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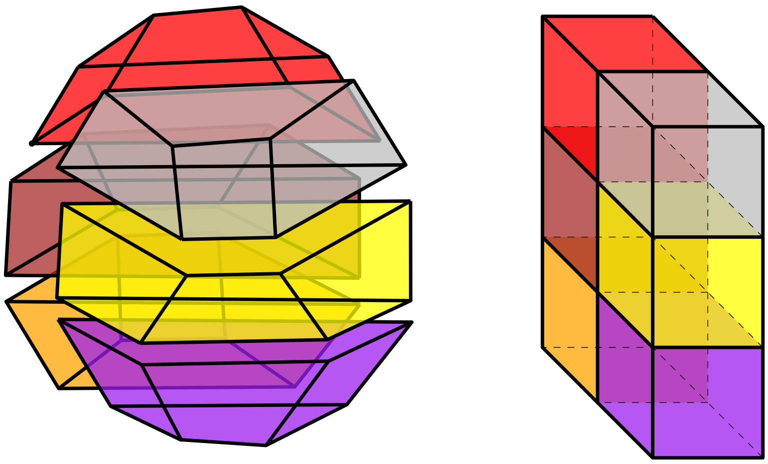
Algebraic combinatorics around a problem in enumerative geometry



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Algebraic combinatorics around a problem in enumerative geometry

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List of publications

Here is a complete list of publications. These are removed from the general bibliography and I used arabic numbers to refer to them in this manuscript, whereas I used classical citations such as [LS82] for the general bibliography.

Works carried out after my PhD:

[1] P. Nadeau, H. Spink, and V. Tewari. The geometry of quasisymmetric coinvariants, *arXiv:2410.12643*, 2024. Chap. 4

[2] P. Nadeau, H. Spink, and V. Tewari. Schubert polynomial expansions revisited, *arXiv:2407.02375*, 2024. Accepted for publication in *Forum Math. Sigma*. Chap. 4

[3] P. Nadeau, H. Spink, and V. Tewari. Quasisymmetric divided differences, *arXiv:2406.01510v2*, 2024. Chap. 4

[4] A. Iraci, P. Nadeau, and A. Vanden Wyngaerd. Smirnov words and the Delta conjectures. *Adv. Math.*, 452:Paper No. 109793, 41 p., 2024.

[5] P. Nadeau and V. Tewari. Forest polynomials and the class of the permutahedral variety. *Adv. Math.*, 453:Paper No. 109834, 33 p., 2024. Chap. 2

[6] P. Nadeau and V. Tewari. P-partitions with flags and back stable quasisymmetric functions. *Comb. Theory*, 4(2):Paper No. 4, 22 p., 2024. Chap. 3

[7] C. Defant, V. Féray, P. Nadeau, and N. Williams. Wiener indices of minuscule lattices. *Electron. J. Combin.*, 31(1):Paper No. 1.41, 23 p., 2024.

[8] W. Fang, É. Fusy, and P. Nadeau. Tamari intervals and blossoming trees. *Comb. Theory*, 5(1):Paper No. 4, 41 p., 2025.

[9] P. Nadeau and V. Tewari. Down-up algebras and chromatic symmetric functions. *Sém. Lothar. Combin.*, 88:Art. B88c, 18 p., [2023–2024]. Chap. 3

[10] P. Nadeau and V. Tewari. The permutahedral variety, mixed Eulerian numbers, and principal specializations of Schubert polynomials. *Int. Math. Res. Not. IMRN*, (5):3615–3670, 2023. Chap. 2

[11] P. Nadeau. Bilateral parking procedures, *hal-04262134*, 2023. Chap. 3

[12] P. Nadeau and V. Tewari. Remixed Eulerian numbers. *Forum Math. Sigma*, 11:Paper No. e65, 26 p., 2023. Chap. 3

[13] M. Josuat-Vergès and P. Nadeau. Koszulity of dual braid monoid algebras via cluster complexes. *Ann. Math. Blaise Pascal*, 30(2):141–188, 2023.

[14] A. Ayyer and P. Nadeau. Combinatorics of a disordered two-species ASEP on a torus. *European J. Combin.*, 103:Paper No. 103511, 20 p., 2022.

[15] P. Nadeau and V. Tewari. Divided symmetrization and quasisymmetric functions. *Selecta Math. (N.S.)*, 27(4):Paper No. 76, 24 p., 2021. Chap. 2

[16] P. Nadeau and V. Tewari. A q-analogue of an algebra of Klyachko and Macdonald's reduced word identity, *arXiv:2106.03828*. Accepted for publication in *Trans. Amer. Math. Soc.* Chap. 3

[17] F. Aigner, I. Fischer, M. Konvalinka, P. Nadeau, and V. Tewari. Alternating sign matrices and totally symmetric plane partitions. Proceedings of FPSAC 2020, *Sém. Lothar. Combin.*, 84B:Art. 77, 12 p., 2020.

[18] O. Bernardi and P. Nadeau. Combinatorial reciprocity for the chromatic polynomial and the chromatic symmetric function. *Discrete Math.*, 343(10):111989, 13, 2020.

[19] H. Mühle, P. Nadeau, and N. Williams. k-indivisible noncrossing partitions. *Sém. Lothar. Combin.*, 81:Art. B81d, 23 p., 2020.

[20] H. Mühle and P. Nadeau. A poset structure on the alternating group generated by 3-cycles. *Algebr. Comb.*, 2(6):1285–1310, 2019.

[21] R. Biagioli, F. Jouhet, and P. Nadeau. 321-avoiding affine permutations and their many heaps. *J. Combin. Theory Ser. A*, 162:271–305, 2019.

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[37] O. Bernardi, B. Duplantier, and P. Nadeau. A bijection between well-labelled positive paths and matchings. *Sém. Lothar. Combin.*, 63:Art. B63e, 13 p., 2010.

[38] L. Lapointe, Y. Le Borgne, and P. Nadeau. A normalization formula for the Jack polynomials in superspace and an identity on partitions. *Elec. J. Combin.*, 16(1):Research Paper 70, 24 p., 2009.

[39] M. Albenque and P. Nadeau. Growth function for a class of monoids. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, volume AK of *Discrete Math. Theor. Comput. Sci. Proc.*, pages 25–38. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.

Short versions of the works [3],[4],[5],[8],[15],[17][20],[24],[29],[32],[34] appear in the proceedings of FPSAC conferences of various years. Short versions of the works [11] and [21] appear in the proceedings of the conferences GASCOM 2016 (Bordeaux) and Discrete Mathematics Days 2022 (Santander) respectively.

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[40] D. Gouyou-Beauchamps and P. Nadeau. Signed enumeration of ribbon tableaux: an approach through growth diagrams. *J. Algebraic Combin.*, 36(1):67–102, 2012.

[41] S. Corteel and P. Nadeau. Bijections for permutation tableaux. *European J. Combin.*, 30(1):295–310, 2009.

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Avant-propos

Introduction

Je vais évoquer ici brièvement les travaux de cette thèse et leur genèse. Une présentation plus détaillée est donnée dans l'introduction en anglais dans la section suivante.

Dans cette thèse d'habilitation, je présente certains des résultats que j'ai obtenus depuis ma thèse de doctorat en 2007. Depuis lors, j'ai travaillé sur une variété de sujets en combinatoire énumérative et algébrique. Je me focalise dans ce manuscrit exclusivement sur des résultats obtenus après 2018, qui sont nés de l'étude d'un problème particulier en géométrie énumérative.

Commençons par expliquer la genèse de ce travail. Tout a commencé au premier semestre de 2018, alors que j'étais en mission longue à l'Université de Washington (UW) à Seattle dans le groupe dirigé par la professeure Sara Billey. Alex Woo, de l'Université de l'Idaho, était également là pour le semestre dans le cadre d'un congé sabbatique. À leur initiative, un groupe de travail sur les variétés de Hessenberg a démarré dès janvier 2018, auquel participaient régulièrement Sean Griffin, Jake Levinson, Josh Swanson et Vasu Tewari, tous doctorants ou postdoctorants à l'époque.

Alex Woo a commencé par quelques exposés sur les aspects géométriques et combinatoires de ces variétés, puis le séminaire s'est concentré sur le calcul de certains coefficients a_w. Plus précisément, w est ici une permutation dans S_n, et a_w compte le nombre de points d'intersection de la variété permutaédrale Perm_n avec une variété de Schubert X_w générique. Le but était de donner une formule manifestement positive pour ces nombres.

Initialement le groupe était impliqué au complet, puis Vasu Tewari et moi-même avons finalement collaboré plus spécifiquement après quelques semaines. Nous avons obtenu quelques résultats préliminaires en avril-mai 2018.

Après avoir quitté Seattle cet été-là, j'ai poursuivi la collaboration avec Vasu – menant in fine aux travaux de cette habilitation. Nous avons en particulier résolu le problème initial de manière satisfaisante, en donnant une interprétation combinatoire des nombres a_w publiée en 2024. En travaillant sur le problème pendant ces années, nous sommes tombés sur de nombreuses questions de combinatoire algébrique qui ont été utiles pour résoudre le problème initial et/ou intéressantes en elles-mêmes. Plus récemment, avec Hunter Spink, une reformulation d'une partie de notre travail a conduit à un point de vue nouveau dans l'étude des polynômes quasisymétriques, tout en apportant un nouvel éclairage sur le problème initial.

Je vais raconter cette histoire dans ce document, qui est structuré en quatre chapitres. Le chapitre 1 rassemble diverses notations et définitions notamment autour des permutations. Nous décrivons également brièvement la théorie des (P,ω)-partitions de Stanley et des fonctions quasisymétriques; donnons quelques détails sur comment la cohomologie permet une approche algébrique de certains problèmes d'intersection; et listons enfin plusieurs propriétés des polynômes de Schubert. Le chapitre 2 est consacré au problème original de géométrie énumérative, et présente deux formules pour a_w, conduisant à plusieurs propriétés de ces nombres ainsi qu'à l'interprétation combinatoire déjà évoquée. Le chapitre 3 énumère les différentes contributions qui ont émergé lors de la résolution du problème original: la théorie des P-partitions avec bornes, celle des procédures de stationnement bilatérales, et enfin une q-déformation de l'approche algébrique originale, menant notamment à l'étude des polynômes eulériens mixtes. Enfin, je présente au chapitre 4 une nouvelle approche de la théorie des polynômes quasisymétriques, soulignant le rôle particulier des polynômes “forêt”.

L'exposition dans ce manuscrit est intentionnellement informelle : les preuves seront au mieux esquissées, et je me référerai aux publications originales pour les détails.

Remerciements

Je souhaite remercier l'ensemble des membres du jury, en particulier Philippe Biane, Ilse Fischer et Vic Reiner qui ont accepté de consacrer du temps à la lecture de ce manuscrit.

Je voudrais également remercier toutes les personnes avec qui j'ai eu l'occasion de travailler au cours des très nombreuses années depuis ma thèse: mes co-auteur.ices (au premier rang desquels se trouve Vasu Tewari), les collègues mathématicien.ne.s et le personnel administratif d'abord de l'Université de Vienne, puis de l'Institut Camille Jordan.

Merci enfin à mes amis et ma famille.

Introduction

I present in this habilitation thesis a part of the results I obtained since my PhD thesis in 2007. I have worked on a variety of topics in enumerative and algebraic combinatorics since then, but I will focus in this manuscript exclusively on a series of works obtained post 2018 that stemmed from studying a particular problem in enumerative geometry.

Let me start by explaining the genesis of this work. In the first semester of 2018, I was hosted at the University of Washington (UW) in Seattle in the group headed by Prof. Sara Billey. Alex Woo from the University of Idaho was also there for the semester, on sabbatical, so a working seminar about Hessenberg varieties was started in January 2018 with –in addition to the three of us– Sean Griffin, Jake Levinson, Josh Swanson, and Vasu Tewari, all of whom were PhD or postdoctoral students at the time.

Alex Woo first gave a few lectures on geometric and combinatorial aspects of these varieties, but the seminar quickly focused on the computation of certain intersection coefficients a_w. Explicitly w is here a permutation in S_n, and a_w counts the number of points in the intersection of the permutahedral variety Perm_n with a generic Schubert variety X_w; the goal was to find a manifestly positive rule for these coefficients. While the whole group was involved in the first few weeks, Vasu Tewari and I focused more specifically in the end, and we obtained some preliminary results around April-May 2018.

After I left Seattle that summer, the collaboration with Vasu continued, all the way to the time of writing this very habilitation thesis. We ended up solving satisfyingly the original problem by giving a combinatorial interpretation for the numbers a_w, published in 2024. While working on the problem, we came up with several contributions to the algebraic combinatorics literature that were both helpful in solving the problem and interesting on their own, leading to several other articles. Finally, together with Hunter Spink –who moved to Toronto in 2023, as did Vasu–, a reinterpretation of some of the earlier work led to a nice breakthrough in the study of quasisymmetric polynomials, while shedding new light on the original problem.

I will detail many of these results in this document, which is structured in four chapters. Chapter 1 gathers various notations and definitions in particular around permutations. We briefly sketch Stanley's theory of (P,ω)-partitions and quasisymmetric functions, give some details about how cohomology theory provides an algebraic approach to certain intersection problems, and finally list several properties of Schubert polynomials. Chapter 2 is devoted to the original problem in enumerative geometry, presents two formulas for a_w as well, leading to several properties of these numbers as well as the mentioned combinatorial interpretation. Chapter 3 lists various contributions that emerged while solving the original problem: the theory of flagged P-partitions, bilateral parking procedures, and a q-deformation of the original algebraic approach, leading to remixed Eulerian numbers. Lastly, I present in Chapter 4 a new approach to the theory of quasisymmetric polynomials, highlighting the special role of forest polynomials parallel to the case of Schubert polynomials

The exposition in this manuscript is intentionally informal. I will either sketch proofs or refer to the original publications, namely **[1]**,**[2]**,**[3]**,**[5]**,**[6]**,**[9]**,**[10]**,**[11]**,**[12]**,**[15]**,**[16]** in my list of publications. I will stress how various ideas and structures can be found in several places, and how we understand their interconnections.

Our plan is not to tell all of the results in the above papers either, as we try to keep a certain focus in this manuscript. We have thus prioritized the results that connect to the original problem regarding the coefficients a_w. Also, as we want to deal mostly with enumerative and algebraic combinatorics, we will make almost no mention of our results of a more geometric nature: this concerns most of **[1]**, as well as some sections of **[10]** and **[12]** for instance.

We now present the main results from each chapter. Many of the definitions of the various notions and structures are found in the main body of this manuscript.

Chapter 2: Schubert coefficients for the permutahedral variety

The results presented here come from the works **[5]**,**[10]** and **[15]**, in collaboration with Vasu Tewari.

Let us give some motivation for the original problem. Hessenberg varieties[1] are a relatively recent family of subvarieties of the flag variety Fl_n with inspiration from numerical analysis [DMPS92]. Recall that Fl_n is the space of complete flags

$$V_\bullet = (V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n),$$

where each V_i is a linear subspace of $\mathbb{C}^n$ of dimension i; Hessenberg varieties will be defined in Section 2.1 Their study has also revealed a rich interplay between geometry, representation theory and combinatorics [AT10, HT17, Tym07], and the last decade has witnessed increasing interest with impetus coming from the study of chromatic quasisymmetric functions and its ramifications for the Stanley-Stembridge conjecture [HP19, SW12, SW16] (recently proved by Hikita [Hik24]). The study of the cohomology rings of Hessenberg varieties has been linked to the study of hyperplane arrangements and representations of the symmetric group [AHM19, AHM$^+$, BC18, HHMP19]. We refer the reader to Abe and Horiguchi's e survey article [AH20] and references therein for more details on mathematics surrounding Hessenberg varieties.

The permutahedral variety Perm_n plays a special role in the theory. It is the lowest dimensional of the (regular, semisimple, irreducible) Hessenberg varieties, and is in fact naturally contained in all of them. The permutahedral variety is a smooth projective complex toric variety of dimension $n-1$: it is determined by the braid fan, the normal fan of the usual permutahedron of dimension $n-1$.

As it is a subvariety of Fl_n, its cohomology class $[\mathrm{Perm}_n] \in H^*(\mathrm{Fl}_n)$ expands in terms of Schubert classes $[X_w]$ as follows:

$$[\mathrm{Perm}_n] = \sum_{w \in S'_n} a_w [X_w],$$

where S'_n denotes the symmetric group on n letters with length $n-1$.

The coefficients a_w are nonnegative integers as they are intersection numbers. We use two different formulas to study them, each one based on results in algebraic geometry.

The first formula is (**2.2.1**) which follows from work of Klyachko [Kly85]:

$$a_w = \sum_{\mathbf{i} \in \mathrm{Red}(w)} \frac{A_{c(\mathbf{i})}}{(n-1)!}.$$

Here the numbers A_c are mixed Eulerian numbers, and these are known to be positive integers. It follows immediately from this formula that the a_w are positive – but not that they are integers because of the division by $(n-1)!$.

However, this formula allows us to derive several properties of a_w:

- The symmetry $a_w = a_{w^{-1}}$, cf. Corollary 2.2.2.

[1] These have been defined in all Lie types, we focus here only on type A.

- Summation properties, see Propositions 2.2.3 and 2.2.4.
- A combinatorial interpretation for "vexillary" permutations (Proposition 2.2.5).

To obtain a general combinatorial interpretation, we use a second formula (**2.1.5**) which follows from work of Anderson and Tymoczko [AT10]:

$$a_w = \langle \mathfrak{S}_w(x_1, \ldots, x_n) \rangle_n.$$

Here $\mathfrak{S}_w$ is a *Schubert polynomial*, while $\langle \cdot \rangle_n$ is a certain linear form called divided symmetrization, see (**2.1.4**). From this formula we will be able to obtain the following result. There is a certain *parking procedure* Ω for words over $\mathbb{Z}$ defined in Section 2.3.2, which selects certain words to be Ω-parking, and for which we have:

Theorem A (*Theorem 2.3.4*)

Let $w \in S'_n$. Then a_w is the number of reduced words of w^{-1} that are also Ω-parking words.

This is the solution to the original problem from the working seminar in Seattle. There are a number of ingredients in the proof, which is sketched in Section 2.3.

A key one is the definition of a family of *forest polynomials* P_F. Here F is an *indexed forest* (a certain forest of plane binary trees) and P_F counts certain labelings of the nodes of F; see Section 2.3.3 for details. The P_F form a basis of $\mathbb{Q}[x_1, x_2, \cdots]$, as do Schubert polynomials themselves, as well as the *slide polynomials* $\mathfrak{F}_c$ defined in Chapter 1. Say that a polynomial expands positively in a basis if all coefficients in the expansion are nonnegative. In the course of the proof, we will establish the following.

Theorem B

Schubert polynomials expand positively in forest polynomials, and forest polynomials expand positively into slide polynomials.

Another ingredient in the proof is the ideal $\mathrm{QSym}_n^+ \subset \mathbb{Q}[x_1, \ldots, x_n]$ generated by *quasisymmetric polynomials* with zero constant term, as they interact nicely with the P_F and divided symmetrization. We will come back to these objects in Chapter 4 with a very different point of view.

Chapter 3: Miscellany of combinatorics

The results presented here come from the works **[6]**,**[9]**,**[12]**,**[16]** written with Vasu Tewari, and **[11]** by myself. They are all inspired by the methods and results from Chapter 2.

The first contribution in this chapter, Section 3.1 is somewhat technical, and deals with an extension of Stanley's theory of (P, ω)-partitions presented in Chapter 1. Its motivation is to explain, inside a satisfying framework, the expansion of forest polynomials into slide polynomials used in the proof of the combinatorial interpretation of a_w.

It deals with flagged P-partitions, as studied by Assaf and Bergeron [AB20], but gives a more streamlined version of the theory. In particular the slide decomposition (**3.1.4**) will be automatic following Stanley's steps.

The second contribution is about a theory of *bilateral parking procedures*: we give a precise mathematical definition in Section 3.2. The idea is that we have a word $a_1 a_2 \cdots a_r$, representing cars $1, 2, \cdots, r$ arriving successively, and car i wants to park in the spot $a_i \in \mathbb{Z}$. In the classical parking procedure, a car whose desired spot is taken parks in the nearest available spot on the right.

Bilateral parking procedures $\mathcal{P}$ allow rules where one can also park in the nearest available spot on the left. One says a word $a_1 a_2 \cdots a_r$ is $\mathcal{P}$-parking if cars are parked in spots $\{1, 2, \cdots, r\}$. We define a notion of *local* procedure for which the following result holds:

Theorem C (*Theorem 3.2.3*)

Let $\mathcal{P}$ be a bilateral, local parking procedure. Then the number of $\mathcal{P}$-parking functions of length r is given by $(r+1)^{r-1}$.

