Big(\overline{p}_n^{(n)}(t) - \overline{p}_n^{(n)}(0)\Big) = \int_0^t \Big(\overline{\tau}_+ - \overline{r}_n^{(n)}(s)\Big)\mathrm{d}s - (2\gamma + \widetilde{\gamma})\int_0^t \overline{p}_n^{(n)}(s)\mathrm{d}s. \tag{7.34}$$

Using the energy bound (4.7) we conclude that the right hand side is of order of magnitude $n^{-3/2}$ as $n \to +\infty$. Thanks to Proposition 4.2 we conclude (5.7). The proof for $x = 1$ is analogous. □

Proof of Lemma 5.1, estimate (5.8). We show the proof for $x = n$ only, as the argument for $x = 0$ is analogous. Note that,

$$\mathbb{E}_n\left[\int_0^t \left(p_n^2(s) - T_+\right) \mathrm{d}s\right] = T_+ \int_0^t \mathrm{d}s \int_{\Omega_n} p_n \partial_{p_n} \widetilde{f}_n(s) \mathrm{d}\widetilde{\nu}.$$

Thus, by the Cauchy-Schwarz inequality

$$\left|\mathbb{E}_n\left[\int_0^t \left(p_n^2(s) - T_+\right) \mathrm{d}s\right]\right|$$

$$\leqslant T_+ \left\{\int_0^t \mathrm{d}s \int_{\Omega_n} p_n^2 \widetilde{f}_n(s) \mathrm{d}\widetilde{\nu}\right\}^{1/2} \left\{\int_0^t \mathrm{d}s \int_{\Omega_n} \frac{\left(\partial_{p_n} \widetilde{f}_n(s)\right)^2}{\widetilde{f}_n(s)} \mathrm{d}\widetilde{\nu}\right\}^{1/2}. \quad (7.35)$$

From (7.21), Lemma 5.1–(5.5) and (5.6), which have been proved above, we get

$$\left|\mathbb{E}_n\left[\int_0^t \left(r_n^2(s) - \overline{\tau}_+^2 - p_n^2(s)\right) \mathrm{d}s\right]\right| \lesssim \frac{1}{\sqrt{n}}, \quad n \geqslant 1. \quad (7.36)$$

Then, by Lemma 5.1–(5.9) and (7.36), for any $t > 0$

$$\int_0^t \mathrm{d}s \int_{\Omega_n} p_n^2 \widetilde{f}_n(s) \mathrm{d}\widetilde{\nu} \lesssim 1, \quad n \geqslant 1. \quad (7.37)$$

Using this estimate together with Proposition 4.5–(4.7) we conclude (5.8). □

Proof of Lemma 5.1, estimate (5.9). From (7.21) and Lemma 5.1–(5.5) we get

$$\int_0^t \mathbb{E}_n\left[r_n^2(s)\right] \mathrm{d}s \leqslant \int_0^t \mathbb{E}_n\left[p_n^2(s)\right] \mathrm{d}s + |\overline{\tau}_+| \left|\int_0^t \mathbb{E}_n\left[r_n(s)\right] \mathrm{d}s\right|$$

$$+ (\widetilde{\gamma} + 2\gamma) \left|\int_0^t \mathbb{E}_n\left[p_n(s) r_n(s)\right] \mathrm{d}s\right| + o_n(1), \quad (7.38)$$

where $o_n(1) \to 0$, as $n \to +\infty$. Using the Young inequality we conclude that

$$\mathbb{E}_n\left[\int_0^t r_n^2(s) \mathrm{d}s\right] \lesssim \mathbb{E}_n\left[\int_0^t p_n^2(s) \mathrm{d}s\right] + 1, \quad n \geqslant 1. \quad (7.39)$$

We use Lemma 5.1–(5.8) to conclude that

$$\mathbb{E}_n\left[\int_0^t r_n^2(s)\, \mathrm{d}s\right] \lesssim 1, \quad n \geqslant 1.$$

An analogous estimate on $\mathbb{E}_n\left[\int_0^t r_1^2(s)\, \mathrm{d}s\right]$ follows from the same argument, using the relation

$$n^{-2} L(p_0 r_1) = (p_1 - p_0) p_0 + r_1^2 - (\widetilde{\gamma} + 2\gamma) p_0 r_1 \quad (7.40)$$

and the entropy production bound at $x = 0$. □

Proof of Lemma 5.1, estimate (5.10). For the right boundary current $j_{n-1,n}$, the equality follows from the definition (5.4), thanks to: Proposition 4.2, Lemma 5.1–(5.8), and the energy estimate (4.7). An analogous argument, using (5.3) instead, works for left boundary current. □

Proof of Lemma 5.1, estimate (5.11)–(5.12). From (7.21), combined with the above (7.36), and Lemma 5.1–(5.8), the result follows. □

Proof of Lemma 5.1, estimate (5.13). The derivative of $\mathbf{H}_{n,T}(t)$ can be computed similarly to (7.8) as

$$\mathbf{H}'_{n,T}(t) = -n^2\mathbf{D}_T(f_n(t)) - n^2\sum_{x=0,n}\left(T^{-1} - T_x^{-1}\right)\left(T_x - \int_{\Omega_n} p_x^2(t) f_n(t)\mathrm{d}\nu_T\right) + n^2 T^{-1}\overline{\tau}_+\overline{p}_n^{(n)}(t), \quad (7.41)$$

where

$$\mathbf{D}_T(f_n(t)) := \gamma\sum_{x\in\mathbb{I}_n}\mathcal{D}_{x,T}(f_n(t)) + \widetilde{\gamma}\sum_{x=0,n} T_x \int_{\Omega_n}\frac{[\partial_{p_x}(f_n(t)/h_{x,T})]^2}{f_n(t)/h_{x,T}}\mathrm{d}\nu_{T_x}$$

and $h_{x,T} = g_{T_x}/g_T$. Using (7.41) we conclude that for any $T > 0$

$$n^2\sum_{x=0,n}\left(T^{-1} - T_x^{-1}\right)\int_0^t\left(T_x - \int_{\Omega_n} p_x^2(s) f_n(s)\mathrm{d}\nu_T\right)\mathrm{d}s + n^2\int_0^t \mathbf{D}_T(f_n(s))\mathrm{d}s + \mathbf{H}_{n,T}(t) = \mathbf{H}_{n,T}(0) + n^2 T^{-1}\int_0^t \overline{\tau}_+\overline{p}_n^{(n)}(s)\mathrm{d}s. \quad (7.42)$$

Since the entropy $\mathbf{H}_{n,T}(t)$ and the form $\mathbf{D}_T(f_n(s))$ are both non-negative (from a similar argument as in (7.11)), and the right hand side of (7.42) grows at most linearly in n (from Proposition 4.2) we conclude

$$\sum_{x=0,n}\left(T^{-1} - T_x^{-1}\right)\int_0^t\left(T_x - \mathbb{E}_n\left[p_x^2(s)\right]\right)\mathrm{d}s \lesssim \frac{1}{n}. \quad (7.43)$$

Let $T^{-1} := T_0^{-1} + T_n^{-1}$. From (7.43) we get

$$(T_0 T_n)^{-1}\sum_{x=0,n} T_x\int_0^t\left(T_x - \mathbb{E}_n\left[p_x^2(s)\right]\right)\mathrm{d}s \lesssim \frac{1}{n} \quad (7.44)$$

and (5.13) follows. □

8. Energy balance identity and equipartition

The main result which is left to be proved is Proposition 4.6, which describes an equipartition phenomenon between the mechanical and thermal energies. To prove that result, we will use the *Fourier-Wigner distributions* which permit to control the energy profiles over various frequency modes, and have been successfully used in previous works. The major difficulty here is the presence of boundary terms, which all need to be controlled. In Section 8.1 we introduce definitions and write down the evolution equation satisfied by wave functions. In Section 8.2 we obtain an *energy balance identity* (Proposition 8.1). The proof of Proposition 4.6 is achieved in Section 8.3.

