ENERGY MOMENT BOUNDS

M. SIMON

1. Harmonic system perturbed with the flip noise

In these notes we are giving a few elements to fill the gap in the proof of the ernegy moment bounds in *Hydrodynamic limits for the velocity-flip model* (SPA **123**, 2013 [3]). Unfortunately the problem is harder than expected and remains open, as we will see in the next paragraphs.

1.1. **Towars the control of energy moments.** We introduce the one dimensional harmonic chain of n oscillators, all of mass 1, and with periodic boundary conditions (meaning that configurations are indexed by the discrete torus $\mathbb{T}_n$), and we follow the notations of [3].

The configurations are sequences $(\mathbf{r},\mathbf{p}) := \{r_i, p_i\}_{i\in\mathbb{T}_n}$, where p_i stands for the momentum of the oscillator at site i, and r_i represents the distance between oscillator i and oscillator $i+1$. The equations of the determistic motion are given by

$$\begin{cases} \dfrac{\mathrm{d}p_i}{\mathrm{d}t} = r_i - r_{i-1}, \\ \dfrac{\mathrm{d}r_i}{\mathrm{d}t} = p_{i+1} - p_i, \end{cases} \tag{1.1}$$

so that the dynamics conserves the total energy

$$\mathcal{E} := \sum_{i\in\mathbb{T}_n} \left\{ \frac{p_i^2}{2} + \frac{r_i^2}{2} \right\}.$$

At independently distributed random Poissonian times, the momentum p_i is flipped into $-p_i$. This noise still conserves the total energy $\mathcal{E}$. The generator of this diffusion is given by

$$\mathcal{L}_n := n^2\mathcal{A}_n + n^2\gamma\mathcal{S}_n.$$

Here the Liouville operator $\mathcal{A}_n$ is given by

$$\mathcal{A}_n = \sum_{i\in\mathbb{T}_n} \big(p_i - p_{i-1}\big)\frac{\partial}{\partial r_i} + \sum_{i\in\mathbb{T}_n} (r_{i+1} - r_i)\frac{\partial}{\partial p_i},$$

while, for $f : (\mathbb{R}\times\mathbb{R})^n \to \mathbb{R}$,

$$\mathcal{S}_n f(\mathbf{r},\mathbf{p}) = \sum_{i\in\mathbb{T}_n} \big(f(\mathbf{r},\mathbf{p}^i) - f(\mathbf{r},\mathbf{p})\big)$$

Date: October 2015.

where $(\mathbf{p}^i)_j = p_j$ if $j \neq i$ and $(\mathbf{p}^i)_i = -p_i$. The system has a family of stationary measures given by the canonical Gibbs distributions

$$d\mu^n_{\lambda,\beta}(\mathbf{r},\mathbf{p}) = \prod_{i\in\mathbb{T}_n} \frac{e^{-\beta\mathcal{E}_i - \lambda r_i}}{Z(\lambda,\beta)}\, dr_i\, dp_i, \qquad \beta > 0, \tag{1.2}$$

where we denote

$$\mathcal{E}_i = \frac{p_i^2}{2} + \frac{r_i^2}{2},$$

the energy that we attribute to the particle i, and

$$Z(\lambda,\beta) = \frac{2\pi}{\beta} \exp\Big(\frac{\lambda^2}{2\beta}\Big). \tag{1.3}$$

We assume the initial condition to be distributed according to a local Gibbs equilibrium $\mu^n_{\lambda_0,\beta_0}$ associated to continuous profiles $\lambda_0, \beta_0 : \mathbb{T} \to (0,+\infty)$, written as

$$d\mu^n_{\lambda_0,\beta_0}(\mathbf{r},\mathbf{p}) := \frac{1}{Z(\lambda_0(\cdot),\beta_0(\cdot))} \prod_{i\in\mathbb{T}_n} \exp\Big(-\beta_0\Big(\frac{i}{n}\Big)\mathcal{E}_i - \lambda_0\Big(\frac{i}{n}\Big) r_i\Big)\, dr_i\, dp_i.$$

The configuration at time t is denoted by $\eta^n_t := \{r_i(t), p_i(t)\}_{i\in\mathbb{T}_n}$, and the law of the Markov process $(\eta^n_t)_{t\geqslant 0}$ is denoted by μ^n_t.

The statement of [3] whose proof has to be corrected is the following:

Theorem 1.1. *For any $t \geqslant 0$, $n \in \mathbb{N}$, and any positive integer $k \geqslant 1$, there exists a positive constant $C > 0$, such that*

$$\mu^n_t\Big[\sum_{i\in\mathbb{T}_n} \mathcal{E}_i^k\Big] \leqslant (Ck)^k \times n. \tag{1.4}$$

To keep the notation simple, we let C denote constants (that do not depend on n, k, t) that may change from line to line.

We first recall the sketch of the proof followed in [3]. Since the chain is harmonic, Gibbs states are Gaussian. Remarkably, all Gaussian moments can be expressed in terms of variances and covariances. We start with a graphical representation of the dynamics of the process given by the generator $\mathcal{L}_n/n^2$. Notice that time is not accelerated in the diffusive scale. To avoid any confusion, the law of this new process is denoted by ν^n_t. Then, we recover the diffusive time accelerated process by:

$$\mu^n_t = \nu^n_{tn^2}.$$

In the following, we always respect the decomposition of the space $\mathbb{R}^n \times \mathbb{R}^n$, where the first n components stand for $\mathbf{r}$ and the last n components stand for $\mathbf{p}$. All vectors and matrices are written according to this decomposition.