This can be seen as a universal enumeration result: we do not need to know the details of the procedure to determine the number of parking functions. The procedure Ω mentioned above is such a procedure, and we give other examples.

The section comprises in addition:

- A natural lift $\mathcal{P} \mapsto \widehat{\mathcal{P}}$ of parking procedures to parking correspondences. There is a map $\Omega \mapsto \Omega^{\bullet}$ used in the proof of Theorem A which has this form.
- A probabilistic version of bilateral procedures. In particular we consider the procedure where cars perform a biased random walk until they find an empty spot. The corresponding probability of a word to be parking under this procedure leads to the definition of the *remixed Eulerian numbers* $A_c(q)$, a q-deformation of the A_c.

The third and last contribution is about a q-analogue of the algebraic notions encountered in Chapter 2, which lead to the two formulas for a_w stated above: behind the first formula is a certain Klyachko algebra that we deform into a new *q-Klyachko algebra*; and the divided symmetrization of the second formula is replaced by a *q-divided symmetrization*.

We then show that the connection between these two notions for $q = 1$ (illustrated by the existence of the two formulas for a_w) has a q-analogue, which we state in Theorem 3.3.2.

This connection is proved by finding several alternative definitions for the remixed Eulerian numbers $A_c(q)$. These turn out to be polynomials with pleasing properties:

Theorem D (*Theorem 3.3.5*)

The remixed Eulerian numbers $A_c(q)$ are polynomials in $\mathbb{N}[q]$, whose coefficients form a symmetric unimodal sequence.

We emphasize also several aspects of the classical *q-hit numbers*, which turn out to be a natural subfamily of $A_c(q)$.

Chapter 4: A new approach to quasisymmetric polynomials

The results presented here come from the works **[1]**,**[2]**,**[3]** in collaboration with Hunter Spink and Vasu Tewari.

We begin by defining operators R_i and T_i on $\mathbb{Q}[x_1, x_2, \ldots]$: R_i is the substitution $\mathsf{R}_i f = f(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots)$ while $\mathsf{T}_i f = \frac{\mathsf{R}_{i+1} f - \mathsf{R}_i f}{x_i}$, that is,

$$\mathsf{T}_i f(x_1, x_2, \ldots) = \frac{f(x_1, \ldots, x_{i-1}, x_i, 0, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots)}{x_i}.$$

These are the *quasisymmetric divided differences*, from which we will build, in the main Section 4.1, a theory that parallels the classical one for *divided differences* recalled in Chapter 1.4.

We show first that $f \in \mathrm{QSym}_n$ if and only if $\mathsf{T}_i f = 0$ for $i = 1, \ldots, n-1$. We then consider composites of the T_i, and show that they have the form T_F for F an indexed forest as in Chapter 2. It is associated with a monoid structure on Forest, from which one is led to define a set $\mathrm{LTer}(F)$ and certain trimmed forests F/i for $i \in \mathrm{LTer}(F)$. The forest polynomials come up naturally:

Theorem E (*Theorem 4.1.6*)

The family $(\mathsf{P}_F)_F$ *with* $F \in \mathsf{Forest}$ *is the unique family of homogeneous polynomials satisfying* $\mathsf{P}_\varnothing = 1$ *and, for any* $i > 0$ *and* $F \in \mathsf{Forest}$*,*

$$\mathsf{T}_i \mathsf{P}_F = \begin{cases} \mathsf{P}_{F/i} & \text{if } i \in \mathrm{LTer}(F); \\ 0 & \text{otherwise.} \end{cases} \tag{0.0.1}$$

This is an analogue of Theorem 1.4.1 for Schubert polynomials. We have then the following complementary theorem, in which $\mathsf{Forest}_n \subset \mathsf{Forest}$ are certain subsets with bounded support.

Theorem F (*Theorem 4.1.7*)

Let $B \subset \mathsf{Forest}$*. The forest polynomials* $(\mathsf{P}_F)_{F \in B}$ *form a basis of*

a) $\mathbb{Q}[x_1, x_2, \ldots]$ *when* $B = \mathsf{Forest}$*.*

b) $\mathbb{Q}[x_1, \ldots, x_n]$ *when* B *consists of forests* F *with* $\mathrm{LTer}(F) \subset \{1, \ldots, n\}$*.*

c) QSym_n^+ *when* B *consists of forests* F *with* $\mathrm{LTer}(F) \subset \{1, \ldots, n\}$ *and* $F \notin \mathsf{Forest}_n$*.*

d) The quotient space $\mathrm{QSCoinv}_n := \mathrm{Pol}_n / \mathrm{QSym}_n^+$ *when* $B = \mathsf{Forest}_n$*.*

This is an analogue of Theorems 1.4.2 for Schubert polynomials.

We give three applications of this theory in Section 4.2: a method to extract coefficients in the basis of fundamental quasisymmetric polynomials; a proof of positivity for several expansions in the forest basis (related to Theorem B); and finally we come back to a_w, and explain a connection with divided symmetrization that allows one to give a combinatorial proof of $a_w > 0$.

In Section 4.3, the last one of this manuscript, we want to understand the parallel between Theorem E and F, and their Schubert polynomial analogues. The setup is quite involved and we will be brief here.

We introduce the notion of a divided difference pair (X, M), which describes a sequence of operators X_i and a monoid of relations that they satisfy under compositions. We then want to have conditions to ensure the existence of a basis of polynomials *dual* to this setting.

A key idea is the notion of *creation operators* Y_i that give some pseudo-inverse to the X_i, and allow one to construct explicitly such a dual basis, see Theorem 4.3.3. Applied to the classical case of Schubert polynomials, we obtain in particular a *new manifestly positive formula* for these polynomials; we also sketch how this is related to the classical pipe dream expansion.

Chapter 1

Preliminaries

Let us fix some notations and classical combinatorial structures.

We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers, and $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$ the set of positive integers. We often write $[a, b]$ for $\{a, a+1, \ldots, b\}$ and $[k] = [1, k]$. Let $\mathbf{x}_+$ denote the set of variables $\{x_i \mid i \in \mathbb{Z}_{>0}\}$. We let $\mathrm{Pol} = \mathbb{Q}[\mathbf{x}_+]$ be the space of polynomials in these variables, and $\mathrm{Pol}_n = \mathbb{Q}[x_1, \ldots, x_n]$ its subspace of polynomials in n variables.

Let Codes denote the set of sequences $c = (c_1, c_2, \ldots)$ of nonnegative integers where only finitely many entries are nonzero; these are sometimes known as *weak compositions*, or $\mathbb{N}$-vectors. Let Codes_n be the subset where $c_i = 0$ for $i > n$. We will write $|c| = c_1 + c_2 + \cdots \in \mathbb{N}$. A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence in $\mathbb{Z}_{>0}$. We write $\alpha \vDash N$ if $\sum_i \alpha_i = N$ and say that α is a composition of N. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence in $\mathbb{Z}_{>0}$. We write $\lambda \vdash N$ if $\sum_i \lambda_i = N$ and say that λ is partition of N.

1.1 Permutations

We denote by S_n the group of permutations of $\{1, \ldots, n\}$. We usually write an element w of S_n in one line notation, that is, as the word $w(1)w(2)\cdots w(n)$. The permutation w_o^n is the element $n(n-1)\cdots 21$. We write simply w_o if n is clear from the context.

Descents: An index $1 \leq i < n$ is a *descent* of $w \in S_n$ if $w(i) > w(i+1)$. The set of such indices is the *descent set* $\mathrm{Des}(w) \subseteq [n-1]$ of w. Given a subset $S \subseteq [n-1]$, define $\beta_n(S)$ to be the number of permutations $w \in S_n$ such that $\mathrm{Des}(w) = S$.

Example 1.1.1

If $n = 4$ and $S = \{1, 3\}$, one has $\beta_4(S) = |\{2143, 3142, 4132, 3241, 4231\}| = 5.$

A permutation is *m-Grassmannian* if it is the identity or has a unique descent m. Equivalently, w is m-Grassmannian if $\mathrm{Des}(w) \subseteq \{m\}$.

Code and length: The (Lehmer) *code* $\mathrm{lcode}(w)$ of a permutation $w \in S_n$ is the sequence $(c_1, c_2, \ldots, c_n)$ given by $c_i = |\{j > i \mid w(j) > w(i)\}|$. The map $w \mapsto \mathrm{lcode}(w)$ is a bijection from S_n to the set $C_n := \{(c_1, c_2, \ldots, c_n) \mid 0 \leq c_i \leq n-i,\ 1 \leq i \leq n\}$. The *shape* $\lambda(w)$ is the partition obtained by rearranging the nonzero elements of the code in nonincreasing order. The *length* $\ell(w)$ of a permutation $w \in S_n$ is the number of inversions, i.e. pairs $i < j$ such that $w(i) > w(j)$. It is therefore equal to $c_1 + c_2 + \ldots + c_n$ if $(c_1, \ldots, c_n)$ is the code of w.

For example, the permutation $w = 3165274 \in S_7$ has code $c(w) = (2, 0, 3, 2, 0, 1, 0)$, shape $\lambda(w) = (3, 2, 2, 1)$ and length 8.

We also let S'_n be the set of permutations of S_n of length $n-1$, which plays a special role in our story.

Pattern avoidance: Let $u \in S_k$ and $w \in S_n$ where $k \leq n$. An occurrence of the pattern u in w is a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $u_r < u_s$ if and only if $w_{i_r} < w_{i_s}$. We say that w

avoids the pattern u if it has no occurrence of this pattern and we refer to w as *u-avoiding*. For example, 35124 has two occurrences of the pattern 213 at positions $1 < 3 < 5$ and $1 < 4 < 5$. It is 321-avoiding.

Reduced words: The symmetric group S_n is generated by the elementary transpositions $s_i = (i, i+1)$ for $i = 1, \dots, n-1$. Given $w \in S_n$, the minimum length of a word $s_{i_1} \cdots s_{i_l}$ in the s_i's representing w is the length $\ell(w)$ defined above, and such a word is called a *reduced expression* for w. We denote by $\mathrm{Red}(w)$ the set of all *reduced words*, where $i_1 \cdots i_l$ is a reduced word for w if $s_{i_1} \cdots s_{i_l}$ is a reduced expression of w. For $w = 3241$, we have $\ell(w) = 4$ and $\mathrm{Red}(w) = \{1231, 1213, 2123\}$.

With these generators, S_n has a well-known *Coxeter* presentation given by the relations

$$s_i^2 = 1 \text{ for } i = 1, \dots, n-1; \tag{1.1.1}$$
$$s_i s_j = s_j s_i \text{ if } |j-i| > 1 \text{ for } i, j \leq n-1; \tag{1.1.2}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i < n-1. \tag{1.1.3}$$

The relations (1.1.2) and (1.1.3) are called the *commutation relations* and *braid relations* respectively. Note that 321-avoiding permutations can be characterized as *fully commutative*, i.e. any two of their reduced expressions can be linked by a series of commutation relations [BJS93].

The limit $\mathbf{S_\infty}$: One has natural monomorphisms $\iota_n : S_n \to S_{n+1}$ given by letting $n+1$ be a fixed point. One can then consider the direct limit of the groups S_n, denoted by S_∞: it is naturally realized as the set of permutations w of $\{1, 2, 3, \dots\}$ such that $\{i \mid w(i) \neq i\}$ is finite. Any group S_n thus injects naturally into S_∞ by restricting to permutations for which all $i > n$ are fixed points.

Most of the notions we defined above for $w \in S_n$ are well defined for S_∞. The code can be naturally extended to $w \in S_\infty$ by defining $c_i = |\{j > i \mid w(j) > w(i)\}|$ for all $i \geq 1$. It is then a bijection between S_∞ and the set Codes of infinite sequences $(c_i)_{i \geq 1}$ such that $\{i \mid c_i > 0\}$ is finite. The length $\ell(w)$ is thus also well defined. Occurrences of a pattern $u \in S_k$ are well defined in S_∞ if $u(k) \neq k$. Reduced words extend naturally.

1.2 (P, ω)-partitions and quasisymmetric polynomials

1.2.1 (P, ω)-partitions

In this section $(P, \leq_P)$ is a finite poset (partially ordered set). We denote the cover relation by $\prec_P$: we have $x \prec_P y$ if $x <_P y$ and no z satisfies $x <_P z <_P y$. The Hasse diagram is the graph whose edges represent cover relations.

Definition 1.2.1

A P-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that $f(u) \geq f(v)$ whenever $u \prec_P v$.

An important example is $P = P_\lambda$, a Ferrers poset: here $\lambda \vdash n$ is a partition, and P_λ has as elements the n cells of the Ferrers diagram of λ, where a cell is larger than another one if it is weakly to its northeast. A P_λ-partition is then a filling of the cells that is weakly increasing from left to right and top to bottom: this is usually called a reverse plane partition.

More generally, fix in addition a bijective labeling $\omega : P \to \{1, \dots, \#P\}$. The pair (P, ω) then forms a labeled poset. A *(P, ω)-partition* is a P-partition f such that $f(u) > f(v)$ whenever $u \prec_P v$ and $\omega(u) > \omega(v)$. Let $\mathrm{Part}(P, \omega)$ denote the set of (P, ω)-partitions.

Example 1.2.2

The (P_0, ω_0)-partitions for the example illustrated in Figure 1.1 are the functions $f : P_0 \to \mathbb{Z}_{>0}$ that satisfy $f(a) \geq f(b)$ and $f(c) > f(b)$.

For the poset P_λ, let us pick ω_λ by labeling rows from top to bottom, and from left to right in each row. Then $(P_\lambda, \omega_\lambda)$-partitions are reverse plane partitions that are strictly increasing down rows. This is precisely the definition of a semistandard Young tableau of shape λ.

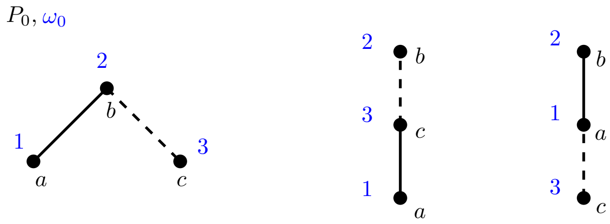

Figure 1.1. A labeled poset and its two linear extensions.

Now for any (P, ω) one can consider the generating function

$$K_{(P,\omega)} = \sum_f \prod_{u \in P} x_{f(u)} \tag{1.2.1}$$

where the sum is over the set of all (P, ω)-partitions. The generating function $K_{(P_\lambda, \omega_\lambda)}$ is the generating function of SSYTs of shape λ: one recognizes the *Schur function s_λ*.

Gessel [Ges84] realized that these series have a particular "quasisymmetric" property.

Definition 1.2.3

Let f be a series with bounded degree in the variables x_i for $i \in I$ where I is an interval in $\mathbb{Z}$. Then f is quasisymmetric if for any composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ and any subsets $\{i_1 < \cdots < i_k\}$ and $\{j_1 < \cdots < j_k\}$ of I, the coefficients of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ and $x_{j_1}^{\alpha_1} \cdots x_{j_k}^{\alpha_k}$ in f are the same.

If $I = \mathbb{Z}_{>0}$, we obtain the space QSym of quasisymmetric functions. When $I = [n]$, one obtains the space of quasisymmetric polynomials $\mathrm{QSym}_n \subset \mathrm{Pol}_n$. It is then immediate that any series $K_{(P,\omega)}$ is in QSym.

We now come to Stanley's fundamental decomposition [Sta72, Theorem 6.2]. A *linear extension* L of P is a linear ordering of P extending $\leq_P$. Thus a linear extension is in particular a totally ordered set with P as its underlying set. Let $\mathrm{Lin}(P)$ be the set of linear extensions of P. Then Stanley showed

$$\mathrm{Part}(P, \omega) = \bigsqcup_{L \in \mathrm{Lin}(P)} \mathrm{Part}(L, \omega). \tag{1.2.2}$$

The proof can be done directly, by showing that any (P, ω)-partition is "compatible" with a unique linear extension. One can also use an induction argument by adding relations to P between incomparable elements.

Example 1.2.4

For P_0 in Figure 1.1, the two linear extensions are shown together with their induced labeling on the right. Then (1.2.2) says that (P_0, ω_0)-partitions are the functions f satisfying either $f(a) \geq f(c) > f(b)$ or $f(c) > f(a) \geq f(b)$, as can be directly checked.

Linear extensions for P_λ are equivalent to *standard Young tableaux (SYTs)*.

Stanley's decomposition (1.2.2) gives the following expansion as an immediate corollary:

$$K_{(P,\omega)} = \sum_{L \in \mathrm{Lin}(P)} K_{(L,\omega)}. \tag{1.2.3}$$

The series $K_{(L,\omega)}$ are the fundamental quasisymmetric functions introduced by Gessel [Ges84], as we now detail. Given $r \geq 0$ a subset S of $\{1, \ldots, r-1\}$, the *fundamental quasisymmetric function* $F_{r,S} \in \mathrm{QSym}$ is defined by

$$F_{r,S} = \sum_{\substack{i_1 \geq \cdots \geq i_r \geq 1 \\ i_j > i_{j+1} \text{ if } j \in S}} x_{i_1} \cdots x_{i_r}. \tag{1.2.4}$$

Note that we have weakly decreasing indices; while this is not the most usual convention, it will fit more naturally with some later developments. For instance $F_{3,\{1\}} = \sum_{i_1 > i_2 \geq i_3 \geq 1} x_{i_1} x_{i_2} x_{i_3}$ and $F_{3,\{2\}} = \sum_{i_1 \geq i_2 > i_3 \geq 1} x_{i_1} x_{i_2} x_{i_3}$.

The $F_{r,S}$ for $r \geq 0$ and S a subset of $\{1, \ldots, r-1\}$ form a basis of the space QSym.

Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vDash r$ be the composition corresponding to $S \subseteq \{1, \ldots, r-1\}$ under the folklore correspondence given by $S = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$. We will then use freely the notation F_α to denote $F_{r,S}$. For instance $F_{(1,2)} = F_{3,\{1\}}$ and $F_{(2,1)} = F_{3,\{2\}}$.

Now if L is a chain $v_1 \prec_L \cdots \prec_L v_r$ with a labeling ω, define $\mathrm{Des}(\omega) = \{i \in [r-1] \mid \omega(v_i) > \omega(v_{i+1})\}$. If we denote the composition of r corresponding to $\mathrm{Des}(\omega)$ by α_ω, then it is easily verified that

$$K_{(L,\omega)} = F_{\alpha_\omega}.$$

This shows that the expansion (1.2.3) expresses any $K_{(P,\omega)}$ positively in the basis of fundamental quasisymmetric functions.

Example 1.2.5

For the poset in Figure 1.1, we get the expansion $K_{(P_0,\omega_0)} = F_{(2,1)} + F_{(1,2)}$.

For $(P_\lambda, \omega_\lambda)$*, say that* $i \in \{1, \ldots, n-1\}$ *is a descent in a tableau* $T \in \mathrm{SYT}(\lambda)$ *if* $i+1$ *is in a higher row than* i *in* T*, and let* $\mathrm{Des}(T)$ *be their set. Then we have the following decomposition*

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(T)}. \tag{1.2.5}$$

Let us finish by two remarks:

- We will extend the notion of (P,ω)-partitions in Section 3.1, with the notion of flags which are upper bounds on the values given by a P-partition at a vertex.
- Quasisymmetric polynomials will be used to get our second formula in Chapter 2, and revisited greatly in Chapter 4.

1.3 The flag variety and its cohomology

Here we review standard material that can be found for instance in [Ful97, Man01, Bri05] and the references therein. Our goal is to give some minimal background to the interested reader on how to go from the geometric problem at heart of our manuscript to an algebraic and combinatorial one.

For E a $\mathbb{C}$-vector space of dimension n, the flag variety $\mathrm{Fl}(E)$ is defined as the set of complete flags

$$V_\bullet = (V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = E).$$

Here each V_i is a linear subspace of E of dimension i. We write simply $\mathrm{Fl}_n = \mathrm{Fl}(\mathbb{C}^n)$. It admits a transitive action of $GL_n(\mathbb{C})$ via $g \cdot V_\bullet = (\{0\} \subset g(V_1) \subset g(V_2) \subset \cdots \subset \mathbb{C}^n)$, and inherits a natural structure of a smooth projective variety of dimension $\binom{n}{2}$.