8.1. **The wave and Wigner functions.** In the present section we restore the superscript n when referring to the mean and fluctuation of the stretch and momentum. We define the *fluctuating wave function* as

$$\widetilde{\psi}_x^{(n)}(t) = \widetilde{r}_x^{(n)}(t) + i\widetilde{p}_x^{(n)}(t), \qquad x \in \mathbb{I}_n, t \geqslant 0, \tag{8.1}$$

and its Fourier transform,

$$\widehat{\widetilde{\psi}}^{(n)}(t,k) = \widehat{\widetilde{r}}^{(n)}(t,k) + i\widehat{\widetilde{p}}^{(n)}(t,k), \qquad k \in \widehat{\mathbb{I}}_n, t \geqslant 0. \tag{8.2}$$

The wave function extends to a periodic function on $\frac{1}{n+1}\mathbb{Z}$, by letting $\widehat{\widetilde{\psi}}^{(n)}(k+1) = \widehat{\widetilde{\psi}}^{(n)}(k)$ for any $k \in \widehat{\mathbb{I}}_n$. In particular $\widehat{\widetilde{\psi}}^{(n)}\left(k + \frac{\eta}{n+1}\right)$ is well defined for any $\eta \in \mathbb{Z}$. Then for $k \in \widehat{\mathbb{I}}_n, \eta \in \mathbb{Z}, t \geqslant 0$ we define the Fourier-Wigner functions:

$$\begin{aligned}
\widetilde{W}_n^+(t,\eta,k) &:= \frac{1}{2(n+1)}\mathbb{E}_n\left[\widehat{\widetilde{\psi}}^{(n)}\left(t, k + \tfrac{\eta}{n+1}\right)\left[\widehat{\widetilde{\psi}}^{(n)}\right]^\star(t,k)\right],\\
\widetilde{W}_n^-(t,\eta,k) &:= \frac{1}{2(n+1)}\mathbb{E}_n\left[\left[\widehat{\widetilde{\psi}}^{(n)}\right]^\star\left(t, -k - \tfrac{\eta}{n+1}\right)\widehat{\widetilde{\psi}}^{(n)}(-k)\right] = (\widetilde{W}_n^+)^\star(t,-\eta,-k),\\
\widetilde{Y}_n^+(t,\eta,k) &:= \frac{1}{2(n+1)}\mathbb{E}_n\left[\widehat{\widetilde{\psi}}^{(n)}\left(t, k + \tfrac{\eta}{n+1}\right)\widehat{\widetilde{\psi}}^{(n)}(t,-k)\right],\\
\widetilde{Y}_n^-(t,\eta,k) &:= \frac{1}{2(n+1)}\mathbb{E}_n\left[\left[\widehat{\widetilde{\psi}}^{(n)}\right]^\star\left(t, -k - \tfrac{\eta}{n+1}\right)\left[\widehat{\widetilde{\psi}}^{(n)}\right]^\star(k)\right] = (\widetilde{Y}_n^+)^\star(t,-\eta,-k).
\end{aligned}$$

As a direct corollary from Proposition 4.5 we conclude the following bound: for any $t > 0$ we have

$$\sum_{\iota=\pm}\left(\sup_{s\in[0,t]}\sup_{\eta\in\mathbb{Z}}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}|\widetilde{W}_n^\iota(s,\eta,k)| + \sup_{s\in[0,t]}\sup_{\eta\in\mathbb{Z}}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}|\widetilde{Y}_n^\iota(s,\eta,k)|\right) < +\infty. \tag{8.3}$$

Note that, similarly to (8.1)–(8.2), we can also define the (full) wave function $\psi_x^{(n)}(t) = r_x^{(n)}(t) + ip_x^{(n)}(t)$ and its Fourier transform $\widehat{\psi}^{(n)}(t,k)$. Using (2.4)–(2.5) we conclude that the fluctuating wave function satisfies

$$\begin{aligned}
\mathrm{d}\widehat{\widetilde{\psi}}^{(n)}(t,k) &= -n^2\left(2i\sin^2(\pi k)\widehat{\widetilde{\psi}}^{(n)}(t,k) + \sin(2\pi k)[\widehat{\widetilde{\psi}}^{(n)}]^-(t,k)\right)\mathrm{d}t \qquad (8.4)\\
&- \gamma n^2\left\{\sum_{\iota=\pm}\iota[\widehat{\widetilde{\psi}}^{(n)}]^\iota(t,k)\right\}\mathrm{d}t - \widehat{\sum_{k'\in\widehat{\mathbb{I}}_n}}\left\{\sum_{\iota=\pm}\iota[\widehat{\psi}^{(n)}]^\iota(t^-,k-k')\right\}\mathrm{d}\widehat{\widetilde{\mathcal{N}}}(t,k') + \mathrm{d}\widehat{\widetilde{\mathcal{R}}}_n(t,k),
\end{aligned}$$

where $\widehat{\widetilde{\mathcal{N}}}(t,k) := \widehat{\mathcal{N}}(t,k) - \gamma n^2 t(n+1)\delta_{k,0}$ is a martingale, with

$$\widehat{\mathcal{N}}(t,k) := \sum_{x\in\mathbb{I}_n}\mathcal{N}_x(\gamma n^2 t)e^{-2i\pi xk},$$

where $[\widehat{\psi}^{(n)}]^\pm(t,k)$ are defined by (2.16) and finally

$$\mathrm{d}\widehat{\widetilde{\mathcal{R}}}_n(t,k) := n^2(\widetilde{p}_n(t) - \widetilde{p}_0(t)) + i\sum_{x=0,n}e^{-2\pi ixk}\left(-\widetilde{\gamma}n^2\widetilde{p}_x(t)\mathrm{d}t + n\sqrt{2\widetilde{\gamma}T_x}\mathrm{d}w_x(t)\right).$$

Here $\delta_{x,y}$ is the usual Kronecker delta function, which equals 1 if $x = y$ and 0 otherwise. The process $\widehat{\mathcal{N}}(t,k)$ is a semi-martingale whose mean and covariation can be computed from the relations $\langle \mathrm{d}\widehat{\mathcal{N}}(t,k)\rangle = \gamma n^2(n+1)\delta_{k,0}\mathrm{d}t$, and $\langle \mathrm{d}\widehat{\mathcal{N}}(t,k), \mathrm{d}\widehat{\mathcal{N}}(t,k')\rangle = \gamma n^2(n+1)t\delta_{k,-k'}\mathrm{d}t$.