Let ν be a measure on $\mathbb{R}^n \times \mathbb{R}^n$. We denote by $\mathbf{m} \in \mathbb{R}^{2n}$ its mean vector and by $\mathbf{C} \in \mathfrak{M}_{2n}(\mathbb{R})$ its covariance matrix. We also denote by ${}^{\mathbf{t}}Z$ the real transpose of the matrix Z. There exist $\rho := \nu[\mathbf{r}] \in \mathbb{R}^n$, $\pi := \nu[\mathbf{p}] \in \mathbb{R}^n$ and $U, V, Z \in \mathfrak{M}_n(\mathbb{R})$ such that

$$\mathbf{m} = (\rho, \pi) \in \mathbb{R}^{2n} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} U & {}^{\mathbf{t}}Z \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2n}(\mathbb{R}). \tag{1.5}$$

Thanks to the trivial convexity inequality $(a+b)^k \leqslant 2^{k-1}(a^k + b^k)$ (for $a, b \geqslant 0$), instead of proving (1.4) we can show

$$\nu_t^n \Big[\sum_{i \in \mathbb{T}_n} p_i^{2k} \Big] \leqslant (Ck)^k \times n \quad \text{and} \quad \nu_t^n \Big[\sum_{i \in \mathbb{T}_n} r_i^{2k} \Big] \leqslant (Ck)^k \times n. \tag{1.6}$$

Start of the proof of Theorem 1.1. (i) Poisson Process and Gaussian Measures – We start by giving a graphical representation of the process, based on the Harris description. Let us define the antisymmetric $(2n, 2n)$-matrix, written by blocks as

$$A := \begin{pmatrix} 0_n & \mathfrak{A}_n \\ -{}^{\mathbf{t}}\mathfrak{A}_n & 0_n \end{pmatrix} \quad \text{where} \quad \mathfrak{A}_n := \begin{pmatrix} 1 & & & -1 \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix} \in \mathfrak{M}_n(\mathbb{R}).$$

Above 0_n is the null (n, n)-matrix, and empty spaces in $\mathfrak{A}_n$ are filled with 0. Let $\{N_i\}_{i \in \mathbb{T}_n}$ be a sequence of independent standard Poisson processes of intensity γ. At time 0 the process has an initial state $(\mathbf{r}, \mathbf{p})(0)$. Let

$$T_1 = \inf_{t \geqslant 0} \Big\{ \text{there exists } i \in \{1, \dots, n\} \text{ such that } N_i(t) = 1 \Big\}$$

and i_1 the site where the infimum is achieved. During the interval $[0, T_1)$, the process (not accelerated in time) follows the deterministic evolution given by the generator $\mathcal{A}_n$. More precisely, during the time interval $[0, T_1)$, $(\mathbf{r}, \mathbf{p})(t)$ follows the evolution given by the system:

$$y'(t) = A \cdot y(t). \tag{1.7}$$

At time T_1, the momentum p_{i_1} is flipped, and gives a new configuration. Then, the system starts again with the deterministic evolution up to the time of the next flip, and so on. Let $\xi := (i_1, T_1), \dots, (i_q, T_q), \dots$ be the sequence of sites and ordered times for which we have a flip, and let us denote its law by $\mathbb{P}$. Conditionally to ξ, the evolution is deterministic, and the state of the process $(\mathbf{r}, \mathbf{p})^\xi(t)$ is given for all $t \in [T_q, T_{q+1})$ by

$$(\mathbf{r}, \mathbf{p})^\xi(t) = e^{(t - T_q)A} \circ F_{i_q} \circ e^{(T_q - T_{q-1})A} \circ F_{i_{q-1}} \circ \cdots \circ e^{T_1 A}(\mathbf{r}, \mathbf{p})(0) \tag{1.8}$$

where F_i is the map $(\mathbf{r}, \mathbf{p}) \mapsto (\mathbf{r}, \mathbf{p}^i)$. If initially the process starts from $(\mathbf{r}, \mathbf{p})(0)$ which is distributed according to a Gaussian measure ν_0^n, then $(\mathbf{r}, \mathbf{p})^\xi(\mathbf{t})$ is distributed according to a Gaussian measure ν_t^ξ. Finally, the density ν_t^n is given by the convex combination

$$\nu_t^n(\cdot) = \int \nu_t^\xi(\cdot)\, d\mathbb{P}(\xi). \tag{1.9}$$

Moreover, we are able to write the evolution of the mean vector $\mathbf{m}_t^\xi$ and the covariance matrix $\mathbf{C}_t^\xi$ of ν_t^ξ. During the interval $[0, T_1)$, $\mathbf{m}_t$ follows the evolution given by system (1.7). At time T_1, the component $m_{i_1+n} = \pi_{i_1}$ (which corresponds to the mean of p_{i_1}) is flipped, and gives a new mean vector. Then, the deterministic evolution goes on up to the time of the next flip, and so on.

In the same way, during the interval $[0, T_1)$, $\mathbf{C}_t$ follows the evolution given by the (matrix) system:

$$M'(t) = AM(t) - M(t)A. \tag{1.10}$$

At time T_1, all the components $C_{i_1+n,j}$ and C_{i,i_1+n} when $i, j \neq i_1 + n$ are flipped and the matrix $\mathbf{C}_{T_1}$ becomes $\Sigma_{i_1} \cdot \mathbf{C}_{T_1} \cdot {}^{\mathbf{t}}\Sigma_{i_1}$, where Σ_i is defined as

$$\Sigma_i := \begin{pmatrix} I_n & 0_n \\ 0_n & I_n - 2E_{i,i} \end{pmatrix},$$

and so on up to the next flip. Above, I_n is the (n, n)-identity matrix, and $E_{i,i}$ is the (n, n)-matrix composed by the elements $(\delta_{i,k}\delta_{i,\ell})_{1 \leqslant k,\ell \leqslant n}$ where $\delta_{i,k}$ is the Kronecker delta function. More precisely,

$$\mathbf{C}_t^\xi = e^{(t-T_q)A} \cdot \Sigma_{i_q} \cdots \Sigma_{i_1} \cdot e^{T_1 A} \cdot \mathbf{C}_0 \cdot e^{-T_1 A} \cdot {}^{\mathbf{t}}\Sigma_{i_1} \cdots {}^{\mathbf{t}}\Sigma_{i_q} e^{-(t-T_q)A}. \tag{1.11}$$