Fix any reference flag $V_\bullet^{ref}$ in Fl_n. For instance, one may pick $V_\bullet^{std}$ or $V_\bullet^{opp}$ the *standard flag* given by $V_i^{std} = \mathrm{span}(e_1, \ldots, e_i)$ or *opposite flag* $V_i^{opp} = \mathrm{span}(e_{n-i+1}, \ldots, e_n)$ respectively. Then Fl_n has a natural affine paving given by *Schubert cells* $\Omega_w(V_\bullet^{ref})$ indexed by permutations $w \in S_n$. As algebraic varieties one has $\Omega_w(V_\bullet^{ref}) \simeq \mathbb{C}^{\ell(w)}$ where $\ell(w)$ is the length of w. By taking closures of these cells, one gets the family of *Schubert varieties* $X_w(V_\bullet^{ref})$.

Cohomology: The cohomology ring $H^*(\mathrm{Fl}_n)$ with rational coefficients is a well-studied graded commutative ring that we now go on to describe. First of all, to any irreducible subvariety $Y \subset \mathrm{Fl}_n$ of dimension d can be associated a *fundamental class* $[Y] \in H^{n(n-1)-2d}(\mathrm{Fl}_n)$. In particular there are classes $[X_w(V_\bullet^{ref})] \in H^{n(n-1)-2\ell(w)}$. These classes do not in fact depend on $V_\bullet^{ref}$, and we write

$\sigma_w := [X_{w_o w}(V_\bullet^{ref})] \in H^{2\ell(w)}(\mathrm{Fl}_n)$. The affine paving by Schubert cells implies that these *Schubert classes* σ_w form a linear basis of $H^*(\mathrm{Fl}_n)$,

$$H^*(\mathrm{Fl}_n) = \bigoplus_{w \in S_n} \mathbb{Q}\sigma_w. \tag{1.3.1}$$

Now given Y irreducible subvariety of dimension d, we have an expansion of its fundamental class

$$[Y] = \sum_w b_w(Y)\sigma_w, \tag{1.3.2}$$

where the sum is over permutations of length $\ell(w_o) - d$.

Fact 1.3.1

The numbers $b_w(Y)$ are nonnegative integers.

The reason is that $b_w(Y)$ can be interpreted as the number of points in the intersection of Y with $X_{w_o w}(V_\bullet)$, where $V_\bullet$ is a *generic* flag.

This leads to the following major problem: give a manifestly positive formula for these coefficients when $Y = X_u(V_\bullet^{std}) \cap X_{w_o v}(V_\bullet^{opp})$ with $u, v \in S_n$. In that case Y is a *Richardson variety* (if nonempty). Indeed the coefficients $b_w(Y)$ in this case are exactly the generalized Littlewood-Richardson coefficients c_{uv}^w encoding the cup product in cohomology:

$$\sigma_u \cup \sigma_v = \sum_{w \in S_n} c_{uv}^w \sigma_w. \tag{1.3.3}$$

This is only known in some particular cases: the classical case is when u, v are m-Grassmannian permutations. In this case c_{uv}^w is zero unless w is also m-Grassmannian, and the coefficients are the classical Littlewood-Richardson coefficients, which possess a nice combinatorial interpretation [Ful97, Man01]. Recent work involves the case of "separated descents" between u and v [Hua23, KZJ23].

Remark 1.3.2

Fl_n is part of the family of generalized flag varieties G/B, with G a connected reductive group and B a Borel subgroup. In this context, Fl_n corresponds to the type A case, with $G = GL_n$ and $B = B_n$ the group of upper triangular matrices. The partial flag varieties in type A are of the form GL_n/P for P a space of block-triangular matrices with block sizes $k_1, \ldots, k_m$. For instance, the Grassmannian of k-planes in $\mathbb{C}^n$ is obtained by picking $k_1 = k$, $k_2 = n - k$.

Borel presentation and Schubert polynomials. In this paragraph and the next one we explain how one can compute concretely in $H^*(\mathrm{Fl}_n)$. We denote the space of homogeneous polynomials of degree $k \geq 0$ in Pol_n by $\mathbb{Q}^{(k)}$. Let $\mathrm{Sym}_n \subseteq \mathrm{Pol}_n$ be the subring of symmetric polynomials in $x_1, \ldots, x_n$.

Let Sym_n^+ be the ideal of Pol_n generated by the elements $f \in \mathrm{Sym}_n$ such that $f(0) = 0$. Equivalently, Sym_n^+ is generated as an ideal by the elementary symmetric polynomials $e_1, \ldots, e_n$ in n variables. The quotient ring $R_n = \mathrm{Pol}_n / \mathrm{Sym}_n^+$ is the *coinvariant ring*.

Let ∂_i be the *divided difference* operator on Pol_n, given by

$$\partial_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}. \tag{1.3.4}$$

Define the *Schubert polynomials* for $w \in S_n$ as follows:

$$\mathfrak{S}_{w_o} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}, \tag{1.3.5}$$
$$\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i} \text{ if } \ell(ws_i) > \ell(w_i). \tag{1.3.6}$$

These are well defined since the ∂_i satisfy the relations (1.1.2),(1.1.3). For $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ is a homogeneous polynomial of degree $\ell(w)$ in Pol_n.

Remark 1.3.3

In fact Schubert polynomials are well defined for $w \in S_\infty$, as can be shown by a direct computation. Moreover, when $w \in S_\infty$ runs through all permutations whose largest descent is at most n, the Schubert polynomials $\mathfrak{S}_w$ form a basis of Pol_n. We will revisit this in Section 1.4.

Now consider the ring homomorphism

$$j_n : \mathbb{Q}[x_1, \dots, x_n] \to H^*(\mathrm{Fl}_n) \tag{1.3.7}$$

given by $j_n(x_i) = \sigma_{s_i} - \sigma_{s_i - 1}$ for $i > 1$ and $j_n(x_1) = \sigma_{s_1}$ (this is equivalent to the usual definition in terms of "Chern classes"). Then we have the following theorem, grouping famous results of Borel [Bor53] and Lascoux and Schützenberger [LS82] (see [Man01, Section 3.6]).

Theorem 1.3.4

The map j_n is surjective and its kernel is Sym_n^+. Therefore $H^*(\mathrm{Fl}_n)$ is isomorphic as an algebra to R_n. Furthermore,

$$j_n(\mathfrak{S}_w) = \begin{cases} \sigma_w & \text{if } w \in S_n, \\ 0 & \text{if } w \in S_\infty - S_n \text{ has largest descent at most } n. \end{cases}$$

It follows immediately that the product of Schubert polynomials is given by the structure coefficients in (1.3.3): If $u, v \in S_n$, then

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_n} c_{uv}^w \mathfrak{S}_w \mod \mathrm{Sym}_n^+. \tag{1.3.8}$$

It is also possible to work directly with polynomials and not the quotient R_n: the coefficients c_{uv}^w are well defined for $u, v, w \in S_\infty$, and one has

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_\infty} c_{uv}^w \mathfrak{S}_w. \tag{1.3.9}$$

As mentioned above, no manifestly positive formula for these coefficients is known in general.

Expansion in Schubert classes. The goal of this section is to explain a method to extract coefficients in Schubert classes. The reader may want to go directly to Formula (1.3.13), which is what we will use in this work.

Given $\beta \in H^*(\mathrm{Fl}_n)$, let $\int \beta$ be the coefficient of σ_{w_o} in the Schubert class expansion. This can be computed explicitly by

$$\int \beta = \partial_{w_o}(B)(0, \dots, 0), \tag{1.3.10}$$

where B is any polynomial such that $j_n(B) = \beta$. The natural *Poincaré duality* pairing on $H^*(\mathrm{Fl}_n)$ is given algebraically by

$$(\alpha, \beta) \mapsto \int (\alpha \cup \beta).$$

The Schubert classes are known to satisfy $\int \sigma_u \cup \sigma_v = 1$ if $u = w_o v$ and 0 otherwise, illustrating the fact that the pairing is nondegenerate. If $A, B \in \mathrm{Pol}_n$ are such that $j_n(A) = \alpha, j_n(B) = \beta$, then one can compute the pairing explicitly by:

$$\int (\alpha \cup \beta) = \partial_{w_o}(AB)(0, \dots, 0). \tag{1.3.11}$$

Let us fix $\alpha \in H^{n(n-1)-2p}(\mathrm{Fl}_n)$. Our main interest is to consider $\alpha = [Y]$ where Y is an irreducible closed subvariety of Fl_n of dimension p. Associated to α is the linear form $\psi_\alpha : \beta \mapsto \int(\alpha \cup \beta)$ defined on $H^*(\mathrm{Fl}_n)$. It vanishes if β is homogeneous of degree $\neq 2p$, which leads to the following definition: Define the linear form $\phi_\alpha : \mathbb{Q}^{(p)}[\mathbf{x}_n] \to \mathbb{Q}$ by $\phi_\alpha(P) = \psi_\alpha(j_n(P))$ where j_n is the Borel morphism defined earlier. By definition, ϕ_α vanishes on $\mathbb{Q}^{(p)}[\mathbf{x}_n] \cap \mathrm{Sym}_n^+$. For any polynomial $A, P \in \mathrm{Pol}_n$ such that $j_n(A) = \alpha$, we have by (**1.3.11**) the expression

$$\phi_\alpha(P) = \partial_{w_o}(AP)(0). \tag{1.3.12}$$

Since $j_n(\mathfrak{S}_{w_o w}) = \sigma_{w_o w}$ by Theorem 1.3.4, and we use the duality of Schubert classes $\int \sigma_u \cup \sigma_v = 0$ unless $v = w_o u$ where it is 1.The coefficient $b_w(\alpha)$ in the expansion $\alpha = \sum_w b_w(\alpha)\sigma_w$ is given by

$$b_w(\alpha) = \partial_{w_o}(\mathfrak{S}_{w_o w} A)(0, \dots, 0). \tag{1.3.13}$$

1.4 More on Schubert polynomials

The *nil-Coxeter monoid* is the partial monoid[1] whose elements are permutations in S_∞, equipped with the partial monoid structure

$$u \circ v = \begin{cases} uv & \text{if } \ell(u) + \ell(v) = \ell(uv) \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{1.4.1}$$

It is presented by the relations $\mathrm{d}_i \mathrm{d}_{i+1} \mathrm{d}_i = \mathrm{d}_{i+1} \mathrm{d}_i \mathrm{d}_{i+1}$, $\mathrm{d}_i \mathrm{d}_j = \mathrm{d}_j \mathrm{d}_i$ and d_i^2 is undefined (or equal to 0 if seen as a monoid with zero). Divided difference operators ∂_i defined[2] in (**1.3.4**) satisfy these relations, and thus the operators ∂_w are well-defined.

There is a permutation w/i such that $w = (w/i) \circ s_i$ if and only if i is in $\mathrm{Des}\, w$, in which case it is unique and given by the formula $w/i = ws_i$. We have then the following characterization of Schubert polynomials.

Theorem 1.4.1

The Schubert polynomials $(\mathfrak{S}_w)_w$ for $w \in S_\infty$ are the unique family of homogeneous polynomials indexed by permutations w in S_∞ with $\mathfrak{S}_{\mathrm{id}} = 1$ that satisfy

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w/i} & \text{if } i \in \mathrm{Des}\, w \\ 0 & \text{otherwise.} \end{cases}$$

Let us give a sketch of the proof. If a second family $(\mathfrak{R}_w)$ satisfies this, then consider the family $D_w = \mathfrak{S}_w - \mathfrak{R}_w$. We have $D_{\mathrm{id}} = 0$. Now consider $w \neq \mathrm{id}$. By induction, we can assume $D_v = 0$ for all v such that $\ell(v) < \ell(w)$. We get $\partial_i D_w = 0$ for all i, so that D_w is symmetric in all $x_i > 0$. This is only possible if $D_w = 0$, which proves uniqueness. As for existence we have the Definition 1.3.5, which is easily seen to satisfy the characterization. This requires checking that $\mathfrak{S}_w = \mathfrak{S}_{\iota_n(w)}$ for $w \in S_n$, so that $\mathfrak{S}_w$ is well defined for $w \in S_\infty$.

Schubert polynomials form nice bases of subspaces of Pol when indices are restricted to various subsets of S_∞:

Theorem 1.4.2

The Schubert polynomials $(\mathfrak{S}_w)_{w \in B}$ form a basis of

1. Pol *when* $B = S_\infty$.
2. Pol_n *when B is the set of permutations with all descents $\leq n$.*
3. *The coinvariant ideal Sym_n^+ when B is the set of permutations with all descents $\leq n$ but not in S_n.*
4. *The space spanned by monomials x^a where $a_i \leq n - i$ when $B = S_n$; for the same B, the images modulo Sym_n^+ form a basis of the coinvariant space $\mathrm{Pol}_n / \mathrm{Sym}_n^+$.*

[1] this is equivalent to a "monoid with zero", where the zero element 0 satisfies $0m = m0 = 0$ for any m in the monoid.

[2] In this section we consider them as operators on Pol, not on Pol_n.

Moreover we have the following well known special case:

Theorem 1.4.3

If w is an m-Grassmannian permutation, $\mathfrak{S}_w$ coincides with the Schur polynomial $s_\lambda(x_1,\ldots,x_m)$ where λ is the shape of w. These form a basis of the space Sym_m of symmetric polynomials in $x_1,\ldots,x_m$.

Schubert polynomials are known to have positive coefficients, and many combinatorial formulas for Schubert polynomials exist: the algorithmic method of Kohnert [Ass22, Koh91], the balanced tableaux of Fomin–Greene–Reiner–Shimozono [FGRS97], the bumpless pipe dreams of Lam–Lee–Shimozono [LLS21], and the prism tableau model of Weigandt–Yong [WY18].

The most classical one is the pipe dreams of Bergeron–Billey [BB93] and Fomin–Kirillov [FK96], and the related slide expansions of Billey–Jockusch–Stanley [BJS93] and Assaf–Searles [AS17], on which we now focus.

BJS formula. Given a word $\mathrm{a} = a_1a_2\cdots a_k$ with $a_i \in \mathbb{Z}_{>0}$, let $C(\mathrm{a})$ be the set of words $b_1b_2\cdots b_k$ such that $b_1 \geq \cdots \geq b_k$ such that $1 \leq b_j \leq a_j$, and $b_j > b_{j+1}$ whenever $a_j > a_{j+1}$.

$$\mathfrak{F}(\mathrm{a}) = \sum_{b\in C(\mathrm{a})} \mathbf{x}^b \tag{1.4.2}$$

We will see this family occur at several points in this work. Let us note that they comprise all fundamental quasisymmetric polynomials, as can be seen from the respective definitions:

Proposition 1.4.4

Let QSeq_n be the set of sequences $(a_1,\ldots,a_k)$ satisfying $n = a_1$ and $a_i - a_{i+1} \in \{0,1\}$ for $1 \leq i \leq k-1$. If $(a_1,\ldots,a_k) \in \mathrm{QSeq}_n$ then

$$\mathfrak{F}(a) = F_{k,S}(x_1,\ldots,x_n),$$

the fundamental quasisymmetric polynomial where S is the set of i such that $a_i = a_{i+1} + 1$.

We have the *BJS formula* of Billey, Jockusch and Stanley [BJS93]:

$$\mathfrak{S}_w = \sum_{\mathrm{i}\in\mathrm{Red}(w^{-1})} \mathfrak{F}(\mathrm{i}). \tag{1.4.3}$$

As noticed by Assaf and Searles [AS17], this can be seen as an expansion in a certain basis. Indeed, let $c \in$ Codes, and define the word w_c as the unique weakly decreasing word that has c_i occurrences of i for all i. For instance $c = (0,2,0,1,3,0,\ldots)$ corresponds to the word $w_c = 555422$.

Definition 1.4.5

The slide polynomial $\mathfrak{F}_c$ *is defined as* $\mathfrak{F}(w_c)$.

It is clear x^c is the leading monomial in reverse lexicographic order, from which it follows that slide polynomials form a basis of Pol. In fact, any $\mathfrak{F}(\mathrm{a})$ is either equal to $\mathfrak{F}_c$ for a certain $c \in$ Codes, or is 0 if the set $C(\mathrm{a})$ is empty, see [RS95, Section 4] for instance.

Pipe dreams. The expansion (1.4.3) has a nice combinatorial version with *pipe dreams* (also known as rc-graphs), which we now describe. Let $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ be the semi-infinite grid, starting from the northwest corner. Let (i,j) indicate the position at the ith row from the top and the jth column from the left. A *pipe dream* is a tiling of this grid with $+$'s (pluses) and ⌟⌜'s (elbows) with a finite number of $+$'s. The *size* $|\gamma|$ of a pipe dream γ is the number of $+$'s.

Any pipe dream can be viewed as composed of *strands*, which cross at the $+$'s. Strands naturally connect bijectively rows on the left edge of the grid and columns along the top; let $w_\gamma(i) = j$ if the ith row is connected to the jth column, which defines a permutation $w_\gamma \in S_\infty$.

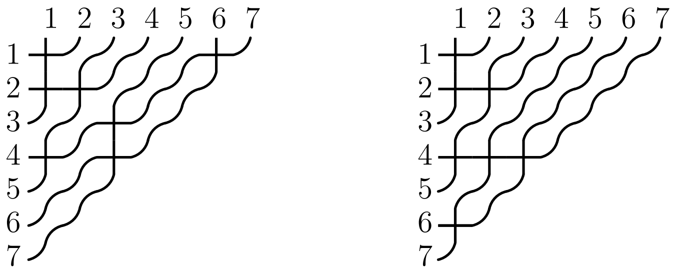

Figure 1.2. Two reduced pipe dreams with permutation $w_\gamma = 2417365$.

Say that γ is *reduced* if $|\gamma| = \ell(w_\gamma)$; equivalently, any two strands of γ cross at most once. We let $\mathrm{PD}(w)$ be the set of reduced pipe dreams γ such that $w_\gamma = w$. Notice that if $w \in S_n$ then the $+$'s in any $\gamma \in \mathrm{PD}(w)$ can only occur in positions (i,j) with $i + j < n$, so we can restrict the grid to such positions.

Given $\gamma \in \mathrm{PD}(w)$, define $c(\gamma) \coloneqq (c_1, c_2, \ldots)$ where c_i is the number of $+$'s on the ith row of γ. Then the BJS expansion (**1.4.3**) can be rewritten as follows [BJS93, Man01]:

$$\mathfrak{S}_w = \sum_{\gamma \in \mathrm{PD}(w)} \mathbf{x}^{c(\gamma)}. \tag{1.4.4}$$

MacDonald reduced word formulas. These are beautiful formulas for principal specializations of Schubert polynomials. The first formula was proved in [Mac91]:

$$\mathfrak{S}_w(1,1,\ldots) = \frac{1}{\ell} \sum_{\mathbf{a} \in \mathrm{Red}(w)} a_1 a_2 \cdots a_\ell. \tag{1.4.5}$$

Note that the left hand side counts pipe dreams $\mathfrak{S}_w(1,1,\ldots) = \#\mathrm{PD}(w)$, and thus is a positive integer, while the right hand side is a sum of positive rational numbers. Macdonald also conjectured the following q-analogue, that his proof method for $q = 1$ was unable to settle, and which was proved shortly after by Fomin and Stanley [FS94].

$$\mathfrak{S}_w(1,q,q^2,\ldots) = \frac{1}{(\ell)_q!} \sum_{\mathbf{a} \in \mathrm{Red}(w)} q^{\mathrm{comaj}(\mathbf{a})} (a_1)_q (a_2)_q \cdots (a_\ell)_q. \tag{1.4.6}$$

Here $\mathrm{comaj}(\mathbf{a})$ is the sum of all $i \in \{1,\ldots,\ell-1\}$ such that $a_i < a_{i+1}$. Note that with some effort, both formulas can be proved bijectively, see [BHY19].

Chapter 2

Schubert coefficients for the permutahedral variety

In this section we will state and prove the result whose research is at the origin of most of this manuscript. The main works addressing this original question are **[10]** and **[5]**.

2.1 The permutahedral variety and the Schubert coefficients a_w

2.1.1 Statement of the problem

To define a Hessenberg variety $\mathcal{H}(X,h)$ in Fl_n, one needs an $n\times n$ matrix X and a *Hessenberg function* $h:[n]\to[n]$. Then $V_\bullet\in\mathcal{H}(X,h)$ if $XV_i\subseteq V_{h(i)}$ for all i.