8.2. **Energy balance for the fluctuating Wigner functions.** After straightforward computations one gets, for $\iota = \pm$:

$$\begin{aligned}\partial_t \widetilde{W}_n^\iota = {} & \iota\Big(-in(\delta_n s)\,\widetilde{W}_n^\iota - n^2\sin(2\pi k)\,\widetilde{Y}_n^\iota - n^2\sin\big(2\pi\big(k+\tfrac{\eta}{n+1}\big)\big)\,\widetilde{Y}_n^{-\iota}\Big) \\ & + \gamma n^2\,\mathbb{L}\Big(\widetilde{W}_n^+ + \widetilde{W}_n^- - \widetilde{Y}_n^+ - \widetilde{Y}_n^-\Big) + \iota\gamma n^2\Big(\widetilde{W}_n^- - \widetilde{W}_n^+\Big) \\ & + \gamma n^2\overline{U}_n(t,\eta) + \frac{\widetilde{\gamma} n^2}{n+1}\sum_{x=0,n} e^{-2\pi i x\frac{\eta}{n+1}}T_x \\ & + \frac{n^2}{2(n+1)}\,\mathbb{E}_n\Big[\widehat{\widetilde{Z}}_n^{-\iota}(t,k)[\widehat{\widetilde{\psi}}^{(n)}]^\iota\big(t,k+\tfrac{\eta}{n+1}\big) + \widehat{\widetilde{Z}}_n^\iota\big(t,k+\tfrac{\eta}{n+1}\big)[\widehat{\widetilde{\psi}}^{(n)}]^{-\iota}(t,k)\Big],\end{aligned} \tag{8.5}$$

$$\begin{aligned}\partial_t \widetilde{Y}_n^\iota = {} & \iota\Big(n^2\sin(2\pi k)\,\widetilde{W}_n^\iota - in^2(\sigma_n s)\,\widetilde{Y}_n^\iota - n^2\sin\big(2\pi\big(k+\tfrac{\eta}{n+1}\big)\big)\,\widetilde{W}_n^{-\iota}\Big) \\ & + \gamma n^2\,\mathbb{L}\Big(\widetilde{Y}_n^+ + \widetilde{Y}_n^- - \widetilde{W}_n^+ - \widetilde{W}_n^-\Big) + \iota\gamma n^2\Big(\widetilde{Y}_n^- - \widetilde{Y}_n^+\Big) \\ & - \gamma n^2\,\overline{U}_n(t,\eta) - \frac{\widetilde{\gamma} n^2}{n+1}\sum_{x=0,n} e^{-2\pi i x\frac{\eta}{n+1}}T_x \\ & + \frac{n^2}{2(n+1)}\,\mathbb{E}_n\Big[\widehat{\widetilde{Z}}_n^\iota(t,-k)\widehat{\widetilde{\psi}}^\iota\big(t,k+\tfrac{\eta}{n+1}\big) + \widehat{\widetilde{Z}}_n^\iota\big(t,k+\tfrac{\eta}{n+1}\big)\widehat{\widetilde{\psi}}^\iota(t,-k)\Big],\end{aligned} \tag{8.6}$$

where $\mathbb{L}$ is defined by $(\mathbb{L}f)(k) := \widehat{\sum_{k'\in\widehat{\mathbb{I}}_n}} f(k') - f(k)$ for any $f : \widehat{\mathbb{I}}_n \to \mathbb{C}$ and we let

$$\begin{aligned}(\delta_n s)(\eta,k) &:= 2n\Big(\sin^2\big(\pi\big(k+\tfrac{\eta}{n+1}\big)\big) - \sin^2(\pi k)\Big), \\ (\sigma_n s)(\eta,k) &:= 2\Big(\sin^2\big(\pi\big(k+\tfrac{\eta}{n+1}\big)\big) + \sin^2(\pi k)\Big), \\ \widehat{\widetilde{Z}}_n(t,k) &:= \widetilde{p}_n(t) - \widetilde{p}_0(t) - \widetilde{\gamma} i\sum_{x=0,n} e^{-2\pi i x k}\widetilde{p}_x(t), \\ \overline{U}_n(t,\eta) &:= \frac{2}{n+1}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\mathbb{E}_n\Big[\widehat{\widetilde{p}}^{(n)}\big(t,k+\tfrac{\eta}{n+1}\big)[\widehat{\widetilde{p}}^{(n)}]^\star(t,k)\Big].\end{aligned} \tag{8.7}$$

We are interested in the time evolution of the following quantity: let us denote

$$\widetilde{\mathfrak{E}}_n(t) := \sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\Big(|\widetilde{W}_n^+|^2 + |\widetilde{W}_n^-|^2 + |\widetilde{Y}_n^+|^2 + |\widetilde{Y}_n^+|^2\Big)(t,\eta,k). \tag{8.8}$$

One can check that

$$\widetilde{\mathfrak{E}}_n(t) = \frac{1}{2(n+1)} \sum_{x,x'} \Big\{ \Big(\mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \widetilde{p}_{x'}^{(n)}(t) \Big] \Big)^2 + \Big(\mathbb{E}_n \Big[\widetilde{r}_x^{(n)}(t) \widetilde{r}_{x'}^{(n)}(t) \Big] \Big)^2 + 2 \Big(\mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \widetilde{r}_{x'}^{(n)}(t) \Big] \Big)^2 \Big\}.$$

After a tedious but direct calculation, using (8.5)–(8.6), we get the following identity

$$\begin{aligned} \frac{1}{2}\partial_t \widetilde{\mathfrak{E}}_n(t) &= \frac{2\widetilde{\gamma} n^2}{n+1} \sum_{x=0,n} \Big(T_x - \mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \Big]^2 \Big) \mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \Big]^2 \\ &+ \frac{4\gamma n^2}{n+1} \sum_{x\in\mathbb{I}_n} \big(\overline{p}_x^{(n)}(t) \big)^2 \mathbb{E}_n \Big[\big(\widetilde{p}_x^{(n)}(t) \big)^2 \Big] \\ &- \frac{2n^2\widetilde{\gamma}}{n+1} \sum_{x=0,n} \sum_{x'\in\mathbb{I}_n} \Big\{ \Big(\mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \widetilde{r}_{x'}^{(n)}(t) \Big] \Big)^2 + \Big(\mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \widetilde{p}_{x'}^{(n)}(t) \Big] \Big)^2 \Big\} \\ &- \frac{4\gamma n^2}{n+1} \Big\{ \sum_{x,x'\in\mathbb{I}_n, x\neq x'} \Big(\mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \widetilde{p}_{x'}^{(n)}(t) \Big] \Big)^2 + \sum_{x,x'\in\mathbb{I}_n} \Big(\mathbb{E}_n \Big[\widetilde{r}_x^{(n)}(t) \widetilde{p}_{x'}^{(n)}(t) \Big] \Big)^2 \Big\}. \end{aligned} \tag{8.9}$$