Finally, the density ν_t^n is equal to

$$\nu_t^n(\cdot) = \int \nu_t^\xi(\cdot)\, d\mathbb{P}(\xi) = \int G_{\mathbf{m},\mathbf{C}}(\cdot)\, d\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\mathbf{m}, \mathbf{C}), \tag{1.12}$$

where $G_{\mathbf{m},\mathbf{C}}(\cdot)$ denotes the Gaussian measure on $(\mathbb{R} \times \mathbb{R})^n$ with mean $\mathbf{m}$ and covariance matrix $\mathbf{C}$, and $\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\cdot,\cdot)$ is the law of the random variable $(\mathbf{m}_t, \mathbf{C}_t)$, knowing that the Markov process $(\mathbf{m}_t, \mathbf{C}_t)_{t \geqslant 0}$ described by the graphical representation above starts from $(\mathbf{m}_0, \mathbf{C}_0)$. We denote by $\mathbb{P}_{\mathbf{m}_0,\mathbf{C}_0}$ the law of the Markov process $(\mathbf{m}_t, \mathbf{C}_t)_{t \geqslant 0}$, and by $\mathbb{E}_{\mathbf{m}_0,\mathbf{C}_0}$ the corresponding expectation. Observe that we have, from (1.12),

$$\nu_t^n[p_i] = \int G_{\mathbf{m},\mathbf{C}}(p_i)\, d\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\mathbf{m}, \mathbf{C}) = \int \pi_i\, d\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\mathbf{m}, \mathbf{C}), \tag{1.13}$$

$$\nu_t^n[r_i] = \int G_{\mathbf{m},\mathbf{C}}(r_i)\, d\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\mathbf{m}, \mathbf{C}) = \int \rho_i\, d\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\mathbf{m}, \mathbf{C}). \tag{1.14}$$

Notice that we conveniently denote by $G_{\mathbf{m},\mathbf{C}}(f)$ the mean of the function f with respect to the Gaussian measure $G_{\mathbf{m},\mathbf{C}}$. Therefore, we rewrite (1.6) as

$$\nu_t^n\left[\sum_{i \in \mathbb{T}_n} \left\{p_i^{2k} + r_i^{2k}\right\}\right] = \int \sum_{i \in \mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\left(p_i^{2k} + r_i^{2k}\right)\, d\theta^t_{\mathbf{m}_0,\mathbf{C}_0}(\mathbf{m}, \mathbf{C}).$$

(ii) Control of the diagonal of the covariance matrix – First, let us recenter and use the previous convexity inequality: notice that

$$G_{\mathbf{m},\mathbf{C}}\left(r_i^{2k}\right) = G_{\mathbf{m},\mathbf{C}}\left([r_i - \rho_i + \rho_i]^{2k}\right) \leqslant 2^{2k-1}\left\{G_{\mathbf{m},\mathbf{C}}\left([r_i - \rho_i]^{2k}\right) + G_{\mathbf{m},\mathbf{C}}\left(\rho_i^{2k}\right)\right\}.$$

Remarkably, we can express all the centered moments of a Gaussian random variable as functions of the variance only:

$$G_{\mathbf{m},\mathbf{C}}\big([r_i-\rho_i]^{2k}\big) = \frac{(2k)!}{k!2^k}\, G_{\mathbf{m},\mathbf{C}}\big([r_i-\rho_i]^2\big)^k \leqslant (Ck)^k\, \big(C_{i,i}\big)^k(t),$$

where $C>0$ is a positive constant. Therefore, after repeating the same argument for $G_{\mathbf{m},\mathbf{C}}\big(p_i^{2k}\big)$ we are reduced to control, for any sequence ξ,

$$\sum_i (C^{\xi}_{i,i})^k(t) \tag{1.15}$$

and besides

$$\sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big(\pi_i^{2k}\big)(t), \qquad \sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big(\rho_i^{2k}\big)(t). \tag{1.16}$$

In the following we treat separately (1.15) and (1.16). In [3] only (1.15) has been treated.

(iii) Control of (1.15) *using the trace* – Let us fix once for all a sequence ξ a sequence of sites and ordered times for which we have a flip. The matrix $C_t := C_t^{\xi}$ is symmetric, hence diagonalizable, and after denoting its eigenvalues by $\lambda_1, \dots, \lambda_{2n}$, we can write

$$\mathrm{Tr}([C_t]^k) = \sum_i \lambda_i^k.$$

We have now to compare $\sum_i \lambda_i^k$ with $\sum_i [C_{i,i}]^k(t)$. If we denote by P_t the orthogonal matrix of the eigenvectors of C_t, then we get $C_t = (P_t)^*\cdot D\cdot P_t$, where D is the diagonal matrix with entries $\lambda_1, \dots, \lambda_{2n}$. Let us denote by $(P_{i,j})$ the components of P_t. Then,

$$[C_{i,i}]^k(t) = \Big(\sum_{j,\ell} P^*_{i,j} D_{j,\ell} P_{\ell,i}\Big)^k = \Big(\sum_j P^*_{i,j}\lambda_j P_{j,i}\Big)^k = \Big(\sum_j P^*_{i,j} P_{j,i}\cdot\lambda_j\Big)^k.$$

Since P is an orthogonal matrix, $\sum_j P^*_{i,j}P_{j,i} = 1$. Consequently, from the standard convexity inequality we obtain

$$\sum_i [C_{i,i}]^k(t) \leqslant \sum_i\sum_j P^*_{i,j}P_{j,i}\lambda_j^k \leqslant \sum_j \lambda_j^k = \mathrm{Tr}([C_t]^k).$$

Since C_0 and C_t are similar (see (1.11)), we have:

$$\mathrm{Tr}([C_t]^k) = \mathrm{Tr}(C_0^k) = \sum_{i\in\mathbb{T}_n} \frac{1}{\beta_0^k(i/n)} + \left\{\frac{1}{\beta_0(i/n)} + \Big(\frac{\lambda_0}{\beta_0}\Big)^2(i/n)\right\}^k \leqslant Cn,$$

for some constant $C>0$. Therefore, the same inequality holds for $\sum_i [C_{i,i}]^k(t)$.