The *permutahedral variety* Perm_n is the *regular semisimple* Hessenberg variety corresponding to the choice of $h=(2,3,\ldots,n,n)$, with X being a diagonal matrix with distinct entries along the diagonal.

Fact 2.1.1

The permutahedral variety is the smooth, toric variety whose associated fan is the braid fan, given by the type A root system arrangement.

It appears in many areas in mathematics [DMS88, Kly85, Pro90], and notably is a key player in the Huh-Katz resolution of the Rota-Welsh conjecture in the representable case [HK12].

We let $\tau_n=[\mathrm{Perm}_n]$. Since Perm_n is an irreducible subvariety of Fl_n of complex dimension $n-1$, the class τ_n lives in cohomological degree $(n-1)(n-2)$. We consider its Schubert class expansion

$$\tau_n=\sum_{w\in S'_n}a_w\sigma_{w_ow},\tag{2.1.1}$$

where S'_n denotes the set of permutations in S_n of length $n-1$. We refer to Table 2.1 for a list of these coefficients for $n=2,3,4,5,6$.

The coefficients a_w are thus the coefficients $b_w(\mathrm{Perm}_n)$ from (**1.3.2**). Given the geometric interpretation for the a_w as intersection numbers, it follows that $a_w\in\mathbb{N}$.

The goal of this section is to develop a concrete understanding of the coefficients a_w.

As we mentioned in the introduction, this goal has been a driving force behind many of the results presented in this thesis.

Table 2.1. The Schubert class expansions $\tau_n = \sum_{w \in S'_n} a_w \sigma_{w_o w}$ for $2 \leq n \leq 6$.

n	Schubert expansions for τ_n
2	$\mathfrak{S}_1$
3	$\mathfrak{S}_{132} + \mathfrak{S}_{21}$
4	$\mathfrak{S}_{1432} + \mathfrak{S}_{2341} + 2\mathfrak{S}_{2413} + 2\mathfrak{S}_{3142} + \mathfrak{S}_{321} + \mathfrak{S}_{4123}$
5	$\mathfrak{S}_{15432} + \mathfrak{S}_{24531} + 2\mathfrak{S}_{25341} + 3\mathfrak{S}_{25413} + \mathfrak{S}_{32541} + \mathfrak{S}_{34251} + 4\mathfrak{S}_{34512} + 5\mathfrak{S}_{35142} + 3\mathfrak{S}_{35214} + 3\mathfrak{S}_{41532} + 2\mathfrak{S}_{42351} + 5\mathfrak{S}_{42513} + 3\mathfrak{S}_{43152} + \mathfrak{S}_{4321} + 4\mathfrak{S}_{45123} + 2\mathfrak{S}_{51342} + \mathfrak{S}_{51423} + \mathfrak{S}_{52143} + 2\mathfrak{S}_{52314} + \mathfrak{S}_{53124}$
6	$\mathfrak{S}_{165432} + \mathfrak{S}_{256431} + 2\mathfrak{S}_{264531} + 3\mathfrak{S}_{265341} + 4\mathfrak{S}_{265413} + \mathfrak{S}_{346521} + \mathfrak{S}_{354621} + 3\mathfrak{S}_{356241} + 5\mathfrak{S}_{356412} + 3\mathfrak{S}_{362541} + 3\mathfrak{S}_{364251} + 10\mathfrak{S}_{364512} + 9\mathfrak{S}_{365142} + 6\mathfrak{S}_{365214} + 2\mathfrak{S}_{426531} + \mathfrak{S}_{435621} + 2\mathfrak{S}_{436251} + 5\mathfrak{S}_{436512} + 3\mathfrak{S}_{452631} + \mathfrak{S}_{453261} + 5\mathfrak{S}_{453612} + 10\mathfrak{S}_{456132} + 10\mathfrak{S}_{456213} + 9\mathfrak{S}_{461532} + 8\mathfrak{S}_{462351} + 16\mathfrak{S}_{462513} + 11\mathfrak{S}_{463152} + 4\mathfrak{S}_{463215} + 10\mathfrak{S}_{465123} + 4\mathfrak{S}_{516432} + 3\mathfrak{S}_{524631} + 8\mathfrak{S}_{526341} + 11\mathfrak{S}_{526413} + 3\mathfrak{S}_{532641} + 2\mathfrak{S}_{534261} + 10\mathfrak{S}_{534612} + 16\mathfrak{S}_{536142} + 9\mathfrak{S}_{536214} + 6\mathfrak{S}_{541632} + 3\mathfrak{S}_{542361} + 9\mathfrak{S}_{542613} + 4\mathfrak{S}_{543162} + \mathfrak{S}_{54321} + 10\mathfrak{S}_{546123} + 10\mathfrak{S}_{561342} + 5\mathfrak{S}_{561423} + 5\mathfrak{S}_{562143} + 10\mathfrak{S}_{562314} + 5\mathfrak{S}_{563124} + 3\mathfrak{S}_{614532} + 2\mathfrak{S}_{615342} + \mathfrak{S}_{615423} + 3\mathfrak{S}_{623541} + 4\mathfrak{S}_{624351} + 8\mathfrak{S}_{624513} + 2\mathfrak{S}_{625143} + 3\mathfrak{S}_{625314} + 3\mathfrak{S}_{631542} + 3\mathfrak{S}_{632451} + 3\mathfrak{S}_{632514} + 8\mathfrak{S}_{634152} + 3\mathfrak{S}_{634215} + 3\mathfrak{S}_{635124} + 3\mathfrak{S}_{641352} + 3\mathfrak{S}_{641523} + 2\mathfrak{S}_{642153} + 2\mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{651243} + \mathfrak{S}_{651324} + \mathfrak{S}_{652134}$

Remark 2.1.2

> *The Peterson variety is the regular nilpotent Hessenberg variety defined with the same h, and with X chosen to be the nilpotent matrix that has ones on the upper diagonal and zeros elsewhere. This variety has also garnered plenty of attention; see [BĬ7, Dre15, HT11, Ins15, IT16, IY12, Rie03]. It is known that for a given h, all regular Hessenberg varieties have the same class in the rational cohomology $H^*(\mathrm{Fl}_n)$, see [ADGH18]. It follows that the class of the Peterson variety is also given by τ_n.*

Anderson and Tymoczko [AT10] give an expansion for $[\mathcal{H}(X, h)]$ for arbitrary h which involves multiplication of Schubert polynomials depending on *length-additive* factorizations of a permutation w_h attached to h. In general, transforming this expression into one in the basis of Schubert polynomials in a combinatorially explicit manner would require understanding generalized LR coefficients 1.3.8. In fact, the special cases in which Anderson and Tymoczko provide explicit expansions in terms of Schubert polynomials are those for which combinatorial rules are indeed known [AT10, Sections 5 and 6].

The case of τ_n appears again in work of Harada et al [HHMP19, Section 6], as well as Kim [Kim20]. In the former, τ_n is expressed as a sum of classes of Richardson varieties [HHMP19, Theorem 6.4]. Translating this into an explicit Schubert expansion once again amounts to understanding certain generalized LR coefficients.

2.1.2 Coefficients a_w via Klyachko's approach

We extract our first expression from the results of [Kly85]. Given $w \in S_\infty$ of length ℓ, consider the polynomial in $\mathbb{Q}[x_1, x_2, \ldots]$:

$$\mathrm{M}_w(x_1, x_2, \ldots) := \sum_{\mathbf{i} = i_1 i_2 \cdots i_\ell \in \mathrm{Red}(w)} x_{i_1} x_{i_2} \cdots x_{i_\ell} = \sum_{\mathbf{i} \in \mathrm{Red}(w)} \mathbf{x}^{c(\mathbf{i})}, \tag{2.1.2}$$

where $c(\mathbf{i}) = (c_1, c_2, \ldots)$ and c_j is the number of occurrences of j in $\mathbf{i}$. If $w \in S_n$, then M_w is a polynomial in $x_1, \ldots, x_{n-1}$. Notice that Macdonald's formula (1.4.5) states that

$$M_w(1, 2, \ldots) = \ell! \cdot \mathfrak{S}_w(1, 1, \ldots).$$

For $n \geq 3$, let the *Klyachko algebra* $\mathcal{K}_n$ be the commutative $\mathbb{Q}$-algebra with generators $u_1, \ldots, u_{n-1}$ and defining relations

$$\begin{cases} 2u_i^2 = u_i u_{i-1} + u_i u_{i+1} \text{ for } 1 < i < n-1; \\ 2u_1^2 = u_1 u_2; \\ 2u_{n-1}^2 = u_{n-1} u_{n-2}. \end{cases}$$

Given $I = \{i_1 < \cdots < i_j\} \subset [n-1]$, define $u_I := u_{i_1} \cdots u_{i_j}$.

Klyachko showed that these 2^{n-1} elements form a basis of $\mathcal{K}_n$. Given $U = \sum_I c_I u_I \in \mathcal{K}_n$, let $\int_{\mathcal{K}_n} U$ be the top coefficient $c_{[n-1]}$.

Theorem 2.1.3 (*First formula*)

For any $w \in S'_n$, we have

$$a_w = \int_{\mathcal{K}_n} M_w(u_1, u_2, \ldots, u_{n-1}) = \int_{\mathcal{K}_n} \sum_{i_1 i_2 \cdots i_\ell \in \mathrm{Red}(w)} u_{i_1} u_{i_2} \cdots u_{i_\ell}.$$

As shown in **[10]**, this is a reformulation of Klyachko's work [Kly85], specialized to type A. We sketch the details: first, the cohomology ring $H^*(\mathrm{Perm}_n)$ is computed by Klyachko. S_n acts on this ring, and the corresponding subring of invariants is shown to be the algebra $\mathcal{K}_n$ above. In this presentation, the fundamental class of Perm_n is represented by the top element $u_{[n-1]}/(n-1)!$. Now the embedding $\mathrm{Perm}_n \to \mathrm{Fl}_n$ gives a pullback morphism $H^*(\mathrm{Fl}_n) \to \mathcal{K}_n$, under which the image of the Schubert class σ_w is $M_w(u_1, u_2, \ldots, u_{n-1})/\ell(w)!$. Explicitly, this means the following in $\mathcal{K}_n$:

$$\mathfrak{S}_w(u_1, u_2 - u_1, u_3 - u_2, \ldots) = \frac{1}{\ell(w)!} M_w(u_1, u_2, \ldots, u_{n-1}). \tag{2.1.3}$$

If $w \in S'_n$, we have by definition $a_w = \int \sigma_w \cup \tau_n = \int \sigma_w \cup [\mathrm{Perm}_n]$ where $\int$ is as in (**1.3.10**). Pulling back the computation to $\mathcal{K}_n$ gets us the result.

We note that Klyachko was particularly interested in the case where w is Grassmannian, for which he gives a formula [Kly85, Theorem 6] that is not manifestly positive, and gives a combinatorial interpretation in subsequent work [Kly95]. We will retrieve this case as Corollary 2.3.2.

2.1.3 Coefficients a_w via Anderson–Tymoczko's approach

We introduce the operator of *divided symmetrization* $\langle \cdot \rangle_n$. This is a linear operator on Pol_n defined as follows:

$$\langle f(x_1, \ldots, x_n) \rangle_n := \sum_{w \in S_n} w \cdot \left(\frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n-1} (x_i - x_{i+1})} \right). \tag{2.1.4}$$

In general, if f is homogeneous of degree d, $\langle f(x_1, \ldots, x_n) \rangle_n$ is a symmetric polynomial of degree $d - (n-1)$ when $d \geq n-1$, and zero otherwise. In particular, if f has degree $n-1$, the result is a scalar. *We will only use it in this case.*

Theorem 2.1.4 (*Second formula*)

For any $w \in S'_n$,

$$a_w = \langle \mathfrak{S}_w(x_1, \ldots, x_n) \rangle_n. \tag{2.1.5}$$

Let us briefly explain how this follows from the results of Anderson and Tymoczko [AT10]. The main result from that work is a polynomial representative for the class of $\mathcal{H}(X, h)$ in the semisimple case. This class Σ_h does not depend on h and is given as follows:

$$\Sigma_h = \prod_{\substack{1\le i<j\le n\\ j>h(i)}} (x_i - x_j) \mod I_n. \tag{2.1.6}$$

Now in the case of $h = (2, 3, \ldots, n, n)$, we have that $\Sigma_h = \tau_n$ by definition and thus

$$\tau_n = \prod_{\substack{1\le i<j\le n\\ j>i+1}} (x_i - x_j) \mod I_n.$$

By the coefficient formula (**1.3.13**), Theorem 2.1.4 follows from the next proposition.

Proposition 2.1.5

For any $P \in \mathrm{Pol}_n$ of degree $n-1$,

$$\partial_{w_o}\Big(P \prod_{\substack{1\le i<j\le n\\ j>i+1}} (x_i - x_j)\Big) = \langle P \rangle_n.$$

This is a simple computation, see Proposition 3.3 in **[10]**. It uses the fact that $\partial_{w_o} = \frac{1}{\Delta_n}\mathrm{Anti}_n$ where Anti_n is the antisymmetrizing operator $\sum_{\sigma\in S_n} \epsilon(\sigma)\sigma$ [Man01, Proposition 2.3.2].

Note that divided symmetrization was introduced by Postnikov when he showed that the volume of a permutahedron with vertices given by permutations of $(\lambda_1, \ldots, \lambda_n)$ is $\langle (\lambda_1 x_1 + \cdots + \lambda_n x_n)^{n-1} \rangle_n$, cf. [Pos09, Theorem 3.2]. The connection with Proposition 2.1.5 is explained in Remark 3.4 of **[10]** via the notion of *degree polynomial* of a class in $H^*(\mathrm{Fl}_n)$.

2.2 First approach and special properties

Here we use Theorem 2.1.3 to compute a_w. As a first step, we need to rewrite the formula slightly.

Postnikov [Pos09, Section 16] introduced the family of *mixed Eulerian numbers* A_c for any $r > 0$ and $c = (c_1, \ldots, c_r)$ a weak composition with $|c| = r$. They are defined as mixed volumes of hypersimplices, from which one knows $A_c \in \mathbb{Z}_{>0}$.

Petrov shows [Pet18] (see also Lemma 4.1 in **[10]**) that for any fixed r, the mixed Eulerian numbers A_c are uniquely determined by the conditions $A_{(1^r)} = (r)!$ and

$$2A_{(c_1,\ldots,c_r)} = A_{(c_1,\ldots,c_{i-1}+1,c_i-1,\ldots,c_r)} + A_{(c_1,\ldots,c_i-1,c_{i+1}+1,\ldots,c_r)}$$

for any $i \le r$ and $c_i \ge 2$. In fact he has a probabilistic interpretation for these numbers from which it also follows that $A_c > 0$ – we will generalize it in Section 3.2.4.

It is clear that $(n-1)! \int_{\mathcal{K}_n} u^c = A_c$ as the characterization above follow precisely from the properties of $\mathcal{K}_n$ given in Section 2.1.2. For any word $\mathbf{a}$ over $\mathbb{Z}_{>0}$, define its *content* $c(\mathbf{a}) = (c_1, c_2, \ldots) \in$ Codes by letting c_i be the number of occurrences of j in $\mathbf{a}$.

It follows from Theorem 2.1.3 that, for any $w \in S'_n$,

$$a_w = \sum_{\mathbf{i}\in\mathrm{Red}(w)} \frac{A_{c(\mathbf{i})}}{(n-1)!} \tag{2.2.1}$$

where $c(\mathbf{i})$ is truncated to have length $n-1$ (note that all remaining entries are zero since $w \in S_n$, and $|c(\mathbf{i})| = n-1$ as it is equal to $\ell(w)$).

Example 2.2.1

Consider $w = 32415 \in S'_5$. It has three reduced words $2123, 1213$ and 1231. Given that $A_{2,1,1,0} = 6$ and $A_{1,2,1,0} = 12$, we obtain $a_w = \frac{1}{24}(12 + 6 + 6) = 1$.

As this example illustrates, the integer numbers a_w are decomposed via this formula as a sum of rational numbers. It is unclear how to see integrality for the formula; we will use a different method in Section 2.3 to obtain a combinatorial interpretation for the numbers. There are however several properties that can be deduced.

First, we have the following direct consequence of (**2.2.1**); the positivity statement answers a question asked in [HHMP19, Problem 6.6].

Corollary 2.2.2

For any $w \in S'_n$, $a_w > 0$ and $a_w = a_{w^{-1}}$.

Indeed all A_c are positive as recalled above, which proves the first part. Also, $\mathbf{i} \in \mathrm{Red}(w)$ if and only if $\mathrm{rev}(\mathbf{i}) \in \mathrm{Red}(w^{-1})$, and obviously $c(\mathrm{rev}(\mathbf{i})) = c(\mathbf{i})$ which proves the second part.

In the rest of this section, we give special properties of the numbers a_w.

Indecomposable permutations Let $w_1, w_2 \in S_m \times S_p$ with $m, p > 0$. The concatenation $w = w_1 \times w_2 \in S_{m+p}$ is defined by $w(i) = w_1(i)$ for $1 \leq i \leq m$ and $w(m+i) = m + w_2(i)$ for $1 \leq i \leq p$. A permutation $w \in S_n$ is called *indecomposable* if it cannot be written as $w = w_1 \times w_2$ for any $w_1, w_2 \in S_m \times S_p$ with $n = m + p$. The indecomposable permutations for $n \leq 3$ are $1, 21, 231, 312, 321$, and their counting sequence is A003319 in [Slo].

Given w in S_n, one has a unique factorization

$$w = w_1 \times w_2 \times \cdots \times w_k, \tag{2.2.2}$$

where each w_i is an indecomposable permutation in S_{m_i} for certain $m_i > 0$. Its cyclic shifts $w^{(1)}, \ldots, w^{(k)}$ are

$$w^{(i)} = (w_i \times w_{i+1} \cdots \times w_k) \times (w_1 \times \cdots \times w_{i-1}). \tag{2.2.3}$$

For instance $w = 53124768 \in S_8$ factors as $w = 53124 \times 21 \times 1$, and its cyclic shifts are $w^{(1)} = w = 53124768$, $w^{(2)} = 21386457$ and $w^{(3)} = 16423587$.

Proposition 2.2.3 (*Cyclic Sum Rule*)

Let $w \in S'_n$, and consider its cyclic shifts $w^{(1)}, \ldots, w^{(k)}$ as above. Then

$$\sum_{i=1}^{k} a_{w^{(i)}} = |\,\mathrm{Red}(w)|. \tag{2.2.4}$$

The proof follows from (**2.2.1**) and a cyclic property satisfied by mixed Eulerian numbers themselves; see Theorem 5.6 in **[10]**. Let $w = 53124768 \in S'_8$ as above. Then one can check that $|\,\mathrm{Red}(w)| = 63$ and $a_{w^{(1)}} + a_{w^{(2)}} + a_{w^{(3)}} = 6 + 21 + 36 = 63$.

We can refine this result for permutations having a unique nontrivial factor in (**2.2.2**). Given a permutation u of length ℓ and $m \geq 0$, consider $\nu_u(m) := \nu_{1^m \times u} = \mathfrak{S}_{1^m \times u}(1, 1, \ldots)$.

By Macdonald's identity (**1.4.5**) we have

$$\nu_u(m) = \frac{1}{\ell!} \sum_{\mathbf{i} \in \mathrm{Red}(u)} (i_1 + m)(i_2 + m) \cdots (i_\ell + m). \tag{2.2.5}$$

It is a polynomial in m of degree ℓ with integer values on $\mathbb{N}$. Therefore (see [Sta97] for instance) there exist integers $h_m^u \in \mathbb{Z}$ for $m = 0, \ldots, \ell$ such that

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\sum_{m=0}^{\ell} h_m^u t^m}{(1-t)^{\ell+1}}. \tag{2.2.6}$$

Proposition 2.2.4

Let $u \in S_{p+1}$ be indecomposable of length $n-1$. Define $n-p$ permutations $u^{[m]} \in S'_n$ for $m = 0, \ldots, n-p-1$ by $u^{[m]} := 1^m \times u \times 1^{n-p-1-m}$. Then

$$h_m^u = \begin{cases} a_{u^{[m]}} \text{ if } m < n-p; \\ 0 \text{ if } m \geq n-p \end{cases}$$

Equivalently,

$$\sum_{j \geq 0} \nu_u(j) t^j = \frac{\sum_{m=0}^{n-p-1} a_{u^{[m]}} t^m}{(1-t)^n}. \tag{2.2.7}$$

Moreover, the numbers h_m^u are known to sum to $\ell!$ times the leading term of $\nu_u(m)$, which is $|\operatorname{Red}(u)|$ by (**2.2.5**). Thus the previous proposition is a refinement of Proposition 2.2.3 when only one indecomposable factor is nontrivial.