The main result of this section is the following

Proposition 8.1 (Energy balance identity)**.** *For any $t_* > 0$ there exists $C > 0$ such that*

$$\widetilde{\mathfrak{E}}_n(t) + \gamma n^2 \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \int_0^t \Big(\big| \widetilde{W}_n^+ - \widetilde{W}_n^- \big|^2 + \big| \widetilde{Y}_n^+ - \widetilde{Y}_n^- \big|^2 \Big)(s,\eta,k) \mathrm{d}s \leqslant \widetilde{\mathfrak{E}}_n(0) + Ct\log^2(n+1), \qquad t \in [0,t_*],\ n \geqslant 1. \tag{8.10}$$

Proof. Thanks to Lemma 5.1–(5.13) we can write

$$\sum_{x=0,n} \Big(T_x - \mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \Big]^2 \Big) \mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \Big]^2 \leqslant \sum_{x=0,n} T_x \Big(T_x - \mathbb{E}_n \Big[\widetilde{p}_x^{(n)}(t) \Big]^2 \Big) \lesssim \frac{1}{n}, \qquad n \geqslant 1.$$

By Proposition 4.5 and Lemma 5.4 for any $t_* > 0$ we also have

$$\begin{aligned} &\sum_{x\in\mathbb{I}_n} \int_0^t \big(\overline{p}_x^{(n)}(s) \big)^2 \mathbb{E}_n \Big[\big(\widetilde{p}_x^{(n)}(s) \big)^2 \Big] \mathrm{d}s \\ &\leqslant \Big(\sup_{s\in[0,t_*]} \sum_{x\in\mathbb{I}_n} \mathbb{E}_n \Big[\big(\widetilde{p}_x^{(n)}(s) \big)^2 \Big] \Big) \Big(\int_0^t \sup_{x\in\mathbb{I}_n} \big(\overline{p}_x^{(n)}(s) \big)^2 \mathrm{d}s \Big) \lesssim \frac{\log^2(n+1)}{n} \end{aligned} \tag{8.11}$$

for $n \geqslant 1$, $t \in [0,t_*]$. Since

$$\sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \Big(\big| \widetilde{W}_n^+ - \widetilde{W}_n^- \big|^2 + \big| \widetilde{Y}_n^+ - \widetilde{Y}_n^- \big|^2 \Big)(t,\eta,k) = \frac{4}{n+1} \sum_{x,x'\in\mathbb{I}_n} \Big\{ \mathbb{E}_n \Big[\widetilde{r}_x^{(n)}(t) \widetilde{p}_{x'}^{(n)}(t) \Big] \Big\}^2$$

we conclude (8.10). □

8.3. **Equipartition of energy: proof of Proposition 4.6.** In this section we prove Proposition 4.6. Let us recall here its statement: under the assumptions of Theorem 3.5 (namely Assumptions 3.1, and 3.4), for any complex valued test function $G \in C_0^\infty([0,+\infty) \times \mathbb{I} \times \mathbb{T})$ we have

$$\lim_{n\to+\infty} \int_0^t \frac{1}{n} \sum_{x\in\mathbb{I}_n} G_x(s)\mathbb{E}_n\Big[\big(\widetilde{r}_x^{(n)}(s)\big)^2 - \big(\widetilde{p}_x^{(n)}(s)\big)^2\Big]\mathrm{d}s = 0. \tag{8.12}$$

Let us introduce

$$\begin{aligned}\widetilde{V}_n(t,\eta,k) &:= \widetilde{Y}_n^+(t,\eta,k) + \widetilde{Y}_n^-(t,\eta,k), \\ \widetilde{R}_n(t,\eta,k) &:= \widetilde{Y}_n^+(t,\eta,k) - \widetilde{Y}_n^-(t,\eta,k).\end{aligned} \tag{8.13}$$

By the Parseval identity,

$$\sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \big|\widetilde{V}_n(t,\eta,k)\big|^2 = \frac{1}{2(n+1)} \sum_{x,x'\in\mathbb{I}_n} \Big(\mathbb{E}_n\big[\widetilde{r}_x^{(n)}(t)\widetilde{r}_{x'}^{(n)}(t)\big] - \mathbb{E}_n\big[\widetilde{p}_x^{(n)}(t)\widetilde{p}_{x'}^{(n)}(t)\big]\Big)^2 \tag{8.14}$$

and for any $G \in C^\infty(\mathbb{I})$, cf. (2.18),

$$\sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \widetilde{V}_n(t,\eta,k)\widehat{G}^\star(\eta) = \frac{1}{n+1} \sum_{x\in\mathbb{I}_n} \mathbb{E}_n\left[\widetilde{r}_x^2 - \widetilde{p}_x^2\right] G_x^\star.$$

To prove Proposition 4.6 we need to show that

$$\lim_{n\to+\infty} \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \int_0^{+\infty} \widetilde{V}_n(t,\eta,k)\widehat{G}^\star(t,\eta,k)\mathrm{d}t = 0 \tag{8.15}$$

for any $G \in C_0^\infty([0,+\infty) \times \mathbb{I} \times \mathbb{T})$. From (8.6) we obtain,

$$\begin{aligned}\partial_t \widetilde{R}_n =\ & n^2(\delta_n\widehat{s})(\eta,k)\ \big(\widetilde{W}_n^+ - \widetilde{W}_n^-\big) - in^2(\sigma_n s)\ \widetilde{V}_n + 2\gamma n^2 \widetilde{R}_n \\ &+ \frac{in^2}{n+1}\Big(\mathbb{E}_n\left[(\widetilde{p}_n(t) - \widetilde{p}_0(t))\widehat{\widetilde{p}}\big(t, k + \tfrac{\eta}{n+1}\big)\right] + \mathbb{E}_n\left[(\widetilde{p}_n(t) - \widetilde{p}_0(t))\widehat{\widetilde{p}}^\star(t,k)\right]\Big) \\ &- \frac{i\widetilde{\gamma}n^2}{n+1} \sum_{x=0,n} \left\{e^{2\pi i x k}\mathbb{E}_n\left[\widetilde{p}_x(t)\widehat{\widetilde{r}}\big(t, k + \tfrac{\eta}{n+1}\big)\right] + e^{-2\pi i x (k + \frac{\eta}{n+1})}\mathbb{E}_n\left[\widetilde{p}_x(t)\widehat{\widetilde{r}}^\star(t,k)\right]\right\},\end{aligned} \tag{8.16}$$

where $\sigma_n s$ is given by (8.7) and

$$(\delta_n\widehat{s})(\eta,k) := \sin(2\pi k) - \sin\Big(2\pi\big(k + \tfrac{\eta}{n+1}\big)\Big).$$

Given $s \in (0,1)$ we let

$$\widehat{\mathbb{I}}_{n,s} := \left\{k \in \widehat{\mathbb{I}}_n : 0 \leqslant k \leqslant (n+1)^{-s}\right\} \qquad \text{and} \qquad \widehat{\mathbb{I}}_n^s := \widehat{\mathbb{I}}_n \setminus \widehat{\mathbb{I}}_{n,s}.$$