(iv) Control of (1.16) *in the diffusive time scale* – In the following it will be convenient to recenter both quantities

$$\sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big(\pi_i^{2k}\big)(tn^2) \quad \text{and} \quad \sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big(\rho_i^{2k}\big)(tn^2).$$

Therefore, we denote

$$\begin{aligned}\mathfrak{p}_i(t) &:= \nu_t^n(p_i),\\ \mathfrak{r}_i(t) &:= \nu_t^n(r_i).\end{aligned}$$

By adding and substracting these two terms, and using the same convexity inequality as before, it is sufficient to control

$$\sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big((\pi_i-\mathfrak{p}_i)^{2k}\big)(tn^2) \quad \text{and} \quad \sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big((\rho_i-\mathfrak{r}_i)^{2k}\big)(tn^2), \tag{1.17}$$

and besides

$$\sum_{i\in\mathbb{T}_n} \mathfrak{p}_i^{2k}(tn^2) \quad \text{and} \quad \sum_{i\in\mathbb{T}_n} \mathfrak{r}_i^{2k}(tn^2). \tag{1.18}$$

From (1.13) and (1.14), the variables $\pi_i-\mathfrak{p}_i$ and $\rho_i-\mathfrak{r}_i$ are centered under the law $G_{\mathbf{m},\mathbf{C}}$. We start with (1.18): the sequences $\{\mathfrak{p}_i\}$ and $\{\mathfrak{r}_i\}$ are completely deterministic and satisfy the following system of differential equations: for $i\in\mathbb{T}_n$ and $t\geqslant 0$,

$$\begin{cases}\mathfrak{p}_i' = \mathfrak{r}_{i+1}-\mathfrak{r}_i - 2\gamma\ \mathfrak{p}_i,\\ \mathfrak{r}_i' = \mathfrak{p}_i-\mathfrak{p}_{i-1},\end{cases}$$

Denote by $\mathfrak{P}$ the column vector ${}^{\mathbf{t}}(\mathfrak{p}_1,\dots,\mathfrak{p}_n,\mathfrak{p}_1',\dots,\mathfrak{p}_n')$. It is not difficult to see that $\mathfrak{P}(t)$ follows a first order ordinary differential equation written as

$$y'(t) = M\cdot y(t), \tag{1.19}$$

where M is the following constant block matrix:

$$M := \begin{pmatrix} 0_n & I_n \\ \\ D & -2\gamma I_n\end{pmatrix} \text{ where } D := \begin{pmatrix} -2 & 1 & & & 1\\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1\\ 1 & & & 1 & -2\end{pmatrix}.$$

Above I_n is the (n,n)-identity matrix. One can easily check that the column vector $\mathfrak{R} :={}^{\mathbf{t}}(\mathfrak{r}_1,\dots,\mathfrak{r}_n,\mathfrak{r}_1',\dots,\mathfrak{r}_n')$ follows the same first order ordinary differential equation. The matrix D represents the discrete Laplacian operator with mixed periodic boundary conditions. The characteristic polynomial of M is $\chi(X) := \det(XI_{2n}-M) = \det(D-X(X+2\gamma)I_n)$. In other words, the eigenvalues of M are exactly equal to the solutions of

$$x(x+2\gamma) = -\lambda,$$

where $-\lambda$ takes any eigenvalue of D. It is well-known that the eigenvalues of D are all negatives. Precisely,

(i) if $\gamma^2>\lambda$, then the two solutions are real negative numbers written as

$$x_\pm = -\gamma \pm \sqrt{\gamma^2-\lambda} < 0,$$

(ii) if $\gamma^2 < \lambda$, then the two solutions are complex numbers written as

$$x_\pm = -\gamma \pm i\sqrt{-\gamma^2 + \lambda},$$

(iii) if $\gamma^2 = \lambda$, then $-\gamma$ is the unique solution.

As a consequence, every eigenvalue of M has a negative real part, and the system (1.19) is hyperbolic. The proof would be done if one could show that there exists a constant $C > 0$ (that does not depend on n) such that, for any $t \geqslant 0$,

$$\big\| \exp(tn^2 M)\, \mathfrak{P}(0) \big\| \leqslant C \big\| \mathfrak{P}(0) \big\|, \tag{1.20}$$

where $\|\cdot\|$ is the standard $2k$-norm. Even if the system is *exponentially stable*, this is not enough to conclude, because we need to control the behavior of this stability when the dimension n of the system becomes very large. This is explained in more details in Subsection 1.2.

However, the result may be obtained in a different way: the system is completely solvable with Fourier transforms. See Subsection 1.3.

Assume that we are able to prove (1.20). We are interested in the quantity $\sum_i \mathfrak{p}_i^{2k}(tn^2)$, which is less or equal than the following norm

$$\big\|\mathfrak{P}(tn^2)\big\|_{2k} := \bigg(\sum_{i \in \mathbb{T}_n} \Big\{ |\mathfrak{p}_i(tn^2)|^{2k} + |\mathfrak{p}_i'(tn^2)|^{2k} \Big\} \bigg)^{\frac{1}{2k}}.$$

Observe that the initial condition writes

$$\big\|\mathfrak{P}(0)\big\|_{2k}^{2k} = \sum_{j \in \mathbb{T}_n} \bigg| \frac{\lambda_0}{\beta_0}\Big(\frac{j+1}{n}\Big) - \frac{\lambda_0}{\beta_0}\Big(\frac{j}{n}\Big) \bigg|^{2k}.$$

Since the initial profiles are smooth, it is clear that $\|\mathfrak{P}(0)\|_{2k}^{2k}$ is of order n^{1-2k}. Therefore, we proved that there exists a constant $C > 0$ such that

$$\sum_{i \in \mathbb{T}_n} |\mathfrak{p}_i(tn^2)|^{2k} \leqslant \big\|\mathfrak{P}(tn^2)\big\|_{2k}^{2k} \leqslant Cn.$$

The same argument is valid for $\mathfrak{R}(tn^2)$, since the initial condition reads

$$\big\|\mathfrak{R}(0)\big\|_{2k}^{2k} = \sum_{j \in \mathbb{T}_n} \bigg| \frac{\lambda_0}{\beta_0}\Big(\frac{j}{n}\Big) \bigg|^{2k},$$

and $\big\|\mathfrak{R}(0)\big\|_{2k}^{2k}$ is of order n (instead of n^{1-2k}). This is enough to conclude.