We can use this theorem to get an explicit combinatorial rule in a particular case: for *vexillary permutations*, which are the permutations avoiding the pattern 2143. Any vexillary permutation can be written as $u^{[m]}$ for u indecomposable and $m \geq 0$ as in Proposition 2.2.4. Now to u one can associate its shape $\lambda(u)$, cd. Section 1.1. Seen as a Ferrers poset P_λ as in 1.2.1, one can construct a certain bijective labeling ω_u on P_λ, as well as an integer N_u: the details are fairly technical, we refer to Section 7 of **[10]**.

Our result is then an explicit combinatorial interpretation:

Proposition 2.2.5

For any vexillary permutation $w = u^{[m]}$, a_w is the number of tableaux $T \in \operatorname{SYT}(\lambda)$ with $m+N_u$ ω_u-descents.

2.3 Second approach and combinatorial interpretation

We obtain a combinatorial interpretation for any $w \in S'_n$ in this section: the main result is stated as Theorem 2.3.4. The proof will have several steps, starting with the formula $a_w = \langle \mathfrak{S}_w \rangle_n$ from Theorem 2.1.4.

2.3.1 Computing with divided symmetrization

In this section we present the main results from **[15]**: Propositions 2.3.1 and 2.3.3.

First, we have the following result that gives us a first glimpse of the connection of our problem to the quasisymmetric world.

Proposition 2.3.1

If $f \in \operatorname{QSym}$ has degree $n-1$, then we have the identity

$$\sum_{j \geq 1} f(1^j) t^j = \frac{\sum_{m=1}^{n} \langle f(x_1, \ldots, x_m) \rangle_n t^m}{(1-t)^n}. \tag{2.3.1}$$

The proof in **[15]** goes by showing that (**2.3.1**) holds for the basis elements M_α, the monomial quasisymmetric functions. It relies ultimately on the following monomial evaluation $\langle x^c \rangle_n$ for $c = (c_1, \ldots, c_{n-1})$ with $|c| = n-1$, due to Postnikov [Pos09, Section 3] (see also [Pet18] and **[15]** for a different proof).

Let $S(c)$ be the set of indices $k \in [n-1]$ such that $c_1 + \cdots + c_k < k$. Then

$$\langle x^c \rangle_n = (-1)^{\#S(c)} \beta_n(S_c) \tag{2.3.2}$$

where $\beta_n(S)$ is the number of permutations of S_n with descent set S.

Grassmannian case[1]**.** When $w \in S'_n$ is m-Grassmannian, the Schubert polynomial $\mathfrak{S}_w$ is a Schur polynomial $s_\lambda(x_1, \cdots, x_m)$, cf. Theorem 1.4.3, where λ is the shape of w. It is symmetric, thus quasisymmetric, and so $a_w = \langle s_\lambda(x_1, \ldots, x_m)\rangle_n$ occurs in the numerator in (**2.3.1**). The right-hand side can be computed via the theory of P-partitions, essentially by specializing (**1.2.5**); see Example 4.6 in **[15]**. The following evaluation recovers a result of Klyachko [Kly95]:

Corollary 2.3.2

If $w \in S'_n$ is m-Grassmannian with shape λ, a_w is equal to the number of $T \in \mathrm{SYT}(\lambda)$ with $m-1$ descents.

As an example, consider the permutations $w_1 = 351246$ and $w_2 = 146235$, which are the two Grassmannian permutations in S'_6 with shape $(3, 2)$. Note that w_1 has descent 2 while w_2 has descent 3. So $a_{w_1} = 2$ and $a_{w_2} = 3$.

To finish this section, we have the following second key result from **[15]**, which will be crucial in the proof of Theorem 2.3.4. Let QSym_n^+ be the ideal of Pol_n generated by quasisymmetric polynomials f with $f(0) = 0$.

Proposition 2.3.3

If $P \in \mathrm{QSym}_n^+$ has degree $n-1$, then $\langle P\rangle_n = 0$.

2.3.2 Combinatorial interpretation for a_w

Parking procedure Ω: Consider parking spots indexed by $\mathbb{Z}$, initially all empty. Cars $1, 2, \ldots$ arrive successively, with car i preferring spot v_i, and want to park at (empty) spots. Assume inductively that $i-1$ cars have already parked. If spot v_i is empty, then car i parks there. Otherwise, v_i belongs to an interval $[a, b] := \{a, a+1, \ldots, b-1, b\}$ of occupied spots with spots $a-1$ and $b+1$ being free[2]. Let $j < i$ be maximal such that $v_j \in [a, b]$: in words, v_j is the preferred spot of the car that parked last in $[a, b]$. The parking rule is then that car i parks in $b+1$ if $v_i \geq v_j$, while it parks in $a-1$ if $v_i < v_j$.

After k cars have parked according to this special rule, they occupy a k-subset denoted $\Omega(v_1 \cdots v_k) \subset \mathbb{Z}$. A preference word $v_1 \cdots v_k$ is called a *Ω-parking word* if $\Omega(v_1 \cdots v_k) = \{1, \ldots, k\}$. We can then state the main result of **[5]**, giving a combinatorial interpretation for the intersection numbers a_w

Theorem 2.3.4

Let $w \in S'_n$. Then a_w is the number of reduced words of w^{-1} that are also Ω-parking words.

Consider $w = 21543 \in S_5$ with $\ell(w) = 4$. To compute a_w, we need to compute Ω-parking words in $\mathrm{Red}(w^{-1}) = \mathrm{Red}(w)$. Here are all reduced words for 21543.

$$\mathrm{Red}(w) = \{1343, 3143, 3413, 3431, 1434, 4134, 4314, 4341\}.$$

Only the first four are Ω-parking words, and therefore $a_{21543} = 4$.

While the statement of Theorem 2.3.4 is succinct, we need some new theory to arrive at it. We start from the formula $a_w = \langle \mathfrak{S}_w\rangle_n$ and want to extract a positive rule from it.

- The rule to compute divided symmetrization on monomials 2.3.2 is signed and complicated. So using it on the pipe dream expansion (**1.4.4**) is in general unwieldy.

[1] This is only to illustrate the previous result. This case is covered by the vexillary case above, and is also a consequence of the general combinatorial interpretation below.

[2] Note that the classical "one-way street" parking rule is to then have the ith car park in $b+1$ at this point.

- One can use the more compact version given by the BJS expansion 1.4.3 into slide polynomials. But computer experimentation shows that $\langle \mathfrak{F}(\mathbf{a}) \rangle_n$ can be also negative, and we could not find any pattern.
- We also thought of coarsening even more, by applying $\langle \cdot \rangle_n$ on the sum of $\mathfrak{F}(\mathbf{i})$ where the reduced words run over a commutation class. Computer experimentation this time seemed to show nonnegativity in all examples, but the values seemed hard to conjecture.

What will turn out to work in the end is a decomposition ((**2.3.5**) below) that sits between the last two, by grouping reduced words according to an equivalence relation that refines commutation classes. The resulting sums of slide polynomials will be easy to evaluate under $\langle \cdot \rangle_n$, and turn out to be an interesting class of polynomials that we now define.

2.3.3 Forest polynomials

Let $S \subset \mathbb{Z}_{>0}$ be a finite set of integers. It decomposes uniquely as $S = I_1 \sqcup \ldots \sqcup I_k$, where each I_j is a maximal subset of consecutive integers in S. An *indexed forest F on S* is the data of a rooted binary tree T_j with $|I_j|$ internal nodes for any $j \in [k]$. Its size $|F|$ is the cardinality of S, and we represent the latter by unit intervals of the integer line.

Figure 3.2 shows an indexed forest (ignoring the labels in red for now) with $|F| = 6$. It comprises three trees supported on the intervals $[2,4]$, $[7]$, and $[11,12]$ from left to right. Note how elements of S are naturally in bijection with internal nodes of F, and we name the label in S corresponding to a node its *canonical label*.

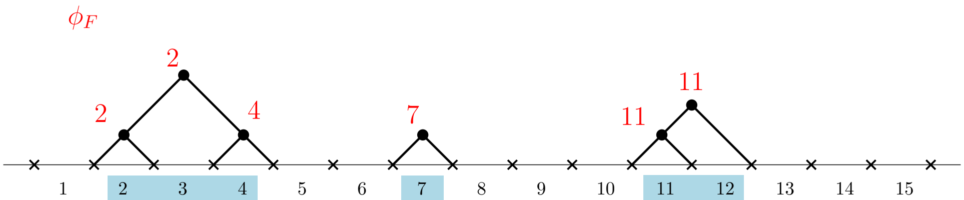

Figure 2.1. An indexed forest with its flag values.

Given an indexed forest, label leaves by the unit interval to its right. For each internal node, let its flag value $\phi_F(v)$ be the label of the leaf at the end of its left branch, see Figure 2.1 for an illustration. Let the *code* $\mathrm{c}(F)$ be the vector $(c_1, c_2, \ldots)$ where c_i is the number of flag values equal to i. It is thus equal to $(0,2,0,1,0,0,1,0,0,0,2,0,0,\ldots)$ in the example. Let $\mathrm{IN}(F)$ be the set of internal nodes.

Definition 2.3.5 (*Forest polynomials* P_F)

Let $F \in$ Forest. The forest polynomials P_F is defined as

$$\mathsf{P}_F = \sum_{f:\mathrm{IN}(F)\to\mathbb{Z}_{>0}} \prod_{v\in\mathrm{IN}(F)} x_{f(v)}$$

where the sum is over all f whose values are weakly increasing down left edges, strictly increasing down right edges, and such that $f(v) \le \phi_F(v)$ for all v.

For example, given the indexed forest F in Figure 2.2 one has

$$\mathsf{P}_F = \sum_{\substack{2\ge a\ge b\\ 4\ge c>b}} x_a x_b x_c = x_2^2 x_4 + x_1x_2x_4 + x_1^2 x_4 + x_2^2 x_3 + x_1x_2x_3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2.$$

More examples are listed in Table A.1. Forest polynomials expand positively in the basis of slide polynomials. Explicitly, consider F as a poset by seeing it as a Hasse diagram on $\mathrm{IN}(F)$. Linear

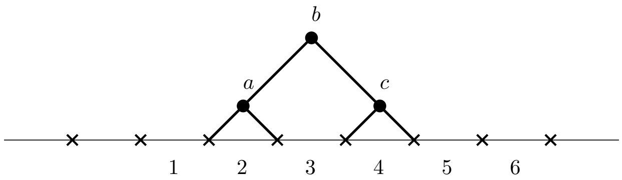

Figure 2.2. An indexed forest F with $\mathrm{c}(F) = (0, 2, 0, 1)$

extensions, which can be seen as decreasing forests, are associated to words over $\mathbb{Z}_{>0}$ by reading the ϕ_F-labels. If $Lin(F)$ is the set of these words, then one has

$$\mathsf{P}_F = \sum_{W \in Lin(F)} \mathfrak{F}(W). \tag{2.3.3}$$

This expansion follows from a natural extension of the theory of (P, ω)-partitions from Section 1.2, which we describe in Section 3.1.

2.3.4 Correspondence $\Omega^\bullet$

We can construct a correspondence $\Omega^\bullet$, of which the Ω-parking procedure is but a shadow. We skip the exact description and most details here, and refer to Section 5 of **[5]** for the interested reader; see also Section 3.2.1 for an explanation in the more general context of bilateral parking procedures. Briefly put, this correspondence associates with any word $\mathbf{a}$ in the alphabet $\mathbb{Z}$ an ordered pair of *labeled* indexed forests $(\mathsf{P}(\mathbf{a}), \mathsf{Q}(\mathbf{a}))$ by a certain insertion procedure

$$\Omega^\bullet : \mathbf{a} \mapsto (\mathsf{P}(\mathbf{a}), \mathsf{Q}(\mathbf{a})),$$

see Figure 2.3 for an illustration.

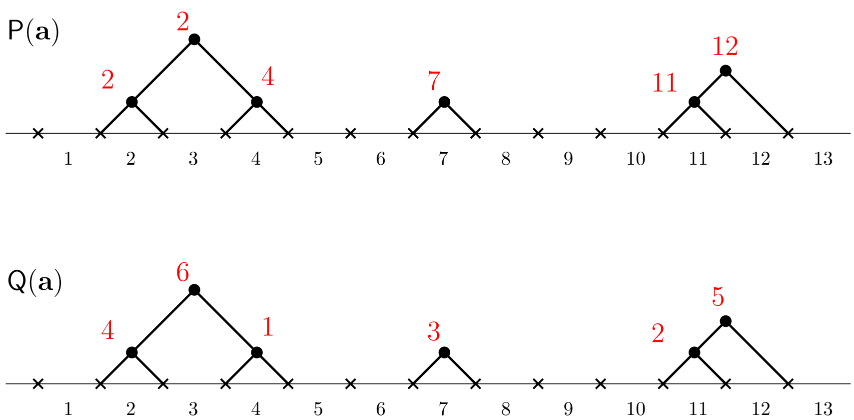

Figure 2.3. The tableaux $\Omega^\bullet(\mathbf{a}) = (\mathsf{P}(\mathbf{a}), \mathsf{Q}(\mathbf{a}))$ for the word $\mathbf{a} = 4.11.7.11.12.2$.

The insertion procedure is such that $\mathsf{P}(\mathbf{a})$ is a local binary search forest whereas $\mathsf{Q}(\mathbf{a})$ is a decreasing forest. Both are labelings of the same underlying indexed forest[3].

Fact 2.3.6 ($\Omega^\bullet$ *lifts* Ω)

Let $\mathbf{a}$ be a word over $\mathbb{Z}$. The indexed forest F underlying $\Omega^\bullet(\mathbf{a})$ has support $\Omega(\mathbf{a})$. In particular, $\mathbf{a} = a_1 \cdots a_k$ is Ω-parking if and only if F consists of a single tree with support $[k]$.

[3]We allow supports of indexed forests to be in $\mathbb{Z}$ instead of $\mathbb{Z}_{>0}$ in this section; if the support of F is not in $\mathbb{Z}_{>0}$, we set $\mathsf{P}_F = 0$.

The $\Omega^\bullet$-correspondence allows us to define the equivalence $\underset{\Omega}{\equiv}$ on words: $\mathbf{a} \underset{\Omega}{\equiv} \mathbf{b}$ if $\mathsf{P}(\mathbf{a}) = \mathsf{P}(\mathbf{b})$. To each class $\mathcal{C}$ we attach the polynomial

$$\mathsf{P}(\mathcal{C}) := \sum_{\mathrm{i}\in\mathcal{C}} \mathfrak{F}(\mathrm{i}). \tag{2.3.4}$$

Each class $\mathcal{C}$ corresponds to a labeled forest of the form $\mathsf{P}(\mathbf{a})$, and thus to its underlying forest: we will simply write it as $F(\mathcal{C})$. Then one can show that

$$\mathsf{P}(\mathcal{C}) = \mathsf{P}_{F(\mathcal{C})}.$$

That is, (2.3.4) always gives a forest polynomial (or zero). For any permutation w, the set $\mathrm{Red}(w)$ is closed[4] under $\underset{\Omega}{\equiv}$. Therefore it decomposes into $\underset{\Omega}{\equiv}$-equivalence classes. Let $\mathcal{G}_w$ denote the set of $\underset{\Omega}{\equiv}$-equivalence classes of $\mathrm{Red}(w^{-1})$. Then (1.4.3) can be rewritten

$$\mathfrak{S}_w = \sum_{\mathcal{C}\in\mathcal{G}_w} \mathsf{P}_{F(\mathcal{C})}. \tag{2.3.5}$$

This is the decomposition hinted at at the end of Section 2.3.2.

2.3.5 The Aval–Bergeron–Bergeron decomposition

The work of Aval–Bergeron–Bergeron [ABB04] is concerned with the quotient $\mathrm{Pol}_n / \mathrm{QSym}_n^+$. They compute a basis of this quotient, showing that its dimension is the nth Catalan number. Explicitly, let ABB_n denote the set of $c \in \mathsf{Codes}_n$ such that

$$\sum_{i=1}^{j} c_{n+1-i} \leq j-1 \text{ for } j = 1, \ldots, n.$$

They show

$$\mathrm{Pol}_n = \mathbb{Q}\{\mathbf{x}^c \mid c \in \mathsf{ABB}_n\} \oplus \mathrm{QSym}_n^+. \tag{2.3.6}$$

For $f \in \mathrm{Pol}_n$, let us write $\pi(f) \in \mathbb{Q}\{\mathbf{x}^c \mid c \in \mathsf{ABB}_n\}$ for the projection parallel to QSym_n^+ according to the above decomposition, that is, $f - \pi(f) \in \mathrm{QSym}_n^+$.

Proposition 2.3.7

Let $F \in$ Forest such that $\mathrm{c}(F) \in \mathsf{Codes}_n$. Then the following holds:

$$\pi(\mathsf{P}_F) = \begin{cases} \mathsf{P}_F & \mathrm{c}(F) \in \mathsf{ABB}_n \\ 0 & \mathrm{c}(F) \notin \mathsf{ABB}_n. \end{cases}$$

For the first case, it is in fact easy to show from Definition 2.3.5 that all monomials occurring in x^c in P_F satisfy $c \in \mathsf{ABB}_n$. The second case means that $\mathsf{P}_F \in \mathrm{QSym}_n^+$; the proof in **[5]** is by induction, using a special recurrence relation for forest polynomials.

Proof of Theorem 2.3.4. We have now all the ingredients to sketch the proof. Let $w \in S'_n$, and we want to compute $a_w = \langle \mathfrak{S}_w \rangle_n$. We apply this to the decomposition (2.3.5). By Proposition 2.3.3, we have

$$a_w = \sum_{\mathcal{C}\in\mathcal{G}'_w} \langle \mathsf{P}_{F(\mathcal{C})} \rangle_n \tag{2.3.7}$$

[4]There is a characterization of $\underset{\Omega}{\equiv}$ as generated by certain local moves $\cdots ij \cdots \mapsto \cdots ji \cdots$ for $|j-i| \geq 2$ under a certain constraint, from which this closure property follows immediately.

where $\mathcal{G}'_w \subset \mathcal{G}_w$ consists of the classes $\mathcal{C}$ with $\mathrm{c}(F(\mathcal{C})) \in \mathsf{ABB}_n$. Since $F(\mathcal{C})$ has size $n-1$, this is easily seen to be equivalent to the support of $F(\mathcal{C})$ be $[n-1]$. For such a class,

$$\langle \mathsf{P}_{F(\mathcal{C})} \rangle_n = \mathsf{P}_{F(\mathcal{C})}(1, 1, \ldots) = \#\mathcal{C}.$$

Indeed, the first equality follows from $\langle x^c \rangle_n = 1$ for all $c \in \mathsf{ABB}_n$ with $\sum_i c_i = n-1$, as can be deduced easily from Postnikov's rule (**2.3.2**). The second one follows from the combinatorics of forest polynomials and is skipped here. Replacing in (**2.3.7**), we get

$$a_w = \sum_{\mathcal{C} \in \mathcal{G}'_w} \#\mathcal{C}. \tag{2.3.8}$$

Using Fact 2.3.6, $\mathcal{C} \in \mathcal{G}'_w$ if all words in $\mathcal{C}$ are Ω-parking (equivalently, $\mathcal{C}$ contains a Ω-parking word). Thus the formula above is precisely the content of Theorem 2.3.4.

Chapter 3

Miscellany of combinatorics

3.1 Flagged P-partitions

We briefly recalled the theory of (P,ω)-partitions in Section 1.2. An extension was given to allow for upper bounds on values ("flags") by Assaf and Bergeron [AB20]. The generating functions will become polynomials (no more quasisymmetry) and the building blocks are now the slide polynomials introduced in(**3.1.5**) instead of fundamental quasisymmetric functions. We present here some results from the article **[6]** which revisits their approach.

3.1.1 Flags

Let $\overline{\mathbb{Z}_{>0}}$ be the ordered alphabet with letters $i^{[j]}$ where $i \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{>0}$. We have a linear order on $\mathbb{Z}_{>0} \sqcup \overline{\mathbb{Z}_{>0}}$ given by $i < i^{[1]} < i^{[2]} < i^{[3]} < \cdots < i+1$ for all i. We define the *value* of $i^{[j]}$ by $\mathrm{val}(i^{[j]}) = i$.