We can write

$$\sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \int_0^t \widetilde{V}_n(s,\eta,k)\widehat{G}^\star(s,\eta,k)\mathrm{d}s = \mathcal{O}_{n,s} + \mathcal{O}_n^s, \tag{8.17}$$

where terms $\mathcal{O}_{n,s}$ and $\mathcal{O}_n^s$ correspond to the summation in k over $\widehat{\mathbb{I}}_{n,s}$ and $\widehat{\mathbb{I}}_n^s$, respectively, and $s \in (0,1)$ is to be determined later on. Denoting $\widehat{G}_1 := -\widehat{G}/(\sigma_n s)$ and $\widehat{G}_2 := \delta_n \widehat{s}\,\widehat{G}/(\sigma_n s)$, and using (8.16) we can write

$$\mathcal{O}_n^s = \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n + \mathrm{IV}_n + \mathrm{V}_n, \tag{8.18}$$

where

$$\begin{aligned}
\mathrm{I}_n &:= \frac{i}{n^2} \sum_{\eta \in \mathbb{I}_n} \widehat{\sum_{k \in \widehat{\mathbb{I}}_n^s}} \int_0^t \partial_s \widetilde{R}_n(s,\eta,k) \widehat{G}_1^\star(s,\eta,k) \mathrm{d}s, \\
\mathrm{II}_n &:= i \sum_{\eta \in \mathbb{I}_n} \widehat{\sum_{k \in \widehat{\mathbb{I}}_n^s}} \int_0^t \left(\widetilde{W}_n^+ - \widetilde{W}_n^- \right)(s,\eta,k) \widehat{G}_2^\star(s,\eta,k) \mathrm{d}s, \\
\mathrm{III}_n &:= -2\gamma i \sum_{\eta \in \mathbb{I}_n} \widehat{\sum_{k \in \widehat{\mathbb{I}}_n^s}} \int_0^t \widetilde{R}_n(s,\eta,k) \widehat{G}_1^\star(s,\eta,k) \mathrm{d}s, \\
\mathrm{IV}_n &:= -\frac{1}{n+1} \sum_{\eta \in \mathbb{I}_n} \widehat{\sum_{k \in \widehat{\mathbb{I}}_n^s}} \int_0^t \Big\{ \mathbb{E}_n \left[\left(\widetilde{p}_n(s) - \widetilde{p}_0(s) \right) \widehat{p}\left(s, k + \tfrac{\eta}{n+1}\right) \right] \\
&\qquad\qquad + \mathbb{E}_n \left[\left(\widetilde{p}_n(s) - \widetilde{p}_0(s) \right) \widehat{p}^\star\,(s,k) \right] \Big\} \widehat{G}_1^\star(s,\eta,k) \mathrm{d}s, \\
\mathrm{V}_n &:= -\frac{1}{n+1} \sum_{x=0,n} \sum_{\eta \in \mathbb{I}_n} \widehat{\sum_{k \in \widehat{\mathbb{I}}_n^s}} \int_0^t \Big\{ e^{2\pi i x k} \mathbb{E}_n \left[\widetilde{p}_x(s) \widehat{p}\left(s, k + \tfrac{\eta}{n+1}\right) \right] \\
&\qquad\qquad + e^{-2\pi i x \left(k + \frac{\eta}{n+1}\right)} \mathbb{E}_n \left[\widetilde{p}_x(s) \widehat{p}^\star(s,k) \right] \Big\} \widehat{G}_1^\star(s,\eta,k) \mathrm{d}s.
\end{aligned}$$

Estimates of $\mathcal{O}_{n,s}$. We show that for any $s \in (0,1)$

$$\lim_{n \to +\infty} \mathcal{O}_{n,s} = 0. \tag{8.19}$$

Choose an arbitrary $\varepsilon > 0$. Since $G \in C_0^\infty([0,+\infty) \times [0,1] \times \mathbb{I})$ we can find a sufficiently large $M > 0$ such that

$$\sum_{|\eta| \geqslant M} \sup_{k \in \mathbb{I}, s \in [0,t]} |\widehat{G}(s,\eta,k)| < \epsilon. \tag{8.20}$$

We can write $\mathcal{O}_{n,s} = \mathcal{O}_{n,s,M} + \mathcal{O}_{n,s}^M$, where the terms $\mathcal{O}_{n,s,M}$ and $\mathcal{O}_{n,s}^M$ correspond in (8.17) to the summation over $|\eta| \leqslant M$ and $|\eta| > M$, respectively. Thanks to (8.19) and (8.3) we conclude that

$$|\mathcal{O}_{n,s}^M| \leqslant t \left\{ \sup_{s \in [0,t], \eta \in \mathbb{I}_n} \widehat{\sum_{k \in \widehat{\mathbb{I}}_n}} \left| \widetilde{V}_n(s,\eta,k) \right| \right\} \left\{ \sum_{|\eta| > M} \sup_{k \in \mathbb{I}, s \in [0,t]} \left| \widehat{G}(s,\eta,k) \right| \right\} < \frac{\varepsilon}{2}, \tag{8.21}$$

provided $M > 0$ is sufficiently large. On the other hand, using the Cauchy-Schwarz inequality and (8.10) we get

$$
\begin{aligned}
|\mathcal{O}_{n,s,M}| &\leqslant \|\widehat{G}\|_\infty \int_0^t \sum_{|\eta|\leqslant M} \widehat{\sum_{k\in\widehat{\mathbb{I}}_{n,s}}} |\widetilde{V}_n(s,\eta,k)| \mathrm{d}s \\
&\leqslant \|\widehat{G}\|_\infty t \sup_{s\in[0,t]} \left\{ \sum_{|\eta|\leqslant M} \widehat{\sum_{k\in\widehat{\mathbb{I}}_{n,s}}} |\widetilde{V}_n(s,\eta,k)|^2 \right\}^{1/2} \frac{M^{1/2}}{n^{s/2}} \lesssim \frac{(M\log^2(n+1))^{1/2}}{n^{s/2}}.
\end{aligned}
$$

Therefore $\lim_{n\to+\infty} |\mathcal{O}_{n,s,M}| = 0$. Combining with (8.21) we obtain (8.19).