(v) Control of (1.17) – The last bounds would be obtained in the same way, as we quickly explain here. Let us introduce a new notation, which consists in rewriting the configurations in a different order: let

$$\omega_{2i} := \pi_i - \mathfrak{p}_i,$$
$$\omega_{2i+1} := \rho_i - \mathfrak{r}_i,$$

for $i \in \{0, ..., n-1\}$. Notice that $\omega_{i+2n} = \omega_i$. With this notation, the quantity (1.17) that we need to control becomes

$$\sum_{i\in\mathbb{T}_{2n}} G_{\mathbf{m},\mathbf{C}}\big(\omega_i^{2k}\big)(tn^2).$$

The idea is to write the time evolution dynamics of $G_{\mathbf{m},\mathbf{C}}\big(\omega_i^{2k}\big)$ in a convenient way such that a dissipative system arises. We now write in detail what happens for small values of k.

a) − Case $k = 2$. For this case, we will have to consider all terms on the form

$$G_{\mathbf{m},\mathbf{C}}\big(\omega_j\omega_k\omega_\ell\omega_m\big). \tag{1.21}$$

Let us define the function $\mathbf{c} : \{j \leqslant k \leqslant \ell \leqslant m\} \to \mathbb{N}$ as

(1) $\mathbf{c}(j, k, \ell, m) = 0$ if one of the following conditions is satisfied:

$$\begin{aligned} (i) &\quad j = k \text{ and } \ell = m,\\ (ii) &\quad j = k \text{ and } \ell, m \text{ are odd},\\ (iii) &\quad k = \ell \text{ and } j, m \text{ are odd},\\ (iv) &\quad k = m \text{ and } j, k \text{ are odd}, \end{aligned}$$

(2) otherwise, $\mathbf{c}(j, k, \ell, m)$ is the number of distinct even integers in $\{j, k, \ell, m\}$ (it can takes any value among $\{0, 1, 2, 3, 4\}$).

Notice the property:

$$\mathbf{c}(j, k, \ell, m) = \mathbf{c}(j + 2p, k + 2p, \ell + 2p, m + 2p), \qquad p \in \mathbb{N}.$$

Let $(j, k, \ell, m) \in \mathbb{T}_{2n}$ and define the operator $\mathbf{T}$ as follows:

$$\begin{aligned} \mathbf{T}\big(G_{\mathbf{m},\mathbf{C}}\big(\omega_j\omega_k\omega_\ell\omega_m\big)\big) =& G_{\mathbf{m},\mathbf{C}}\big((\omega_{j+1} - \omega_{j-1})\omega_k\omega_\ell\omega_m\big)\\ &+ G_{\mathbf{m},\mathbf{C}}\big(\omega_j(\omega_{k+1} - \omega_{k-1})\omega_\ell\omega_m\big)\\ &+ G_{\mathbf{m},\mathbf{C}}\big(\omega_j\omega_k(\omega_{\ell+1} - \omega_{\ell-1})\omega_m\big)\\ &+ G_{\mathbf{m},\mathbf{C}}\big(\omega_j\omega_k\omega_\ell(\omega_{m+1} - \omega_{m-1})\big). \end{aligned}$$

Without loss of generality we can assume $j \leqslant k \leqslant \ell \leqslant m$, and write the time derivative of (1.21) as

$$\frac{d}{dt}\Big(G_{\mathbf{m},\mathbf{C}}\big(\omega_j\omega_k\omega_\ell\omega_m\big)\Big) = \Big(\mathbf{T} - 2\gamma\mathbf{c}(j, k, \ell, m)\mathrm{Id}\Big)\Big(G_{\mathbf{m},\mathbf{C}}\big(\omega_j\omega_k\omega_\ell\omega_m\big)\Big).$$

Let us denote, for $k, \ell, m \in \mathbb{T}_{2n}$,

$$\begin{aligned} S_O(k, \ell, m) &:= \sum_{j=0}^{n-1} G_{\mathbf{m},\mathbf{C}}\big(\omega_{2j+1}\omega_{2j+1+k}\omega_{2j+1+\ell}\omega_{2j+1+m}\big),\\ S_E(k, \ell, m) &:= \sum_{j=0}^{n-1} G_{\mathbf{m},\mathbf{C}}\big(\omega_{2j}\omega_{2j+k}\omega_{2j+\ell}\omega_{2j+m}\big). \end{aligned}$$

At initial time, we can verify that: for all $k, \ell, m \in \mathbb{T}_{2n}$,

$$S_O(k, \ell, m)(0) \leqslant Cn, \qquad S_E(k, \ell, m)(0) \leqslant Cn.$$

Then, without loss of generality we can assume $k \leqslant \ell \leqslant m$, and from above we have

$$\begin{aligned}\frac{d}{dt}S_O(k,\ell,m) = \quad & S_E(k-1,\ell-1,m-1) - S_E(k+1,\ell+1,m+1)\\ & + S_O(k+1,\ell,m) - S_O(k-1,\ell,m)\\ & + S_O(k,\ell+1,m) - S_O(k-1,\ell,m)\\ & + S_O(k,\ell,m+1) - S_O(k,\ell,m-1)\\ & - 2\gamma\mathbf{c}(1,k+1,\ell+1,m+1)S_O(k,\ell,m),\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}S_E(k,\ell,m) = \quad & S_O(k-1,\ell-1,m-1) - S_O(k+1,\ell+1,m+1)\\ & + S_E(k+1,\ell,m) - S_E(k-1,\ell,m)\\ & + S_E(k,\ell+1,m) - S_E(k-1,\ell,m)\\ & + S_E(k,\ell,m+1) - S_E(k,\ell,m-1)\\ & - 2\gamma\mathbf{c}(0,k,\ell,m)S_E(k,\ell,m).\end{aligned}$$