The order is the lexicographic order on $\mathbb{Z} \times \mathbb{Z}_{>0}$: as we will see the values will encode the upper bounds for P-partitions, while the exponents will replace the labeling ω from (P,ω)-partitions.

Let $\mathrm{Inj}(\overline{\mathbb{Z}_{>0}})$ be the set of injective words, which are words with distinct letters over the alphabet $\overline{\mathbb{Z}_{>0}}$. There is a natural way to go from a word w on $\mathbb{Z}_{>0}$ to $\mathrm{Inj}(\overline{\mathbb{Z}_{>0}})$. Namely, one associates a word $W = \mathrm{stan}(w)$ by labeling the occurrences of a given letter i in w from left to right by $i^{[1]}, i^{[2]}, \ldots$. For instance $w = 1221625$ gets mapped to $W = 1^{[1]}2^{[1]}2^{[2]}1^{[2]}6^{[1]}2^{[3]}5^{[1]}$. This process is injective, and will be used as our natural embedding

$$\mathrm{stan} : \mathbb{Z}^* \hookrightarrow \mathrm{Inj}(\overline{\mathbb{Z}}). \tag{3.1.1}$$

3.1.2 (P,Φ)-partitions

We now define our notion of (P,Φ)-partitions with restrictions.

Definition 3.1.1

Let P be a poset. A labeled flag on P is an injective function $\Phi : P \to \overline{\mathbb{Z}_{>0}}$.

Let us note that if $P = (p_1 < p_2 < \ldots < p_k)$ is a chain, then a labeled flag Φ can be identified with the injective word $\Phi(p_1)\Phi(p_2)\ldots\Phi(p_k)$.

Definition 3.1.2 (*(P,Φ)-partitions*)

Let (P,Φ) be a poset with a labeled flag. A (P,Φ)-partition is a function $f : P \to \mathbb{Z}_{>0}$ such that for any $u, v \in P$:

- $f(u) \geq f(v)$ *if* $u \prec_P v$;
- $f(u) > f(v)$ *if* $u \prec_P v$ *and* $\Phi(u) > \Phi(v)$;
- $f(u) \leq \mathrm{val}(\Phi(u))$.

The first two conditions define (P, ω)-partitions for ω ordering elements as Φ does. Now the last condition forces upper bounds, in a way that is "compatible" with the ordering.

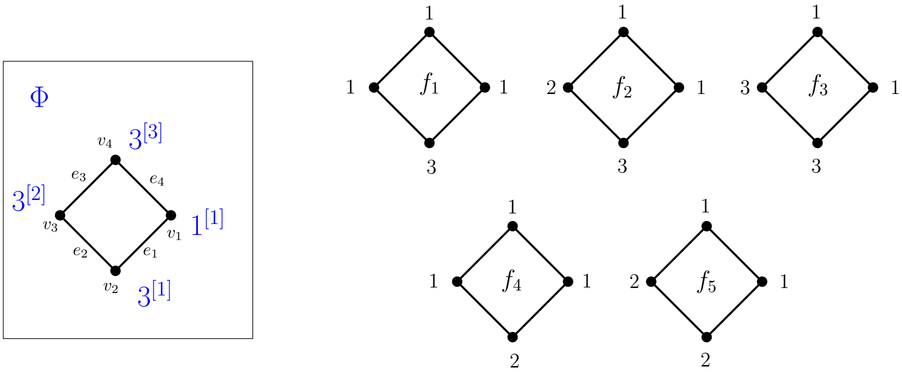

Figure 3.1. A poset with labeled flag (P, Φ) and its five (P, Φ)-partitions

We denote the set of all (P, Φ)-partitions by $\mathrm{Part}(P, \Phi)$. An example is given in Figure 3.1: on the left is the Hasse diagram of the *diamond* poset with the labeled flag in blue, while on the right are the five (P, Φ)-partitions.

3.1.3 Slide decompositions

Let (P, Φ) be a poset with a labeled flag. The proof of the partition (**1.2.2**) goes through to show

$$\mathrm{Part}(P, \Phi) = \bigsqcup_{L \in \mathrm{Lin}(P)} \mathrm{Part}(L, \Phi). \tag{3.1.2}$$

Introducing the generating polynomials

$$K_{(P,\Phi)} = \sum_{f \in \mathrm{Part}(P,\Phi)} \prod_{u \in P} x_{f(u)}. \tag{3.1.3}$$

which gets us then immediately

$$K_{(P,\Phi)} = \sum_{L \in \mathrm{Lin(P)}} K_{(L,\Phi)}. \tag{3.1.4}$$

Recall that (L, Φ) can be encoded as an injective word $W = a_1 \ldots a_r \in \mathrm{Inj}(\overline{\mathbb{Z}_{>0}})$ where $\#P = r$. Then any (L, Φ)-partition f can be encoded in a sequence $(i_1, \ldots, i_r)$ with $i_j = f(v_j)$, and we get $K_{(L,\Phi)}$ is thus given by the explicit series

$$K_{(L,\Phi)} = \mathfrak{F}(W) := \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_r > 0 \\ i_j > i_{j+1} \text{ if } a_j > a_{j+1} \\ i_j \leq \mathrm{val}(a_j)}} x_{i_1} x_{i_2} \cdots x_{i_r}. \tag{3.1.5}$$

Note that we had defined $\mathfrak{F}(w)$ when w is a word on $\mathbb{Z}_{>0}$; it is directly checked that $\mathfrak{F}(w) = \mathfrak{F}(\mathrm{stan}(w))$, which makes the two definitions compatible. We have moreover

Proposition 3.1.3

For any $W \in \mathrm{Inj}(\overline{\mathbb{Z}_{>0}})$, the polynomial $\mathfrak{F}(W)$ is either zero or is equal to $\mathfrak{F}_c$ for a unique c.

This extends the case where W has the form $W = \mathrm{stan}(w)$, and the proof is essentially the same: see Section 3.2 of **[6]**.

Application to Forest polynomials. We can now explain (**2.3.3**) which gives the expansion of forest polynomials into slide polynomials. Indeed, one can identify forest polynomials as the generating $K_{(F,\Phi_F)}$, where F is an indexed forest and Φ_F has values ϕ_F from Section 2.3.3, with exponents increasing going up left edges; see Figure 3.2.

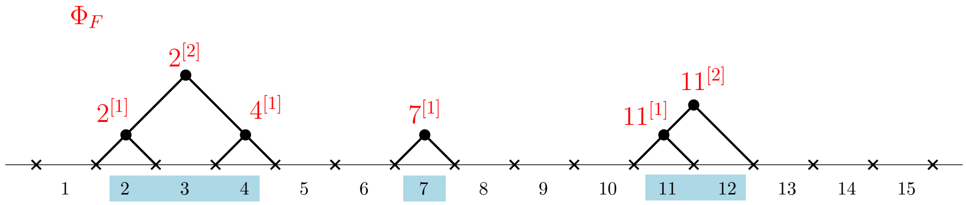

Figure 3.2. An indexed forest with its flag.

3.2 Bilateral parking procedures

Part of the work **[6]** presented above was motivated by giving a nice framework for the expansion of forest polynomials into slide polynomials. The work presented in this section taken from **[11]** was aimed at understanding the parking procedure Ω and correspondence $\Omega^\bullet$ in a larger context, cf. Section 3.2.2. It turns out to also connect to mixed Eulerian numbers, cf. Section 3.2.4, and to give rise to a surprising "universal" enumeration result (Theorem 3.2.3).

3.2.1 Bilateral parking

Abstractly, a parking procedure on $\mathbb{Z}$ will be encoded as a function that takes as entry a word $a_1 \cdots a_r$ (where a_i represents the desired spot of the ith car) and outputs the finite set of spots in $\mathbb{Z}$ where cars are parked. We focus here on a particular class; we let $\mathrm{Fin}(\mathbb{Z})$ be the set of all finite subsets of $\mathbb{Z}$, and given $S \in \mathrm{Fin}(\mathbb{Z})$, we call block of S a maximal interval in S.

A function $\mathcal{P} : \mathbb{Z}^* \to \mathrm{Fin}(\mathbb{Z})$ is a *bilateral parking procedure* if $\mathcal{P}(\epsilon) = \varnothing$ and if, for any $r \geq 1$ and any word $a_1 \cdots a_r$, it satisfies the following conditions:

1. If $a_r \notin \mathcal{P}(a_1 \cdots a_{r-1})$, then $\mathcal{P}(a_1 \cdots a_r) = \mathcal{P}(a_1 \cdots a_{r-1}) \sqcup \{a_r\}$;
2. If $a_r \notin \mathcal{P}(a_1 \cdots a_{r-1})$, let $I = \{t, t+1, \ldots, u\}$ be the block of $\mathcal{P}(a_1 \cdots a_{r-1})$ containing a. Then $\mathcal{P}(a_1 \cdots a_r) = \mathcal{P}(a_1 \cdots a_{r-1}) \sqcup \{a\}$ where $a \in \{t-1, u+1\}$.

The first condition expresses that one parks at their favorite spot if it is available, while the second says that otherwise, one needs to park in the next available spot on the left or on the right. In particular, everyone manages to park: the subset $\mathcal{P}(a_1 \cdots a_r)$ has cardinality r and $\mathcal{P}(a_1 \cdots a_{r-1}) \subset \mathcal{P}(a_1 \cdots a_r)$. A bilateral procedure is determined by the choice of $t-1$ or $u+1$ in the second case. We will say simply "park to the left" or "park to the right" when describing procedures.

Definition 3.2.1

Let $\mathcal{P}$ be a bilateral parking procedure. A word $a_1 \cdots a_r$ is $\mathcal{P}$-parking if $\mathcal{P}(a_1 \cdots a_r) = [r]$.

Let us give some more examples:

- The classical parking procedure $\mathcal{P}^{right}$ consists simply of always parking to the right. It is clear that $\mathcal{P}^{right}$-parking words are the usual parking functions.
- The procedure Ω from Section 2.3.4 also fits this definition: When the desired spot is unavailable, let $j \in \{1, \ldots, r-1\}$ be maximal such that $a_j \in I$. Then $\Omega(W a_r)$ is defined by parking to the left if $a < a_j$, and to the right of if $a \geq a_j$.
- $\mathcal{P}^{prime}$: If the size of the interval I from condition 2. is a *prime number*, park to the right, otherwise park to the left.

3.2.2 Labeled Indexed forests

The following explains the connection between Ω and $\Omega^\bullet$ in a wider context.

To any bilateral parking procedure $\mathcal{P}$ one can attach a correspondence $\widehat{\mathcal{P}}$ that associates to a word $\mathbf{a}$ a pair of labeled indexed forests $(\mathsf{P}(\mathbf{a}), \mathsf{Q}(\mathbf{a}))$ with the same underlying forest $F(\mathbf{a})$ whose support is $\mathcal{P}(\mathbf{a})$. Here $\mathsf{Q}(\mathbf{a})$ is a decreasing forest, while $\mathsf{P}(\mathbf{a})$ are certain labelings by the letters in $\mathbf{a}$.

The definition is inductive, let us sketch it; we refer to Section 5 of **[11]** for more details. Suppose that P and Q are the trees obtained after reading a word $\mathbf{a} = a_1 \cdots a_k$, and $F \in$ Forest be their common shape with support $\mathcal{P}(\mathbf{b})$. Let $b \in \mathbb{Z}$, we want to determine the trees $\mathsf{P}' = \mathsf{P}(\mathbf{a}b)$ and $\mathsf{Q}' = \mathsf{Q}(\mathbf{a}b)$. We know that $\mathcal{P}(\mathbf{a}b)$ contains one more element than $\mathcal{P}(\mathbf{b})$. There is then a unique way to add a root to F so that the new forest F' has support $\mathcal{P}(\mathbf{a}b)$. The labelings P' and Q' are obtaining by labeling the new root in F' by b in P', and by $k+1$ in Q'; all other labels remain the same.

This mapping $\mathbf{a} \mapsto \widehat{\mathcal{P}}(\mathbf{a})$ is injective: one can get back the original by reading labels in $\mathsf{P}(\mathbf{a})$. It is harder to describe its image in general. Note finally that the decreasing forest $\mathsf{Q}(\mathbf{a})$ encodes precisely the order in which the spots were occupied during the procedure, that is the flag of sets

$$\varnothing \subset \mathcal{P}(a_1) \subset \mathcal{P}(a_1 a_2) \subset \cdots \subset \mathcal{P}(\mathbf{a}).$$

3.2.3 Local procedures and universal enumeration

Let $\tau : i \mapsto i+1$ be the shift on $\mathbb{Z}$. It extends to subsets of $\mathbb{Z}$ or words in $\mathbb{Z}^*$ naturally: for instance $\tau(\{2,3,5\}) = \{3,4,6\}$ and $\tau(523) = 634$. If I is a block of $S = \mathcal{P}(W)$, let W_I be the subword of W given by the letters a_i such that the ith car parked in S (this depends on the procedure $\mathcal{P}$).

Definition 3.2.2

A bilateral parking procedure $\mathcal{P}$ is said to be shift invariant if $\mathcal{P}(\tau(W)) = \tau(\mathcal{P}(W))$ for any W. A parking procedure is called locally decided if for any W, I a block of $\mathcal{P}(W)$ and $a \in I$, then the decision to park left or right is the same when reading a after W or W_I.

A procedure is called local if it is both shift invariant and locally decided.

The notion of shift invariance is fairly clear; the notion of local decision can be understood as encoding the fact that the decision to park left or right of a block does not depend on the other blocks. The procedures $\mathcal{P}^{right}, \mathcal{P}^{prime}, \Omega$ are all shift invariant. A procedure that is not shift invariant is the following procedure $\mathcal{P}^{evenodd}$: if the desired spot a_i is occupied, park right if a_i is even, and left if a_i is odd.

The following may appear as a fairly surprising result at first sight, especially thinking of the procedure $\mathcal{P}^{prime}$ for instance.

Theorem 3.2.3

Let $\mathcal{P}$ be a bilateral, local parking procedure. Then the number of $\mathcal{P}$-parking functions of length r is given by $(r+1)^{r-1}$.

The proof is based on Pollak's cyclic lemma argument for the classical case, as given in [FR74]. Very briefly, one uses the "locally decided" property of $\mathcal{P}$ to show that there is a well defined notion of $\mathcal{P}$-parking on a cycle $\mathbb{Z}/(r+1)\mathbb{Z}$ for words of length r with letters in $[r+1]$. The shift-invariance then entails that a fraction $1/(r+1)$ of these are $\mathcal{P}-$parking, leading to the desired enumeration.

3.2.4 Probabilistic parking

There is a natural way to insert randomness in our bilateral parking procedures: replace the deterministic rule to decide to park to the right or to the left by a probabilistic one. A simple example is already given at the end of [KW66], the seminal article introducing the usual parking functions. Fix p a real in $[0,1]$. Then in case one's spot is occupied, park to the right with probability p and to the left with probability $1-p$. Of course $p = 1$ correspond to the procedure $\mathcal{P}^{right}$.

We still keep the notation $\mathcal{P}$ for such procedures; for a given word $\mathbf{a}$, we get a random finite set $\mathcal{P}(\mathbf{a})$ of parking spots, which is supported only on a finite number of sets in $\mathrm{Fin}(\mathbb{Z})$. It is natural to define the *$\mathcal{P}$-parking probability* of $\mathbf{a} = a_1 \cdots a_k$ as the probability that $\mathcal{P}(\mathbf{a})$ is equal to $[k]$. Let us give an example that we will revisit in Section 3.3.

Fix $q \in \mathbb{R}^+$. Recall the q-integers $(n)_q = 1 + q + \ldots + q^{n-1}$ and q-factorials $(n)_q! = (n)_q(n-1)_q \cdots (1)_q$. The procedure $\mathcal{P}^q$ is defined as follows: if one wants to park at a spot $i \in [a, b]$ where $[a, b]$ is a block of occupied spots, then one parks to the right with probability $(i+1-a)_q/(b-a+2)_q$, and to the left with probability $q^{i+1-a}(b-i+1)_q/(b-a+1)_q$. These are the probabilities to exit right and left of the block $[a, b]$ respectively, when doing a biased random walk starting from i with jump probabilities $1/(1+q)$ and $q/(1+q)$.

This has a natural *abelian* property: the probability distribution $\mathcal{P}^q(\mathbf{a})$ (and in particular the $\mathcal{P}^q$-parking probability) is invariant under permuting letters in $\mathbf{a}$. It thus only depends on the content $c(\mathbf{a}) = (c_1, c_2, \ldots) \in$ Codes where c_i counts occurrences of i in $\mathbf{a}$ for all i.

Definition 3.2.4 (*Remixed Eulerian Numbers*)

Given $c = (c_1, \ldots, c_r)$ with sum r, we define the remixed Eulerian number

$$A_c(q) := (r)_q! p_c(q)$$

where $p_c(q)$ is the $\mathcal{P}^q$-parking probability of a word with content c.

When $q = 1$ –which corresponds to an unbiased random walk above– we will see in Section 3.3.2 that $A_c(1)$ is exactly the *mixed Eulerian number* introduced in Section 2.2, and that $A_c(q)$ is a natural q-deformation.

There is a notion of a *memoryless* (deterministic or probabilistic) procedure: it means that $\mathcal{P}(\mathbf{a}b)$ depends only on b and the occupied spots $\mathcal{P}(\mathbf{a})$. While $\mathcal{P}^{right}, \mathcal{P}^{prime}$ are memoryless, the procedure Ω from Section 2.3.2 is not: it requires information about previous drivers' choices. We have the following result (Proposition 4.7 in **[11]**)

Proposition 3.2.5

The procedures $\mathcal{P}^q$ are the only local, memoryless, abelian procedures.

To finish, the following notions generalize to the random setting:

- There is a natural notion of probabilistic local procedures. The proof of Theorem 3.2.3 can be adapted to show the following: the sum of $\mathcal{P}$-parking probabilities of all words of length r is given by $(r+1)^{r-1}$.
- The correspondence with labeled indexed forests can also be extended, by associating to a word $\mathbf{a}$ a probability measure on pairs of bilabeled trees.

3.2.5 Generalization

Instead of starting with preference words over $\mathbb{Z}$, one can define procedures for words over $\mathbb{Z} \times A$: here A is any set adding some information for the parking rules, while $\mathbb{Z}$ is still used to encode the desired spot.

The augmented alphabet $\overline{\mathbb{Z}} \simeq \mathbb{Z} \times \mathbb{Z}_{>0}$ is such an example. The parking procedure Ω from Section 2.3.2 is naturally defined over injective words $\mathrm{Inj}(\overline{\mathbb{Z}})$ over this alphabet, as follows.The parking rules are the same as in the original procedure using only values, except when $v_i = v_j$ in the case where one cannot park at their desired spot (we use the indices as in Section 2.3.2). In that case, the corresponding letters are $v_i^{[e_i]}$ and $v_j^{[e_j]}$, and one parks to the right if $e_i > e_j$ and to the left if $e_i < e_j$. The original parking procedure Ω is obtained by "standardization": we can inject $\mathbb{Z}$ into $\overline{\mathbb{Z}}$ by having all occurrences of the same letter i replaced by $i^{[1]}, i^{[2]}, \ldots$ from left to right.

3.3 q-analogues

While working on **[10]**, we had the idea of deforming some of our notions and introduce a parameter q. The resulting work **[16]** shows how fruitful that idea is, and we present some of its results here. It led also to the definition of remixed Eulerian numbers, an interesting family of polynomials whose combinatorics were studied in the separate work **[12]**.

3.3.1 The Klyachko algebra and Macdonald's reduced word identity

Let $\mathbf{k}$ be a field of characteristic zero, and $q \in \mathbf{k}$ not equal to a nontrivial root of unity. We introduce the *Klyachko algebra* $\mathcal{K}^q$, by considering a natural q-deformation of Klyachko's presentation [Kly85] given in Section 2.1.2. More precisely $\mathcal{K}^q$ is the commutative algebra over $\mathbf{k}$ on the generators $\{u_i \mid i \in \mathbb{Z}\}$ subject to the following relations for $i \in \mathbb{Z}$:

$$(q+1)u_i^2 = qu_iu_{i-1} + u_iu_{i+1}. \tag{3.3.1}$$

We let $\mathcal{K}^q_+$ be the algebra obtained by setting $u_i = 0$ for $i < 0$, and $\mathcal{K}^q_n$ if we set $u_i = 0$ for $i \geq n$ in addition. We recognize that $\mathcal{K}^1_n = \mathcal{K}_n$ as introduced in Section 2.1.2.