Estimates of $\mathcal{O}_n^s$. To estimate $\mathcal{O}_n^s$ we use the decomposition (8.18). By integration by parts formula we get $\mathrm{I}_n = \mathrm{I}_{n,1} + \mathrm{I}_{n,2}$, where

$$
\begin{aligned}
\mathrm{I}_{n,1} &:= -\frac{i}{n^2} \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \int_0^{+\infty} \widetilde{R}_n(t)\partial_t \widehat{G}_1^\star(t)\mathrm{d}t, \\
\mathrm{I}_{n,2} &:= \frac{i}{n^2} \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \widetilde{R}_n(0)\widehat{G}_1^\star(0).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality and $\widehat{G}_1(t,\eta,k) \equiv 0$, for $t \geqslant t_*$, we obtain

$$
\begin{aligned}
&\left| \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \int_0^{+\infty} \widetilde{R}_n(t)\partial_t \widehat{G}_1^\star(t)\mathrm{d}t \right| \\
&\qquad \leqslant \left\{ \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \int_0^{t_*} \left|\widetilde{R}_n(t)\right|^2 \mathrm{d}t \right\}^{1/2} \left\{ \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \int_0^{t_*} \left|\partial_s \widehat{G}_1^\star(t)\right|^2 \mathrm{d}t \right\}^{1/2}.
\end{aligned}
$$

Let $\phi(\eta) := 1/(1+\eta^2)$. We have

$$
\sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \int_0^t \left|\partial_s \widehat{G}_1^\star(s)\right|^2 \leqslant \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \frac{\phi(\eta)}{\left(k^2 + (k+\eta/n)^2\right)^2} \lesssim \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \frac{1}{k^4} \lesssim n^{3s}. \tag{8.22}
$$

Thanks to (8.10) we conclude that

$$
\frac{1}{n^2} \left| \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \int_0^{+\infty} \widetilde{R}_n(t)\partial_t \widehat{G}_1^\star(t) \right| \lesssim n^{3s/2-3}\log(n+1), \qquad n \geqslant 1.
$$

Thus, for any $s \in (0,2)$ we get $\lim_{n\to+\infty} \mathrm{I}_{n,1} = 0$. The argument to prove that $\lim_{n\to+\infty} \mathrm{I}_{n,2} = 0$ is analogous.

Concerning II_n, by the Cauchy-Schwarz inequality we get

$$
|\mathrm{II}_n| \leqslant \left\{ \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \int_0^t \left|\left(\widetilde{W}_n^+ - \widetilde{W}_n^-\right)(s,\eta,k)\right|^2 \mathrm{d}s \right\}^{1/2} \left\{ \sum_{\eta\in\mathbb{I}_n} \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \int_0^t |\widehat{G}_2(s,\eta,k)|^2 \mathrm{d}s \right\}^{1/2}.
$$

Using (8.10) we can estimate the right hand side by an expression of the form

$$\frac{\log(n+1)}{n}\Big\{\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\int_0^{t_*}\Big(\frac{\delta_n\widehat{s}}{\sigma_n s}\Big)^2|\widehat{G}(s,\eta,k)|^2\mathrm{d}s\Big\}^{1/2}$$

$$\lesssim\frac{\log(n+1)}{n}\Big\{\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\frac{1}{(1+\eta^2)k^2}\Big\}^{1/2}\lesssim\frac{\log(n+1)}{n^{1-s/2}}\xrightarrow[n\to\infty]{}0+,\quad\text{provided } s\in(0,2).$$

Estimates of III_n. By the Cauchy-Schwarz inequality we can write

$$|\mathrm{III}_n|\leqslant 2\gamma\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\int_0^{t_*}\big|\widetilde{R}_n(s,\eta,k)\widehat{G}_1^\star(s,\eta,k)\big|\mathrm{d}s$$

$$\leqslant 2\gamma\Big\{\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}_n}}\int_0^{t_*}\big|\widetilde{R}_n(s,\eta,k)\big|^2\mathrm{d}s\Big\}^{1/2}\Big\{\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\int_0^{t_*}\frac{\big|\widehat{G}(s,\eta,k)\big|^2}{(\sigma_n s)^2}\mathrm{d}s\Big\}^{1/2}.$$

Using (8.10), together with (8.22), we can estimate the right hand side by

$$\frac{Cn^{3s/2}\log(n+1)}{n}\xrightarrow[n\to\infty]{}0+,\quad\text{provided } s\in(0,\tfrac{2}{3}).$$

Estimates of IV_n *and* V_n. The argument in both cases is the same, so we only consider IV_n. We can write $\mathrm{IV}_n=\mathrm{IV}_{n,1}+\mathrm{IV}_{n,2}$, where

$$\mathrm{IV}_{n,1}:=-\frac{1}{n+1}\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\int_0^t\mathbb{E}_n\Big[(\widetilde{p}_n(s)-\widetilde{p}_0(s))\,\widehat{\widetilde{p}}\big(s,k+\tfrac{\eta}{n+1}\big)\Big]\widehat{G}_1^\star(s,\eta,k)\mathrm{d}s,$$

$$\mathrm{IV}_{n,2}:=-\frac{1}{n+1}\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\int_0^t\mathbb{E}_n\Big[(\widetilde{p}_n(s)-\widetilde{p}_0(s))\,\widehat{\widetilde{p}}^\star(s,k)\Big]\widehat{G}_1^\star(s,\eta,k)\mathrm{d}s.$$

By the Plancherel identity we can write

$$\mathrm{IV}_{n,2}=-\frac{1}{n+1}\sum_{x\in\mathbb{I}_n}\int_0^t\mathbb{E}_n\Big[(\widetilde{p}_n(s)-\widetilde{p}_0(s))\,\widetilde{p}_x(s)\Big]\widetilde{G}_{1,x}^\star(s)\mathrm{d}s,$$

where

$$\widetilde{G}_{1,x}^\star(s):=\sum_{\eta\in\mathbb{I}_n}\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}e^{2\pi ikx}\widehat{G}_1^\star(s,\eta,k).$$

As a result, invoking (4.7), the Cauchy-Schwarz inequality and Plancherel identity, we can find a constant $C>0$, independent of n, and such that

$$|\mathrm{IV}_{n,2}|\leqslant\frac{1}{n+1}\sum_{x\in\mathbb{I}_n}\int_0^t\Big\{\mathbb{E}_n\big[|\widetilde{p}_n(s)|+|\widetilde{p}_0(s)|\big]^2\Big\}^{1/2}\Big\{\mathbb{E}_n\big[\widetilde{p}_x^2(s)\big]\Big\}^{1/2}|\widetilde{G}_{1,x}(s)|\mathrm{d}s$$

$$\leqslant\frac{1}{n+1}\Big\{\int_0^t\mathbb{E}_n\big[|\widetilde{p}_n(s)|+|\widetilde{p}_0(s)|\big]^2\sum_{x\in\mathbb{I}_n}\mathbb{E}_n\big[\widetilde{p}_x^2(s)\big]\mathrm{d}s\Big\}^{1/2}\Big\{\int_0^t\sum_{x\in\mathbb{I}_n}|\widetilde{G}_{1,x}(s)|^2\mathrm{d}s\Big\}^{1/2}$$

$$\leqslant\frac{C}{(n+1)^{1/2}}\Big\{\int_0^t\mathbb{E}_n\big[\widetilde{p}_n^2(s)+\widetilde{p}_0^2(s)\big]\mathrm{d}s\Big\}^{1/2}\Big\{\int_0^t\widehat{\sum_{k\in\widehat{\mathbb{I}}^s_n}}\Big|\sum_{\eta\in\mathbb{I}_n}\widehat{G}_1^\star(s,\eta,k)\Big|^2\mathrm{d}s\Big\}^{1/2}.$$

Invoking Lemma 5.1, see (5.8), we conclude that

$$|\mathrm{IV}_{n,2}| \lesssim \frac{1}{(n+1)^{1/2}} \left\{ \left(\widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \frac{1}{k^4} \right) \left(\sum_{\eta\in\mathbb{I}_n} \frac{1}{1+\eta^2} \right) \right\}^{1/2} \lesssim \frac{n^{3s/2}}{(n+1)^{1/2}} \xrightarrow[n\to\infty]{} 0+,$$

provided $s \in (0, \frac{1}{3})$. The proof of the fact that $\lim_{n\to+\infty} \mathrm{IV}_{n,1}$ follows the same lines as the argument presented above.