With our notations, (1.17) becomes

$$\sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big((\pi_i - \mathfrak{p}_i)^4\big)(t) = S_E(0,0,0)(t)$$
$$\sum_{i\in\mathbb{T}_n} G_{\mathbf{m},\mathbf{C}}\big((\rho_i - \mathfrak{r}_i)^4\big)(t) = S_O(0,0,0)(t).$$

A simple computation shows that

$$\begin{aligned}&\frac{d}{dt}\Big[\sum_{k,\ell,m} S_E^2(k,\ell,m) + S_0^2(k,\ell,m)\Big] \qquad (1.22)\\ &= -2\gamma\sum_{k,\ell,m}\Big\{\mathbf{c}(1,k+1,\ell+1,m+1)S_O^2(k,\ell,m) + \mathbf{c}(0,k,\ell,m)S_E^2(k,\ell,m)\Big\}\\ &< 0,\end{aligned}$$

so that we have found a Lyapunov function which ensures that the equilibrium value 0 is asymptotically stable. This is not enough to conclude: this implies that, for all $t \geqslant 0$, all $k,\ell,m \in \mathbb{T}_{2n}$,

$$\big|S_E(k,\ell,m)(t)\big| \leqslant Cn^{3/2}, \qquad \big|S_O(k,\ell,m)(t)\big| \leqslant Cn^{3/2}.$$

We have to go into further investigation: let us denote by $\mathbf{S} \in (\mathbb{R}^{\mathbb{T}_{2n}^3} \times \mathbb{R}^{\mathbb{T}_{2n}^3})$ the vector with components $\{S_E(k,\ell,m), S_O(k,\ell,m)\ ;\ (k,\ell,m) \in \mathbb{T}_{2n}^3\}$. Then, notice that one can rewrite

$$\frac{d\mathbf{S}}{dt} = (\mathbf{A} - 2\gamma\mathbf{D})\cdot\mathbf{S},$$

where $\mathbf{D}$ is the diagonal matrix with diagonal elements

$$\{\mathbf{c}(0,k,\ell,m), \mathbf{c}(1,k+1,\ell+1,m+1)\ ;\ (k,\ell,m) \in \mathbb{T}_{2n}^3\}$$

which are all non-negative, and can be zero. Straightforward computations similar to (1.22) show that $\mathbf{A}$ is skew-symmetric. Therefore, the eigenvalues of $\mathbf{A}$ are pure imaginary.

More precisely, let us consider the standard complexification of $\mathbb{R}^{\mathbb{T}_{2n}^3}$, endowed with the Hermitian structure: we denote the inner product by $\langle\cdot\rangle$ and its associated norm by $\|\cdot\|^2$. Let $\mathbf{v} \in \mathbb{C}^{\mathbb{T}_{2n}^3}$ be a complex eigenvector associated to the complex eigenvalue $\lambda \in \mathbb{C}$ such that $\|\mathbf{v}\| = 1$. We have

$$(\mathbf{A} - 2\gamma\mathbf{D})\mathbf{v} = \lambda\mathbf{v}, \tag{1.23}$$

and therefore, after multiplying by $\mathbf{v}$ we get

$$\langle \mathbf{A}\mathbf{v}, \mathbf{v}\rangle = \lambda + 2\gamma\langle \mathbf{D}\mathbf{v}, \mathbf{v}\rangle.$$

In the same way, we have

$$\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle = \overline{\lambda} + 2\gamma\langle \mathbf{v}, \mathbf{D}\mathbf{v}\rangle.$$

Since $\mathbf{A}$ is skew-symmetric and then $\langle \mathbf{A}\mathbf{v}, \mathbf{v}\rangle = -\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle$, we deduce that

$$\mathrm{Re}(\lambda) = -2\gamma\, \mathrm{Re}\big(\langle \mathbf{D}\mathbf{v}, \mathbf{v}\rangle\big).$$

Since $\mathbf{D}$ is diagonal with non-negative values, $\langle \mathbf{D}\mathbf{v}, \mathbf{v}\rangle$ is real and non-negative. It remains to show that this can not be zero. Assume that

$$\langle \mathbf{D}\mathbf{v}, \mathbf{v}\rangle = \sum_i d_{i,i}|v_i|^2 = 0.$$

This implies that $v_i = 0$ for all indexes i such that $d_{i,i} > 0$, namely $\mathbf{v} \in \mathrm{Ker}(\mathbf{D})$. In (1.23) this gives $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, hence $\mathbf{A}\mathbf{v} \in \mathrm{Ker}(\mathbf{D})$. This is not possible, and follows from an easy observation: by definition, $\mathrm{Ker}(\mathbf{D})$ is generated by

$$\begin{aligned}
&S_E(0, 2\ell, 2\ell), S_E(2\ell, 0, \ell), S_E(2\ell, 2\ell, 0)\\
&S_E(0, 2\ell+1, 2m+1), S_E(2\ell+1, 0, 2m+1), S_E(2\ell+1, 2m+1, 0)\\
&S_O(2k, 2\ell+1, 2\ell+1), S_O(2\ell+1, 2k, 2\ell+1), S_O(2\ell+1, 2\ell+1, 2k)\\
&S_O(2k, 2\ell, 2m), \qquad k, \ell, m \in \mathbb{T}_{2n}.
\end{aligned}$$

Applying operator $\mathbf{A}$ to each of these elements gives extra elements which are not in $\mathrm{Ker}(\mathbf{D})$. As a consequence,

$$\mathrm{Re}(\lambda) = -2\gamma\, \mathrm{Re}\big(\langle \mathbf{D}\mathbf{v}, \mathbf{v}\rangle\big) \geqslant -2\gamma \min_i\{d_{i,i}\ ;\ d_{i,i} > 0\} = -2\gamma.$$

In particular, the real parts of the eigenvalues are bounded from below by a constant which does not depend on n. We are reduced to the same question as in the previous case. The key point is to control the behavior with respect to n of a sequence of n-dimensional hyperbolic systems. The next subsection gives a few elements in this direction.