For an $\mathbb{N}$-vector c, let $u^c = \prod_{i\in\mathbb{Z}} u_i^{c_i}$. For any finite subset $I \subset \mathbb{Z}$, define the squarefree monomials u_I by $u_I := \prod_{i\in I} u_i \in \mathcal{K}^q$. The following generalizes Klyachko's basis in the case of $\mathcal{K}^1_n$:

Proposition 3.3.1

The family of squarefree monomials $(u_I)_I$ for I finite subset of $\mathbb{Z}$ (resp. $\mathbb{Z}_{>0}$, resp. $[n-1]$) is a $\mathbf{k}$-basis of the algebra $\mathcal{K}^q$ (resp. $\mathcal{K}^q_+$, res. $\mathcal{K}^q_n$).

The proof is given in Section 3.3 of **[16]**.

q-divided symmetrization Recall that Klyachko's algebra $\mathcal{K}_n = \mathcal{K}^1_n$ is used in the first formula, Theorem 2.1.3, for the numbers a_w. Now the second formula, Theorem 2.1.4, is in terms of divided symmetrization. Up to the evaluation (**2.1.3**), the fact that these two formulas coincide is equivalent to the following algebraic fact: for any polynomial $f \in \mathrm{Pol}_n$ of degree $n-1$, one has

$$\langle f \rangle_n = (n-1)! \int_{\mathcal{K}_n} f(u_1, u_2 - u_1, \ldots, u_{n-1} - u_{n-2}, -u_{n-1}).$$

(*Proof sketch:* The fact that this holds for all $\mathfrak{S}_w$ with $w \in S'_n$ is precisely the formula coincidence for a_w. Now in degree $n-1$, these polynomials span a subspace of Pol_n in direct sum with Sym^+_n, as seen in Section 1.4. It is then easy to show that both sides of the above equality vanish on the degree $n-1$ component of Sym^+_n.)

We can define a notion of $q-$divided symmetrization that will extend this identity to $\mathcal{K}^q_n$. It will have the added advantage of giving a direct proof of the identity for $q = 1$ without using geometry arguments.

The Demazure-Lusztig operators $T_i = (q-1)\partial_i x_i + s_i$ act on Pol. They satisfy braid and commutation relations, and thus the operators T_w for $w \in S_\infty$ are well-defined.

The *q-divided symmetrization* of $f \in \mathrm{Pol}_n$ is defined as

$$\langle f \rangle^q_n := \sum_{w \in S_n} T_w \left(\frac{f}{\prod_{1 \leq i \leq n-1}(qx_i - x_{i+1})} \right),$$

which clearly specializes to divided symmetrization for $q = 1$. Now we have

Theorem 3.3.2

For any polynomial $f \in \mathrm{Pol}_n$ of degree $n-1$, one has

$$\langle f \rangle^q_n = (n-1)_q! \int_{\mathcal{K}^q_n} f(u_1, u_2 - u_1, \ldots, u_{n-1} - u_{n-2}, -u_{n-1}).$$

Here $\int_{\mathcal{K}_n^q} P(u_1,\dots,u_{n-1})$ is the coefficient of $P(u_1,\dots,u_{n-1}) \in \mathcal{K}_n^q$ on the basis element $u_{[n-1]}$.

The idea of the proof is to use the polynomials $f = y_c$ defined by

$$y_c = x_1^{c_1}(x_1+x_2)^{c_2}\dots(x_1+\cdots+x_n)^{c_n},$$

for $\mathbb{N}$-vectors $c = (c_1,\dots,c_n)$ with $|c| = n-1$. One must show that both sides in Theorem 3.3.2 coincide on such polynomials. In fact, both sides vanish if $c_n > 0$, and one is left to show that for $c = (c_1,\dots,c_{n-1})$, :

$$\langle y_c\rangle_n^q = (n-1)_q! \int_{\mathcal{K}_n^q} u_1^{c_1}u_2^{c_2}\cdots u_{n-1}^{c_{n-1}}. \tag{3.3.2}$$

The proof in Section 4 of **[16]** then shows that both sides are equal to the remixed Eulerian numbers $A_{c_1,\dots,c_{n-1}}(q)$, by giving the following characterization of these numbers: the remixed Eulerian numbers $A_c(q)$ are uniquely determined for any r by the conditions $A_{(1^r)} = (r)_q!$ and

$$(1+q)A_{(c_1,\dots,c_r)} = qA_{(c_1,\dots,c_{i-1}+1,c_i-1,\dots,c_r)} + A_{(c_1,\dots,c_i-1,c_{i+1}+1,\dots,c_r)}$$

for any $i \leq r$ and $c_i \geq 2$.

q-Klyachko–Macdonald identity. We give a q-analogue of Klyachko's specialization (**2.1.3**).

Theorem 3.3.3

Let $w \in S_\infty$ of length ℓ. The following equality holds in $\mathcal{K}_+^q$.

$$\mathfrak{S}_w(u_1, u_2-u_1,\dots) = \frac{1}{(\ell)_q!}\sum_{\mathbf{a}\in\mathrm{Red}(w)} q^{\mathrm{comaj}(\mathbf{a})}u_{a_1}u_{a_2}\cdots u_{a_\ell}.$$

We skip the proof: it is given in Section 7.1 of **[16]**. It differs from Klyachko's proof of (**2.1.3**), as presented in Section 8 of **[10]**. In fact the latter proof does not seem to adapt so as to incorporate the parameter q. Instead we adapt the argument in [FS94] using the NilCoxeter algebra, via a "Yang–Baxter relation" that works particularly well with the relations in $\mathcal{K}_+^q$.

Notice that $\mathrm{sp}: \mathcal{K}_+^q \to \mathbf{k},\ u_i \mapsto (i)_q$ is clearly a morphism of algebra, and we retrieve in particular q-Macdonald identity (**1.4.6**).

3.3.2 Remixed Eulerian numbers

In this section we consider q as an indeterminate. We introduced the family $A_c(q)$ first in Definition 3.2.4 with a probabilistic interpretation (and $q \geq 0$ a real number). We then stated two alternative definitions, given by both sides of (**3.3.2**). Note that none of these definitions immediately implies the following result, whose proof consists in a recurrence for these numbers:

Proposition 3.3.4 (*Proposition 5.4 of* [16])

For any $c = (c_1,\dots,c_r)$ such that $|c| = r$, we have $A_c(q) \in \mathbb{N}[q]$.

Combinatorics. This family has nice combinatorial properties. Already for $q = 1$ many were noticed by Postnikov in [Pos09]; all of these naturally have q-analogues as we show in **[12]**.

Subfamilies of $A_c(q)$ recover several classical numbers: (shifted) q-binomial coefficients for $c = (k, 0,\dots,r-k)$; q-analogues of Eulerian numbers if $c_i = r$ and $c_j = 0$ otherwise; more generally, *q-hit numbers* for c connected, that is if the support $\{i \mid c_i \neq 0\}$ is an interval.

Let us focus on q-hit numbers. This family was first introduced by Garsia and Remmel [GR86] as a transformation of the more classical q-rook numbers. They can be studied fruitfully via the probabilistic definition of $A_c(q)$, as can various extensions of these numbers: this is the PhD topic of Solal Gaudin, a current student of Sylvie Corteel and myself, see in particular [Gau24].

Recently (q-)hit numbers have resurfaced in algebraic contexts, related to matroids [BST23, KK24] as well as certain chromatic symmetric functions [AN21, CMP23]. This last connection was intriguing, as a key role in this context is played by certain *modular relations*

$$(1+q)\mathsf{ene} = q\mathsf{een} + \mathsf{nee}$$
$$(1+q)\mathsf{nen} = q\mathsf{enn} + \mathsf{nne}.$$

which are reminiscent of the relation (**3.3.1**).

We explained this connection satisfyingly in Section 3.1 of **[9]**, by defining a morphism from the algebra[1] given by the modular relations to the q-Klyachko algebra $\mathcal{K}^q$.

Polynomial properties. Let $c = (c_1, \ldots, c_r) \in \mathsf{Codes}_r$ with $|c| = r$. Let D_c denote the *degree* of $A_c(q)$, and d_c its *valuation*, i.e. the smallest exponent of q with a nonzero coefficient in $A_c(q)$.

For $t \in \mathbb{R}$, we use $t^+ := \max(0, t)$. Let $h_i(c) = \sum_{1\leq j\leq i}(c_j - 1)$, so that $h_r(c) = 0$ in particular.

$$H(c) := \sum_{1\leq i\leq r-1} h_i(c) \text{ and } H^-(c) := \sum_{1\leq i\leq r-1} (-h_i(c))^+. \tag{3.3.3}$$

Then we have the following formulas

$$d_c = H^-(c), \tag{3.3.4}$$
$$D_c = \binom{r}{2} - \sum_{1\leq i\leq r-1} (h_i(c))^+. \tag{3.3.5}$$

The following pictorial perspective for c is useful. Attach a *Łukasiewicz path* P_c to c by starting at the origin and translating by $(1, c_i - 1)$ as c is read from left to right; see Figure 3.3. For $i \in [r]$, $h_i(c)$ is then the ordinate on P_c after the ith step.

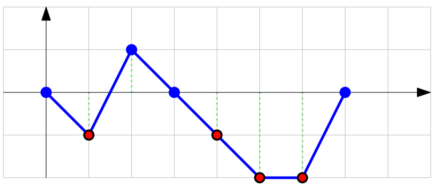

Figure 3.3. P_c when $c = (0, 3, 0, 0, 0, 1, 3)$

Finally, let us say that a polynomial $P(q) = \sum_i p_i q^i \in \mathbb{Z}[q]$ is $\mathrm{psu}(N)$ if it has *p*ositive coefficients; is *s*ymmetric with respect to $N/2$, by which we mean that the coefficients satisfy $p_i = p_{N-i}$; and is *u*nimodal, which means coefficients weakly increase then weakly decrease. Then

Theorem 3.3.5

For any c, $A_c(q)$ is $\mathrm{psu}\left(\binom{r}{2} - H(c)\right)$.

The proof is based on the same recurrence as the one used to prove Proposition 3.3.4.

Let us illustrate these results on $c = (0, 3, 0, 0, 0, 1, 3)$. the formulas give $d_c = 1 + 1 + 2 + 2 = 6$ and $D_c = \binom{7}{2} - 1 = 20$. As a matter of fact, the full polynomial $A_c(q)$ is given by

$$2q^{20}+6q^{19}+11q^{18}+18q^{17}+27q^{16}+35q^{15}+40q^{14}+42q^{13}+40q^{12}+35q^{11}+27q^{10}+18q^{9}+11q^{8}+6q^{7}+2q^{6}.$$

which also illustrates Theorem 3.3.5 since $A_c(q)$ is $\mathrm{psu}(26)$.

[1] More accurately, from the subalgebra spanned by monomials with equal numbers of n and e.

Chapter 4

A new approach to quasisymmetric polynomials

We know that quasisymmetric polynomials play a role in the proof of the combinatorial interpretation of a_w from Theorem 2.3.4. The key is that the space QSym_n^+ interacts well with divided symmetrization, cf. Proposition 2.3.3, and with the forest polynomials, cf. Theorem 2.3.7.

In this section we will develop an operator approach to quasisymmetric polynomials, which leads to a simple derivation of several properties, as well as highlight the special role played by forest polynomials. It is parallel to the one for symmetric polynomials and Schubert polynomials. We will in fact give a general framework explaining the existence of these special families. The works **[2]** and **[3]** are the sources for this chapter, as well as a result from **[1]** connecting to the numbers a_w.

4.1 Quasisymmetric polynomials revisited

The starting point is the special role of Schubert polynomials with respect to divided differences, as stated in Theorem 1.4.1. Note that a polynomial is symmetric in $x_1, \ldots, x_n$ if and only if it is fixed by s_i for all $i < n$, if and only if it is in the common kernel of the operators ∂_i for all $i < n$.

We define similar operators T_i for the space of quasisymmetric polynomials, and develop a theory from the combinatorics of these operators. The results from this section are from[1] **[3]**.

4.1.1 Operators T_i, R_i and quasisymmetric polynomials

Bergeron and Sottile [BS98] were the first to introduce the following operator in the context of Schubert calculus (see also [BS02, LRS06]). For $f \in \mathrm{Pol}$ we define the ith *Bergeron-Sottile map*

$$\mathsf{R}_i(f) = f(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots). \tag{4.1.1}$$

In other words, $\mathsf{R}_i(f)$ sends x_i to 0 and shifts $x_j \mapsto x_{j-1}$ for all $j \geq i+1$. It is an algebra endomomorphism of QSym.

Theorem 4.1.1

$f \in \mathrm{Pol}_n$ has $f \in \mathrm{QSym}_n$ if and only if $\mathsf{R}_1 f = \cdots = \mathsf{R}_n f$.

The proof is a simple verification that $\mathsf{R}_i f = \mathsf{R}_{i+1} f$ holds precisely when f is quasisymmetric in x_i, x_{i+1}, seeing other indeterminates as scalars. Let us mention that Hivert's fundamental work [Hiv00] uses a similar characterization, but in terms of a certain symmetric group action.

[1] A parameter m is present throughout **[3]** to deal with "colored" m-quasisymmetric polynomials; we will only deal with $m = 1$ corresponding to the classical quasisymmetric case.

Corollary 4.1.2

QSym_n *is a subring of* Pol_n.

The proof is immediate from Theorem 4.1.1: the space of $f \in \mathrm{Pol}_n$ such that $\mathsf{R}_i f = \mathsf{R}_{i+1} f$ is a subring since $\mathsf{R}_i, \mathsf{R}_{i+1}$ are algebra morphisms, and the intersection of subrings is a subring. All other proofs of this fact (that I'm aware of) involve identifying an explicit basis (monomial,fundamental) whose multiplication can be explicitly computed, while here we have a more illuminating and streamlined explanation.

We now define the operator $\mathsf{T}_i : \mathrm{Pol} \to \mathrm{Pol}$ by the following

$$\mathsf{T}_i f := \frac{\mathsf{R}_{i+1} f - \mathsf{R}_i f}{x_i}, \text{ which expands to} \tag{4.1.2}$$

$$\mathsf{T}_i f(\mathbf{x}_+) := \frac{f(x_1, \ldots, x_{i-1}, x_i, 0, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots)}{x_i}. \tag{4.1.3}$$

We call T_i the *quasisymmetric divided difference*; later we will see it also as a "trimming operator". Let us note already the following easy factorizations:

$$\mathsf{T}_i f = \mathsf{R}_i \partial_i f = \mathsf{R}_{i+1} \partial_i f. \tag{4.1.4}$$

The following corollary is a direct rephrasing of Theorem 4.1.1.

Corollary 4.1.3

$f \in \mathrm{Pol}_n$ *is quasisymmetric if and only if* $\mathsf{T}_1 f = \cdots = \mathsf{T}_{n-1} f = 0$.

So the operators T_i detect quasisymmetry in the same way that the operators ∂_i detect symmetry. Recall that the operators ∂_i satisfy the relations of the nilCoxeter monoid. The operators T_i satisfy the following:

$$\mathsf{T}_i \mathsf{T}_j = \mathsf{T}_j \mathsf{T}_{i+1} \text{ for all } i > j. \tag{4.1.5}$$

These relations define a monoid structure on some familiar objects, as we now explain.

4.1.2 Forests and the Thompson monoid

Recall the indexed forests of Section 2.3.3. The following is a slightly different definition of these objects that is better suited for this chapter:

Definition 4.1.4

An indexed forest F is an infinite sequence $T_1, T_2, \ldots$ of binary trees where all but finitely many of the trees are trivial.

Given a forest[2] F, its tree T_i above will be represented with a root labeled i. Also, the union of all leaves of these trees can be naturally labeled "from left to right", leading to a bijection[3] between leaves and $\mathbb{Z}_{>0}$: see Figure 4.1.

The equivalence of this definition of indexed forests with the original one is also clear: in particular, nontrivial binary trees correspond to maximal intervals of the support in the original definition.

Monoid structure We define a monoid structure on Forest by taking for $F, G \in$ Forest the composition $F \cdot G \in$ Forest to be obtained by identifying the ith leaf of F with the ith root node of G. The empty forest $\varnothing \in$ Forest is the identity element. This is illustrated in Figure 4.1.

[2]We will often say simply forest instead of indexed forest.

[3]This is the same leaf labeling used in the Definition 2.3.5 of forest polynomials.

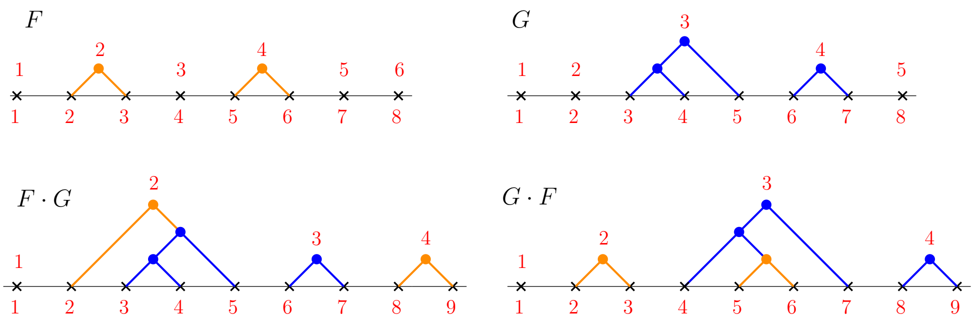

Figure 4.1. The products $F \cdot G$ and $G \cdot F$ for $F, G \in$ Forest, with both roots and leaves labeled

Definition 4.1.5

The *Thompson monoid* *is defined by the presentation*

$$\mathsf{ThMon} := \langle 1, 2, \ldots \mid i \cdot j = j \cdot (i+1) \text{ for all } i > j \rangle.$$

The group of fractions of this monoid is the famous Thompson group F, although we do not know any connection between the work in this section and the vast literature on this famous group. We show[4] in **[3]** that we have an isomorphism of monoids

$$\mathsf{ThMon} \cong \mathsf{Forest} \tag{4.1.6}$$

under the map sending j to the indexed forest $\underline{j}$, whose only nontrivial tree is T_j of size 1. Because of the relation (**4.1.5**), we can then define composite operators

$$\mathsf{T}_F := \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_k} \text{ for any factorization } F = \underline{i_1} \cdots \underline{i_k}. \tag{4.1.7}$$

We now give a few definitions. First, we have the *left terminal set* $\mathrm{LTer}(F)$ which is the analogue of the descent set of a permutation. We say that $i \in \mathrm{LTer}(F)$ if there is a terminal node with left child i. This happens if and only if we can write $F = (F/i) \cdot i$ for a (necessarily unique) forest F/i, and so

$$\mathrm{LTer}(F) = \{i \mid F/i \text{ exists}\}.$$

The set of *trimming sequences* for F is defined as

$$\mathrm{Trim}(F) := \{(i_1, \ldots, i_{|F|}) \mid F = \underline{i_1} \cdots \underline{i_{|F|}}\}.$$

Recall the code map $F \mapsto \mathrm{c}(F) = (a_1, a_2, \ldots)$ where a_j is the number of internal nodes whose leftmost leaf descendant is j. Via this map, $i \in \mathrm{LTer}(F)$ is equivalent to $a_i > 0$ and $a_{i+1} = 0$, and in this case $\mathrm{c}(F/i) = (a_1, \ldots, a_{i-1}, a_i - 1, 0, a_{i+1}, \ldots)$

4.1.3 Forest polynomials revisited

We have defined and used forest polynomials in Section 2.3.3 mainly as an *ad hoc* tool to prove our combinatorial interpretation for a_w. The following fundamental result shows that they have a very different *raison d'être*:

[4] This isomorphism seems to be a folklore fact in the theory of Thompson groups, cf. [BB05] for instance, although the language is slightly different.

Theorem 4.1.6

The family $(\mathsf{P}_F)_F$ with $F \in \mathsf{Forest}$ is the unique family of homogeneous polynomials satisfying $\mathsf{P}_\varnothing = 1$ and, for any $i > 0$ and $F \in \mathsf{Forest}$,

$$\mathsf{T}_i\mathsf{P}_F = \begin{cases} \mathsf{P}_{F/i} & \text{if } i \in \mathrm{LTer}(F); \\ 0 & \text{otherwise.} \end{cases} \quad (4.1.8)$$

The relations in the theorem are not hard to show, following the combinatorial Definition 2.3.5. Uniqueness proceeds in the same way as for Schubert polynomials in Theorem 1.4.1.