8.3.1. *Conclusion.* Therefore, for $s \in (0, \frac{1}{3})$ we have proved that both $\mathcal{O}_{n,s}$ and $\mathcal{O}_n^s$ vanish as $n \to \infty$, and we conclude (8.15).

This ends the proof of Proposition 4.6.

Appendix A. Proof of (6.22)

Clearly

$$|a_n(\eta)| \leqslant \frac{2}{\eta^2 - 4}, \quad \eta^2 > 8. \tag{A.1}$$

Suppose that $\eta^2 \in (0, 8)$. Let $\Phi(u, k) := [4\sin^2(\pi k) - u]^2 + 4\gamma^2 u$. From (6.20) we see that

$$|a_n(\eta)| \lesssim \widehat{\sum_{k\in\widehat{\mathbb{I}}_n^s}} \frac{\sin^2(\pi k)}{\Phi(\eta^2, k)}.$$

After a simple calculation one concludes that $\gamma^2 \sin^4(\pi k) \lesssim \Phi(u, k)$ for $u \in (0, 8)$, $k \in \mathbb{I}_n$. Hence $|a_n(\eta)| \lesssim 1$, $\eta^2 \in (0, 8)$. This together with (A.1) yield (6.22).

Appendix B. Proof of (6.21)

Recall that

$$e_{\mathrm{d},n}(\eta) = \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \frac{i\eta + 2\gamma + 2\widetilde{\gamma}\sin^2(\pi k)}{-\eta^2 + 4\sin^2(\pi k) + 2i\gamma\eta} = \frac{\Xi(\eta, k)}{|\Theta(\eta, k)|^2}, \tag{B.1}$$

where

$\Xi(\eta, k) := \sin^2(\pi k)\big[- 2\eta^2\widetilde{\gamma} + 8\gamma + 8\widetilde{\gamma}\sin^2(\pi k)\big] + i\eta[-\eta^2 + 4\sin^2(\pi k)(1 - \widetilde{\gamma}\gamma) - 4\gamma^2]$,
$\Theta(\eta, k) := -\eta^2 + 4\sin^2(\pi k) + 2i\gamma\eta$.

We have

$$\frac{1}{|\Theta(\eta, k)|^2} \geqslant \frac{1}{4\gamma^2\eta^2}. \tag{B.2}$$

Let $\rho := \sin^2(\pi k)$. Let Γ be the parabola in the (x, y) plane described by the system of equations

$$f(\rho) = \rho\big[- 2\eta^2\widetilde{\gamma} + 8\gamma + 8\widetilde{\gamma}\rho\big],$$
$$g(\rho) = \eta[-\eta^2 + 4\rho(1 - \widetilde{\gamma}\gamma) - 4\gamma^2], \quad \rho \in \mathbb{R}.$$

By a direct calculation one can check that there exist two tangent lines to Γ passing through $(0, 0)$. Hence $(0, 0) \notin \mathrm{Conv}(\Gamma)$ - the closed region bounded by Γ.

Denote by $\mathrm{d}_* > 0$ the distance between $(0,0)$ and $\mathrm{Conv}(\Gamma)$ and P_* the respective nearest neighbor projection of $(0,0)$. Note that

$$D^2(\rho) := f^2(\rho) + g^2(\rho) = \rho^2[-2\eta^2\widetilde{\gamma} + 8\gamma + 8\widetilde{\gamma}\rho]^2 + \eta^2[\eta^2 + 4\gamma^2 - 4\rho(1-\widetilde{\gamma}\gamma)]^2.$$

When $\widetilde{\gamma}\gamma \geqslant 1$ then $D^2(\rho) \geqslant 4\gamma^2\eta^2$. If, on the one hand, $\widetilde{\gamma}\gamma < 1$, then for $\rho \leqslant \gamma^2 + \eta^2/4$, we have

$$D(\rho) \geqslant \widetilde{\gamma}\gamma|\eta|\left(\eta^2 + 4\gamma^2\right)^{1/2} \geqslant 2\widetilde{\gamma}\gamma^2|\eta|.$$

If, on the other hand, $\rho > \gamma^2 + \eta^2/4$, then,

$$-2\eta^2\widetilde{\gamma} + 8\gamma + 8\widetilde{\gamma}\rho \geqslant -2\eta^2\widetilde{\gamma} + 8\gamma + 2\widetilde{\gamma}(4\gamma^2 + \eta^2) \geqslant 8\gamma + 8\widetilde{\gamma}\gamma^2 > 0.$$

Therefore,

$$D(\rho) \geqslant \rho\left|-2\eta^2\widetilde{\gamma} + 8\gamma + 8\widetilde{\gamma}\rho\right| \geqslant 2(\gamma + \widetilde{\gamma}\gamma^2)(4\gamma^2 + \eta^2) \geqslant 8\gamma^3(1 + \widetilde{\gamma}\gamma).$$

We conclude that

$$\mathrm{Re}\left(\Xi(\eta,k)\cdot\frac{P_*^\star}{|P_*|}\right) \geqslant \mathrm{d}_* \geqslant 2\gamma^2\min\left\{4\gamma(1+\widetilde{\gamma}\gamma), \widetilde{\gamma}|\eta|\right\}.$$

Therefore, by virtue of (B.1) and (B.2) we conclude

$$|e_{\mathrm{d},n}(\eta)| \geqslant 2\gamma^2\,\frac{\min\{4\gamma(1+\widetilde{\gamma}\gamma), \widetilde{\gamma}|\eta|\}}{4\gamma^2\eta^2} \tag{B.3}$$

and (6.21) follows.

Appendix C. Proof of (6.34)

Note that

$$e_{\mathrm{s},n}(\eta) = 1 + 4\widetilde{\gamma}\gamma\eta^2\widehat{\sum_{k\in\mathbb{I}_n}}\frac{\cos^2(\pi k)}{|4\sin^2(\pi k) - \eta^2 + 2i\gamma\eta|^2} + 2\widetilde{\gamma}i\eta\widehat{\sum_{k\in\mathbb{I}_n}}\frac{\cos^2(\pi k)[4\sin^2(\pi k) - \eta^2]}{|4\sin^2(\pi k) - \eta^2 + 2i\gamma\eta|^2}$$

and (6.34) follows. □

Appendix D. Proof of (6.23) and (6.36)

We only prove (6.23), as the argument for (6.36) follows the same lines. It is clear from (6.19) that

$$|\pi_{\mathrm{d},n}(\eta)| \lesssim \frac{1}{\eta^2+1}, \qquad \eta^2 > 8. \tag{D.1}$$

By an elementary calculation one gets

$$\left\{\frac{1}{2}\left[(4\sin^2(\pi k) - \eta^2)^2 + 4\gamma^2\eta^2\right]\right\}^{1/2} \geqslant \left(8\gamma^2\sin^4(\pi k) + 4\gamma^2\eta^2\right)^{1/2}$$

for $\eta^2 < 8$. Hence,

$$|\pi_{\mathrm{d},n}(\eta)| \lesssim \widehat{\sum_{k\in\mathbb{I}_n}}\frac{|\sin(\pi k)|}{|\eta| + \sin^2(\pi k)} \lesssim \log\left(1 + \frac{1}{|\eta|}\right), \qquad \eta^2 < 8$$

and combining with (D.1) we conclude (6.23).