□

1.2. **Exponential stability and Jordan normal form.** We state a property which would be the key point in the previous argument, but is absolutely not obvious. Our focus is on the stability of n-dimensional linear systems, when n is very large. It is already well-known that internal stability of linear systems depends on the structural properties of the state matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ that governs the dynamics via $\dot{x} = \mathbf{A}x$. For instance, *exponential stability* ensures that the state $x(t)$ remains within an arbitrarily small neighborhood of the initial state whenever the time span is sufficiently large: there exists $c, \lambda > 0$ such that

$$\|x(t)\| \leqslant ce^{-\lambda t}\|x(0)\|, \tag{1.24}$$

where $\|\cdot\|$ denotes the usual euclidean norm in $\mathbb{R}^n$. This property is satisfied if and only if all eigenvalues of $\mathbf{A}$ have negative real parts (see for example [2]).

Let us consider a sequence of n-dimensional linear systems, given by $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$, which are all exponentially stable. Both constants c, λ in (1.24) depend *a priori* on n. What we need is a bound that does not depend on n, even if we have to pay the price of the exponential decay. With the diffusion of energy in mind, we focus our analysis on the diffusive time scale tn^2.

Lemma 1.2 (not proved!). *Let $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ be a sequence of matrices in $\mathbb{R}^{n\times n}$. We denote by $\{a_{i,j}^{(n)}\}$ the coefficients of $\mathbf{A}_n$, and we assume that:*

(i) there exists $a_{\sup} > 0$ such that, for any $n \in \mathbb{N}$, $i, j \in \{1, \dots, n\}$,

$$|a_{i,j}^{(n)}| \leqslant a_{\sup},$$

(ii) all eigenvalues of $\mathbf{A}_n$ have negative real parts, and there exists $\gamma_0 > 0$ such that, for any $n \in \mathbb{N}$, and any eigenvalue λ of $\mathbf{A}_n$

$$\mathrm{Re}(\lambda) \leqslant -\gamma_0.$$

Let $x_n \in \mathbb{R}^n$ be the solution to the linear differential equation

$$\dot{x}_n(t) = \mathbf{A}_n x_n(t). \tag{1.25}$$

There exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\big\|x_n(tn^2)\big\| \leqslant C\big\|x_n(0)\big\|,$$

where $\|\cdot\|$ is the norm for which we have a good control at initial time, namely:

$$\|x_n(0)\| \leqslant Cn.$$

Remark 1.3. *As we have seen in the previous sections, for (1.17) it should be the supremum norm $\|\cdot\|_\infty$.*

Try of proof. Step 1: Jordan canonical form.

We first recall some well-known facts coming from Linear Algebra: there exists an invertible matrix $\mathbf{P}_n \in \mathbb{C}^{n\times n}$ such that $\mathbf{P}_n^{-1}\mathbf{A}_n\mathbf{P}_n = \mathbf{J}_n \in \mathbb{C}^{n\times n}$ has the form

$$\mathbf{J}_n = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$$

where each block J_k has the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}.$$

In our case, every λ_k has negative real part, and for any $n \in \mathbb{N}$ and any k, $\mathrm{Re}(\lambda_k) \leqslant -\gamma_0$. The solution to (1.25) writes

$$x_n(tn^2) = \mathbf{P}_n \exp(tn^2 \mathbf{J}_n) \mathbf{P}_n^{-1} \cdot x_n(0).$$

Therefore,

$$\big\| x_n(tn^2) \big\| \leqslant \big\| \mathbf{P}_n \big\| \big\| \mathbf{P}_n^{-1} \big\| \big\| \exp(tn^2 \mathbf{J}_n) \big\| \cdot \big\| x_n(0) \big\|.$$

The number $\kappa(\mathbf{P}) := \|\mathbf{P}\| \|\mathbf{P}^{-1}\|$ is the *condition number* and can be arbitrarily large, if $\mathbf{P}$ is “almost” singular. Some remarks:

(1) If $\|\cdot\|$ is the usual euclidean norm, then $\kappa(\mathbf{P}) = \sigma_{\max}(\mathbf{P})/\sigma_{\min}(\mathbf{P})$, where $\sigma_{\max}(\mathbf{P})$ and $\sigma_{\min}(\mathbf{P})$ are maximal and minimal singular values of $\mathbf{P}$, respectively. It is known for example (see [1]) that

$$\kappa(\mathbf{P}) \leqslant \frac{2}{|\det(\mathbf{P})|} \left(\frac{\|\mathbf{P}\|_F}{\sqrt{n}} \right)^{n/2},$$

where $\|\cdot\|_F$ is the Frobenius norm, whose squared is the sum of the squared coefficients.

The problem is that $\det(\mathbf{P})$ can be arbitrarily small.

(2) The norm $\big\| \exp(T\mathbf{J}_n) \big\|$ is not hard to compute, for example in the case of the supremum norm:

$$\big\| \exp(T\mathbf{J}_n) \big\|_\infty = \sum_{k=0}^{n-1} \frac{T^k}{k!} \times \exp(-T\gamma_0) \leqslant \exp\big(T(1+\gamma_0)\big).$$

Therefore, for $T = tn^2$ it can be made exponentially small (of order $\exp(-cn^2)$) if $\gamma_0 > 1$ (this can be ensured by increased the value of the flip intensity γ).

We need $\|\mathbf{P}\|_\infty \|\mathbf{P}^{-1}\|_\infty$ to be of order less than $\exp(cn^2)$.