Appendix E. Proofs of (6.24) and (6.37)

Estimate (6.37) follows straightforwardly from the definition (6.32) and assumption (3.10) on $\widehat{\overline{p}}(0,k)$.

Concerning (6.24) we estimate first in the case $\eta^2 \leqslant 8$. Hypothesis (3.10) allows us then to estimate $|\rho_{\mathrm{d},n}(\eta)| \lesssim 1/|\eta|$ for $n \geqslant 1$, cf. (6.19). Combing with (6.34) we conclude $|\rho_{\mathrm{d},n}(\eta)/e_{\mathrm{d},n}(\eta)| \lesssim 1$.

In the case $\eta^2 > 8$, using the fact that $\widehat{\sum_{k\in\mathbb{I}_n}} \widehat{\overline{r}}(0,k) = r_0 = 0$, we write

$$\begin{aligned} \left|\rho_{\mathrm{d},n}(\eta)\right| &= \left| \widehat{\sum_{k\in\mathbb{I}_n}} \frac{(i\eta+2\gamma)\widehat{\overline{r}}(0,k)}{4\sin^2(\pi k) - \eta^2 + 2i\gamma\eta} - \widehat{\sum_{k\in\mathbb{I}_n}} \frac{(i\eta+2\gamma)\widehat{\overline{r}}(0,k)}{-\eta^2+2i\gamma\eta} \right| \\ &= \left| \widehat{\sum_{k\in\mathbb{I}_n}} \frac{4\sin^2(\pi k)(i\eta+2\gamma)\widehat{\overline{r}}(0,k)}{[4\sin^2(\pi k)-\eta^2+2i\gamma\eta][-\eta^2+2i\gamma\eta]} \right| \lesssim \frac{1}{1+|\eta|^3}. \end{aligned} \tag{E.1}$$

and (6.24) follows.

Appendix F. Proof of (6.35)

From the definition of c_n we conclude that

$$|c_n(\eta)| \lesssim \frac{1}{|\eta|+\eta^2}, \qquad \eta\in\mathbb{R},\ n\geqslant 1. \tag{F.1}$$

By an elementary calculation one gets, for any $w \in \mathbb{C}\smallsetminus[-1,1]$

$$\widehat{\sum_{k\in\mathbb{I}_n}} \frac{e^{2\pi ik}}{e^{4\pi ik} - 2(1+w)e^{2\pi ik} + 1} = \frac{(\Phi_+^{n+1}(1+w)+1)\Phi_+(1+w)}{(\Phi_+^2(1+w)-1)(1-\Phi_+^{n+1}(1+w))}, \tag{F.2}$$

where $\Phi_+ : \mathbb{C}\smallsetminus[-1,1] \to \mathbb{D}\smallsetminus\{0\}$ is the inverse to the function

$$J(z) := \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad z\in\mathbb{D}\smallsetminus\{0\}.$$

Here $\mathbb{D} := \{z\in\mathbb{C} : |z|<1\}$. Using the branch of the square root that maps $\mathbb{C}\smallsetminus(-\infty,0]$ into $\mathbb{C}_+ := \{w : \operatorname{Re} w > 0\}$ we can write

$$\Phi_+(w) = w - \sqrt{w^2-1}, \qquad w\in\mathbb{C}_+\cap(\mathbb{C}\smallsetminus[-1,1]). \tag{F.3}$$

Using (6.33) and (F.2) we can write

$$\begin{aligned} c_n(\eta) &= -\frac{1}{2} - \frac{4-\eta^2+2i\gamma\eta}{2} \widehat{\sum_{k\in\mathbb{I}_n}} \frac{e^{2\pi ik}}{e^{4\pi ik} - 2(1+w)e^{2\pi ik}+1} \\ &= -\frac{1}{2} - \left(2 - \frac{\eta^2}{2} + i\gamma\eta\right) b_n(\eta), \end{aligned} \tag{F.4}$$

where $w(\eta) := -\eta^2/2 + i\gamma\eta$ and

$$b_n(\eta) := \frac{(\Phi_+^{n+1}(1+w)+1)\Phi_+(1+w)}{(\Phi_+^2(1+w)-1)(1-\Phi_+^{n+1}(1+w))}.$$

Suppose now that $\eta^2 \leqslant \delta$ for a sufficiently small $\delta > 0$ so that

$$\Big|\Phi_+\Big(1 - \frac{\eta^2}{2} + i\gamma\eta\Big)\Big| < 1 - \frac{1}{2}(\gamma|\eta|)^{1/2}, \quad \eta^2 \leqslant \delta.$$

Let $A_n := \big\{\eta : \eta^2 > 2^{-1}\sin^2(\pi/n)\big\}$. Then, for any $\eta \in A_n$ we have $\eta^2 \geqslant 2/(\pi n)^2$, therefore, thanks to (F.3),

$$\Big|\Phi_+\Big(1 - \frac{\eta^2}{2} + i\gamma\eta\Big)\Big| \leqslant 1 - \frac{1}{2}\Big(\frac{2^{1/2}\gamma}{\pi n}\Big)^{1/2}, \qquad |\eta| < \delta$$

and

$$\limsup_{n\to+\infty} \sup_{\eta\in A_n, |\eta|<\delta} \Big|\Phi_+\Big(1 - \frac{\eta^2}{2} + i\gamma\eta\Big)\Big|^{n+1} \leqslant \limsup_{n\to+\infty}\Big[1 - \frac{1}{2}\Big(\frac{2^{1/2}\gamma}{\pi n}\Big)^{1/2}\Big]^{n+1} = 0.$$

Therefore,

$$|b_n(\eta)| \lesssim \frac{1}{|\eta|^{1/2}}, \quad \text{for } \eta \in A_n, \quad n \geqslant 1.$$

Combining with (F.1) we conclude that (6.35) holds for $\eta^2 \geqslant 2^{-1}\sin^2(\pi/n)$. When $\eta^2 < 2^{-1}\sin^2(\pi/n)$ we conclude directly from (6.33) that

$$|c_n(\eta)| \lesssim \widehat{\sum_{k\in\widehat{\mathbb{I}}_n}} \frac{1}{4\sin^2(\pi k) + |\eta|} \lesssim \int_0^1 \frac{dk}{k^2 + |\eta|} \lesssim \frac{1}{|\eta|^{1/2}}$$

and (6.35) is also in force.

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