(3) Since $\big\| \exp(tn^2 \mathbf{J}_n) \big\|_\infty$ can be made of order $\exp(-cn^2)$ with $c > 0$ (thanks to the fact that the eigenvalues have negative real parts which are bounded from above by a constant independent on n), we are not afraid of using other norms than $\|\cdot\|_\infty$, since the standard p-norms are equivalent with constants that are polynomial in n:

$$\|\cdot\|_\infty \leqslant \|\cdot\|_p \leqslant n^{1/p} \|\cdot\|_\infty.$$

Conclusion: We need more precise information on the matrix $\mathbf{P}_n$ which represents the generalized eigenvectors of the matrix $\mathbf{A}_n$.

□

1.3. **Fourier transforms.** The control of (1.18) may be obtained through Fourier transforms. Let us come back to the dynamical system written in terms of positions q_x and velocities p_x. Let us introduce $\omega^2(k) = 4\sin^2(\pi k)$ the dispersion relation.

Let us denote by $\widehat{q}(k)$ (resp. $\widehat{p}(k)$) the Fourier transforms of the averages defined for $k \in \mathbb{T}$ as

$$\widehat{q}(t,k) = \sum_{x\in\mathbb{T}_n} \mu_t^n[q_x]\exp(-2i\pi kx), \qquad \widehat{p}(t,k) = \sum_{x\in\mathbb{T}_n} \mu_t^n[p_x]\exp(-2i\pi kx).$$

The time-evolution of the latter is given by

$$\begin{aligned} &\frac{\mathrm{d}\widehat{p}}{\mathrm{d}t}(t,k) = -2\gamma n^2\, \widehat{p}(t,k) - n^2\omega^2(k)\, \widehat{q}(t,k), \\ &\frac{\mathrm{d}\widehat{q}}{\mathrm{d}t}(t,k) = n^2\, \widehat{p}(t,k), \end{aligned}$$

with the initial condition, for $k \in \mathbb{T}$, which reads as

$$\widehat{p}(0,k) = 0, \qquad \omega(k)\widehat{q}(0,k) = -ie^{i\pi k}\,\mathrm{sgn}(k) \sum_{x\in\mathbb{T}_n} r_0\Big(\frac{x}{n}\Big) e^{-2i\pi xk},$$

where $\mathrm{sgn}(k) = 1$ if $k \geqslant 0$ and -1 otherwise. This system can be explicitly solved for $\gamma > 2$ as (denoting $\omega^2 = \omega^2(k)$ for the sake of clarity)

$$\begin{aligned} \widehat{p}(t,k) &= -\frac{\omega^2\, \widehat{q}(0,k)}{2\sqrt{\gamma^2-\omega^2}}\, e^{-\gamma n^2 t} \Big(e^{n^2t\sqrt{\gamma^2-\omega^2}} - e^{-n^2t\sqrt{\gamma^2-\omega^2}}\Big) \\ \widehat{q}(t,k) &= \frac{\widehat{q}(0,k)}{2\sqrt{\gamma^2-\omega^2}}\, e^{-\gamma n^2 t} \Big(\big(\gamma + \sqrt{\gamma^2-\omega^2}\big)e^{n^2t\sqrt{\gamma^2-\omega^2}} \\ &\qquad\qquad - \big(\gamma - \sqrt{\gamma^2-\omega^2}\big)e^{-n^2t\sqrt{\gamma^2-\omega^2}}\Big). \end{aligned}$$

Notice that, when $\gamma \gg 1$,

$$\widehat{q}(t,k) \simeq \widehat{q}(0,k)\exp\Big(-\frac{n^2\,\omega^2(k)\,t}{2\gamma}\Big). \tag{1.26}$$

And we have, for $m \in \mathbb{N}$,

$$\sum_{x\in\mathbb{T}_n} \big|\mu_t^n[q_x]\big|^{2m}(tn^2) = \sum_{x\in\mathbb{T}_n} \bigg| \int_{\mathbb{T}} \widehat{q}(t,k)e^{2i\pi kx}\mathrm{d}k\bigg|^{2m}.$$

Using the approximate above (1.26) and inequality $\sin^2(\pi k) \geqslant Ck^2$, we have

$$\begin{aligned}
\sum_{x\in\mathbb{T}_n}\left|\int_{\mathbb{T}}\widehat{q}(t,k)e^{2i\pi kx}\mathrm{d}k\right|^{2m} &= \sum_{x\in\mathbb{T}_n}\left|\sum_{y\in\mathbb{T}_n} r_0\Big(\frac{y}{n}\Big)\int_{\mathbb{T}} e^{2i\pi k(x-y)}e^{-2n^2\, t\sin^2(\pi k)/\gamma}\ \mathrm{d}k\right|^{2m}\\
&\leqslant \sum_{x\in\mathbb{T}_n}\left(\sum_{y\in\mathbb{T}_n} r_0\Big(\frac{y}{n}\Big)\int_{\mathbb{T}} e^{-2n^2\, t\sin^2(\pi k)/\gamma}\ \mathrm{d}k\right)^{2m}\\
&\leqslant \sum_{x\in\mathbb{T}_n}\left(\sum_{y\in\mathbb{T}_n} r_0\Big(\frac{y}{n}\Big)\int_{\mathbb{T}} e^{-Ck^2n^2}\ \mathrm{d}k\right)^{2m}\\
&\leqslant \sum_{x\in\mathbb{T}_n}\left(\sum_{y\in\mathbb{T}_n} r_0\Big(\frac{y}{n}\Big)\frac{1}{n}\int_{\mathbb{R}} e^{-Cu^2}\ \mathrm{d}u\right)^{2m} = O(n).
\end{aligned}$$

Notice that this estimate holds only for $t > 0$, and in the diffusive scale. To get (1.18) precisely, we need to use carefully the exact expressions above, taking advantage of the exponential decreasing.

References

[1] H.W. Guggenheimer, A.S. Edelman, C.R. Johnson, *A simple estimate of the condition number of a linear system*, College Math. J. **26** (1) (1995) 25.

[2] Wilson J. Rugh, *Linear System Theory (2nd Ed.)* Prentice-Hall, Inc., Upper Saddle River, NJ, USA 1996.

[3] M. Simon, Hydrodynamic limit for the velocity-flip model, Stochastic Processes and their Applications **123** (2013) 3623 3662.