From this theorem we can get several very nice properties of forest polynomials, mimicking the one from Theorem 1.4.2. Let Forest_n be the set[5] of forests in Forest such that all leaves of nontrivial trees are $\leq n$. This plays the role that S_n plays for Schubert polynomials. Also, recall that the set ABB_n was defined in Section 2.3.5.

Theorem 4.1.7

Let $B \subset \mathsf{Forest}$. The forest polynomials $(\mathsf{P}_F)_{F\in B}$ form a basis of

a) Pol when $B = \mathsf{Forest}$.
b) Pol_n when B consists of forests F with $\mathrm{LTer}(F) \subset \{1, \dots, n\}$.
c) QSym_n^+ when B consists of forests F with $\mathrm{LTer}(F) \subset \{1, \dots, n\}$ and $F \notin \mathsf{Forest}_n$.
d) The space spanned by monomials x^c with $c \in \mathsf{ABB}_n$ when $B = \mathsf{Forest}_n$, as well as $\mathrm{QSCoinv}_n := \mathrm{Pol}_n / \mathrm{QSym}_n^+$.

We omit the proof and refer to **[3]**; let us simply note that only $c)$ requires some nontrivial work. We also invite the reader to compare this result with Theorem 1.4.2. We refer to Table A.2 for a side-by-side comparison.

4.2 Applications

4.2.1 Extracting coefficients

As a corollary of Theorem 4.1.6, by iteration, one has that every $f \in \mathrm{Pol}$ can be uniquely written as

$$f = \sum_{F \in \mathsf{Forest}} a_F \mathsf{P}_F \text{ where } a_F = \mathsf{T}_F(f)(0, 0, \dots). \quad (4.2.1)$$

In other words, the linear forms $f \to \mathsf{T}_F(f)(0, 0, \dots)$ are "dual" to forest polynomials. This is interesting in particular in the quasisymmetric case below, as no formula for such coefficient extractions seems to have been known.

The set ZigZag_n of forests F with $\mathrm{LTer}(F) \subset \{n\}$ play an analogous role to the n-Grassmannian permutations. Recall the set QSeq_n of slowly decreasing integer sequences from Section 1.4, where we showed that $\mathfrak{F}(a)$ for $a \in \mathrm{QSeq}_n$ are the fundamental quasisymmetric polynomials in QSym_n.

The mapping $(a_1, \dots, a_k) \mapsto F = a_k \cdots a_1$ is easily seen to be a bijection $\mathrm{QSeq}_n \to \mathsf{ZigZag}_n$, and we have:

Theorem 4.2.1

The forest polynomials with $F \in \mathsf{ZigZag}_n$ are a basis for QSym_n. They coincide with fundamental quasisymmetric polynomials: under the bijection above, we have $\mathsf{P}_F = \mathfrak{F}(a)$.

Suppose we want to decompose the quasisymmetric polynomial $f(x_1, x_2, x_3) = 2x_1^2x_2 + 2x_1^2x_3 + 2x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \in \mathrm{QSym}_3$ into fundamental quasisymmetric polynomials. We track in Figure 4.2 the nonzero applications $\mathsf{T}_{i_3}\mathsf{T}_{i_2}\mathsf{T}_{i_1} f$ where $(i_1, i_2, i_3) \in \mathrm{QSeq}_3$, and read off

$$f = \mathfrak{F}(332) + 2\mathfrak{F}(322) - 3\mathfrak{F}(321) = F_{3,\{2\}} + 2F_{3,\{1\}} - 3F_{3,\{1,2\}}$$

[5] In terms of the earlier definition, this means simply that the support of F is included in $\{1, \dots, n-1\}$.

using (**4.2.1**).

$$
\begin{tikzcd}
& & x_1+x_2 \arrow[r, "\mathsf{T}_2"] & 1 \\
f \arrow[r, "\mathsf{T}_3"] & 2x_1^2+2x_2^2+x_1x_3+x_2x_3 \arrow[ur, "\mathsf{T}_3"] \arrow[dr, "\mathsf{T}_2"'] & & 2 \\
& & -x_1+2x_2 \arrow[ur, "\mathsf{T}_2"] \arrow[dr, "\mathsf{T}_1"'] & \\
& & & -3
\end{tikzcd}
$$

Figure 4.2. Trimming $f \in \mathrm{QSym}_3$

4.2.2 Positive forest decompositions

Recall that for Schubert polynomials, it is wide open to establish the nonnegativity of the generalized Littlewood-Richard coefficients arising in the expansion

$$\mathfrak{S}_u\mathfrak{S}_w = \sum_{v\in S_n} c_{u,w}^v \mathfrak{S}_v$$

in a combinatorial manner. A major hurdle is that the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + (s_i \cdot f)\partial_i(g)$$

involves the non-Schubert positive operator $f \mapsto s_i \cdot f$.

For forest polynomials, things are much nicer. The main tool is the following immediate lemma:

Lemma 4.2.2 (*Twisted Leibniz rule*)

For $f, g \in \mathrm{Pol}$ we have

$$\mathsf{T}_i(fg) = \mathsf{T}_i(f)\mathsf{R}_{i+1}(g) + \mathsf{R}_i(f)\mathsf{T}_i(g).$$

There also simple straightening rules for products and T_i's and R_j's, from which one can obtain for instance that in the expansions

$$\mathsf{R}_i\mathsf{P}_F = \sum b_{i,F}^G \mathsf{P}_G,$$

the coefficients $b_{i,F}^G$ are in $\mathbb{N}$.

From these one can obtain for instance the positivity of structure coefficients of forest polynomials in an algorithmic manner.

Theorem 4.2.3

For any F, G we have

$$\mathsf{P}_F\mathsf{P}_G = \sum c_{F,G}^H \mathsf{P}_H \text{ with } c_{F,G}^H \geq 0.$$

The proof goes by applying T_H to the left hand side, using the twisted Leibniz rule above, and proceed by induction using the aforementioned straightening rules. With the same technique one can prove

$$\mathfrak{S}_w = \sum_{F\in\mathsf{Forest}} a_F \mathsf{P}_F \text{ with } a_F \geq 0.$$

The reader may have noticed that we know this last fact already from the combinatorics of the forest correspondence $\Omega^\bullet$, see (**2.3.5**). In fact Theorem 4.2.3 can also be proved with this approach, see Theorem 6.3 in **[3]**. The proofs sketched in this section are arguably more straightforward.

4.2.3 Connection with the main problem

Results from this section can be found in Section 9 of **[1]**. We start with the following factorization: for any $f \in \mathrm{Pol}_n$ of degree $n-1$.

$$\langle f \rangle_n = \mathsf{T}_1(\mathsf{T}_1 + \mathsf{T}_2) \cdots (\mathsf{T}_1 + \cdots + \mathsf{T}_{n-1}) f. \tag{4.2.2}$$

The proof is by induction. We can rewrite it as

$$\langle f \rangle_n = \sum_{\substack{(i_1,\ldots,i_{n-1}) \\ 1 \leq i_j \leq j}} \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_{n-1}} f. \tag{4.2.3}$$

The subset of sequences $(i_1, \ldots, i_{n-1})$ subject to $i_j \leq j$ is precisely the set of trimming sequences for indexed forests F with $|F| = n-1$ all of whose nontrivial leaves are in $\{1, \ldots, n\}$. It then follows that for any $f \in \mathrm{Pol}_n$ of degree $n-1$

$$\langle f \rangle_n = \sum_{\substack{F \in \mathsf{Forest}_n \\ |F| = n-1}} |\operatorname{Trim}(F)| \, \mathsf{T}_F f. \tag{4.2.4}$$

From Theorem 4.1.7, we know that all T_F occurring here vanish on QSym_n^+. This gives therefore a new proof of Proposition 2.3.3.

Let us now pick $w \in S'_n$. Applying (**4.2.3**) to $\mathfrak{S}_w$, we get:

$$a_w = \sum_{\substack{(i_1,\ldots,i_{n-1}) \\ 1 \leq i_j \leq j}} \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_{n-1}} \mathfrak{S}_w. \tag{4.2.5}$$

Now we know $a_w > 0$ from Corollary 2.2.2 as a_w is a sum of positive rational numbers, and we have also a combinatorial interpretation of a_w showing that $a_w \in \mathbb{N}$. Unfortunately strict positivity seems hard to extract from this interpretation.

Theorem 9.5 of **[1]** gives a combinatorial proof of this positivity:

Theorem 4.2.4

*Let $w \in S'_n$. All terms on the right-hand side of (**4.2.5**) are in $\mathbb{N}$, and one can construct an explicit sequence $(i_1, \ldots, i_{n-1})$ for which it is in $\mathbb{Z}_{>0}$. It follows that $a_w \in \mathbb{Z}_{>0}$.*

To conclude, let us remark for historical purposes that the factorization (**4.2.2**) above is in fact at the origin of the introduction of the operators T_i, and thus of all the work from this chapter.

4.3 Schubert and forest polynomials under the same roof

As we saw, Schubert and forest polynomials are obtained in a common way, and the parallel is highlighted in Table A.2. Briefly speaking, we have in each case operators $(X_i)_{i>0}$ on Pol which, under composition, give a representation of a certain (partial) monoid M. Then in each case we had the existence and uniqueness of a family of polynomials indexed by M that satisfy a "duality" statement, namely Theorems 1.4.1 and 4.1.6. In each case, we have also nice "basis" properties of the corresponding polynomials, cf. Theorem 1.4.2 and 4.1.7.

Let us give another elementary example, with the usual partial derivative operators $\frac{d}{dx_i}$. They commute, and thus generate a monoid of operators isomorphic to Codes under componentwise addition. In this case the family of normalized monomials

$$\{S_c = \frac{\mathbf{x}^c}{c!} := \frac{x_1^{c_1} x_2^{c_2} \cdots}{c_1! c_2! \cdots} \mid c = (c_1, c_2, \ldots) \in \mathsf{Codes}\}$$

is "dual", in the sense that it satisfies $S_\varnothing = 1$ and

$$\frac{d}{dx_i} S_c = \begin{cases} S_{c-e_i} & \text{if } c_i \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.1}$$

Here $c - e_i = (c_1, \ldots, c_{i-1}, c_i - 1, c_{i+1}, \ldots)$.

We want to understand this phenomenon more generally. This is what motivates the following setting, which is taken from **[2]**.

General Setting. We fix a linear operator X: Pol $\to$ Pol of degree -1, i.e. X takes degree d homogeneous polynomials to degree $d-1$ homogeneous polynomials for all d. For any $i \geq 1$, we define the shifted operator $X_i : \mathrm{Pol} \to \mathrm{Pol}$ by the composition

$$X_i : \mathrm{Pol} \cong \mathrm{Pol}_{i-1} \otimes \mathrm{Pol} \to \mathrm{Pol}_{i-1} \otimes \mathrm{Pol} \cong \mathrm{Pol}$$

where the first and last isomorphisms are given by the isomorphism

$$\mathrm{Pol}_{i-1} \otimes \mathrm{Pol} = \mathbb{Q}[x_1, \ldots, x_{i-1}] \otimes \mathbb{Q}[x_i, x_{i+1}, \ldots] \cong \mathrm{Pol},$$

and the middle map is given by $\mathrm{id} \otimes X$. Thus in particular $X = X_1$, and $f \in \mathrm{Pol}_n$ implies that $X_{n+1} f = X_{n+2} f = \cdots = 0$. All three examples $\partial_i, \mathsf{T}_i, \frac{d}{dx_i}$ fit in this setting.

Let M be a partial monoid. We assume that M is a graded partial monoid, generated in degree 1 by $\{a_i\}_{i \geq 1}$, so that the length function $\ell(w) = k$ when $w = a_{i_1} \cdots a_{i_k}$ is well-defined. We let $\mathrm{Fac}(w)$ be the set of factorizations $w = a_{i_1} \cdots a_{i_k}$ and $\mathrm{Last}(w)$ be the set of letters that can occur as the last letter in a factorization. We also assume that M is right-cancellative: if $wa = w'a$ in M for any w, w', a then $w = w'$. Thus if $a \in \mathrm{Last}(w)$ the element $u = w/a$ is well-defined as $w = ua$.

Definition 4.3.1

We define a divided difference pair (or a dd-pair) to be the data of (X, M) as above, such that the map $a_i \mapsto X_i$ is a representation of M in the monoid of linear endomorphisms of Pol.

For $w \in M$ we write X_w for the associated endomorphism of Pol, and in particular we have $X_i = X_{a_i}$. One can now state then the notion of dual family of polynomials:

Definition 4.3.2

A family $(S_w)_{w \in M}$ of homogeneous polynomials in Pol *is dual to a dd-pair (X, M) if $S_1 = 1$, and for each $w \in M$ and $i \in \{1, 2, \ldots\}$ we have*

$$X_i S_w = \begin{cases} S_{w/i} & \text{if } i \in \mathrm{Last}(w) \\ 0 & \text{otherwise.} \end{cases}$$

The main questions we ask are: Does such a family exist ? Is it then unique ? Can we find a formula for it ? We want to have positive answers for all of them, as in our examples. This is what motivates the next definitions; detailed argumentation for their introduction can be found in **[2]**.

We define *creation operators* for the operator X to be a collection of degree 1 polynomial endomorphisms $Y_i : \mathrm{Pol} \to \mathrm{Pol}$ such that on the ideal $\mathrm{Pol}^+ \subset \mathrm{Pol}$ of f with $f(0, 0, \ldots) = 0$, we have the identity

$$\sum_{i=1}^{\infty} Y_i X_i = \mathrm{id}. \tag{4.3.2}$$

A *code map* for M is an injective map $c : M \to$ Codes such that $\ell(w) = |c(w)|$ and $\max \mathrm{supp}\, c(w) = \max\{i \mid a_i \in \mathrm{Last}(w)\}$ for all $w \in M$. Given $c \in$ Codes, let $\max \mathrm{supp}\, c$ be the largest i such that $c_i > 0$ (set $\max \mathrm{supp}\, c = 0$ if no scugh i).

Theorem 4.3.3

Suppose that a dd-pair (X, M) has creation operators Y_i and a code map. Then
(1) The code map is bijective.
(2) There is a unique dual family $(S_w)_{w\in M}$ defined by

$$S_w = \sum_{(i_1,\dots,i_k)\in\mathrm{Fac}(w)} Y_{i_k}\cdots Y_{i_1}(1). \tag{4.3.3}$$

(3) The subfamily $(S_w)_w$ where $\max\operatorname{supp} c(w) \le d$ is a basis of Pol_d for any $d \ge 0$.

Applications. Code maps exist for all of our examples. What about creation operators ?

Let us deal first with the case of partial derivatives: then one can choose $Y_i(f) = \frac{1}{k+1}x_i f$ where f has degree $k > 0$, and extend by linearity. Then (4.3.2) holds as it is Euler's famous theorem $\sum x_i \frac{d}{dx_i} = k\,\mathrm{id}$ on homogeneous polynomials of positive degree k. Then a direct computation show that (4.3.3) gives the normalized monomials as expected.

For Schubert polynomials, using $x_i\mathsf{R}_i\partial_i = \mathsf{R}_{i+1} - \mathsf{R}_i$ and telescoping gives

$$\sum_{i\ge1} x_i\mathsf{R}_i\partial_i = \mathrm{id} - \mathsf{R}_1. \tag{4.3.4}$$

Define $\mathsf{Z} = \mathrm{id} + \mathsf{R}_1 + \mathsf{R}_1^2 + \cdots : \mathrm{Pol}^+ \to \mathrm{Pol}^+$.. Then $\sum_{i\ge1}\mathsf{Z}x_i\mathsf{R}_i\partial_i = \mathrm{id}$ and thus the $\mathsf{Z}x_i\mathsf{R}_i$ are creation operators for the dd-pair given by ∂ and the nil-Coxeter monoid.

It follows from Theorem 4.3.3 that Schubert polynomials exist as a dual basis, and have the expansion[6]:

$$\mathfrak{S}_w = \sum_{(i_1,\dots,i_k)\in\mathrm{Red}(w)} \mathsf{Z}x_{i_k}\mathsf{R}_{i_k}\cdots\mathsf{Z}x_{i_1}\mathsf{R}_{i_1}(1). \tag{4.3.5}$$

Corollary 4.3.4

Schubert polynomials have positive coefficients.

Indeed Equation (4.3.5) is clearly positive, as the operators R_i send monomials to monomials or to zero. This is arguably the simplest proof of positivity for Schubert polynomials.

We can also relate this expansion to the pipe dreams from Section 1.4: applying (4.3.4) directly to Schubert polynomials, we get the recursion

$$\mathfrak{S}_w = \mathsf{R}_1\mathfrak{S}_w + \sum_{i\in\mathrm{Des}(w)} x_i\mathsf{R}_i\mathfrak{S}_{ws_i}.$$

Using the pipe dream expansion (1.4.4) for each Schubert polynomial hints at a new decomposition for these objects, which is proved bijectively as Theorem 3.5 in **[2]**: the first term above corresponds to pipe dreams of w without crosses in their first column, while the sum can be understood via a simple transformation

For forest polynomials, recall that $\mathsf{T}_i = \mathsf{R}_i\partial_i$, and so $\sum_{i\ge1} x_i\mathsf{T}_i = \mathrm{id} - \mathsf{R}_1$ from (4.3.4). It follows that the operators $\mathsf{Z}x_i$ are creation operators for the T_i, and we get from Theorem 4.3.3 that forest polynomials exist as a dual basis, and have the following expansion:

$$\mathsf{P}_F = \sum_{(i_1,\dots,i_k)\in\mathrm{Trim}(F)} \mathsf{Z}x_{i_k}\cdots\mathsf{Z}x_{i_1}(1).$$

In that case, the theorem gets us immediately the existence of forest polynomials without having to come up with Definition 2.3.5.

To conclude, we give in Theorem 5.3 of **[2]** another application of Theorem 4.3.3: we show that slide polynomials, which were ubiquitous in the work presented here, occur also naturally as a dual basis for a certain dd-pair .

[6] This is different from the BJS expansion (1.4.3)

Appendix A

Tables

$\mathrm{c}(F)$	P_F
$(0,0,0,0,0)$	1
$(1,0,0,0,0)$	x_1
$(0,1,0,0,0)$	x_1+x_2
$(0,0,1,0,0)$	$x_1+x_2+x_3$
$(0,0,0,1,0)$	$x_1+x_2+x_3+x_4$
$(2,0,0,0,0)$	x_1^2
$(1,1,0,0,0)$	x_1x_2
$(1,0,1,0,0)$	$x_1^2+x_1x_2+x_1x_3$
$(1,0,0,1,0)$	$x_1^2+x_1x_2+x_1x_3+x_1x_4$
$(0,2,0,0,0)$	$x_1^2+x_1x_2+x_2^2$
$(0,1,1,0,0)$	$x_1x_2+x_1x_3+x_2x_3$
$(0,1,0,1,0)$	$x_1^2+2x_1x_2+x_2^2+x_1x_3+x_2x_3+x_1x_4+x_2x_4$
$(0,0,2,0,0)$	$x_1^2+x_1x_2+x_2^2+x_1x_3+x_2x_3+x_3^2$
$(0,0,1,1,0)$	$x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4$
$(3,0,0,0,0)$	x_1^3
$(2,1,0,0,0)$	$x_1^2x_2$
$(2,0,1,0,0)$	$x_1^2x_2+x_1^2x_3$
$(2,0,0,1,0)$	$x_1^3+x_1^2x_2+x_1^2x_3+x_1^2x_4$
$(1,2,0,0,0)$	$x_1x_2^2$
$(1,1,1,0,0)$	$x_1x_2x_3$
$(1,1,0,1,0)$	$x_1^2x_2+x_1x_2^2+x_1x_2x_3+x_1x_2x_4$
$(1,0,2,0,0)$	$x_1^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_1x_3^2$
$(1,0,1,1,0)$	$x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_1^2x_4+x_1x_2x_4+x_1x_3x_4$
$(0,3,0,0,0)$	$x_1^3+x_1^2x_2+x_1x_2^2+x_2^3$
$(0,2,1,0,0)$	$x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_2^2x_3$
$(0,2,0,1,0)$	$x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1^2x_4+x_1x_2x_4+x_2^2x_4$
$(0,1,2,0,0)$	$x_1x_2^2+x_1x_2x_3+x_1x_3^2+x_2x_3^2$
$(0,1,1,1,0)$	$x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4$
$(4,0,0,0,0)$	x_1^4
$(3,1,0,0,0)$	$x_1^3x_2$
$(3,0,1,0,0)$	$x_1^3x_2+x_1^3x_3$
$(3,0,0,1,0)$	$x_1^3x_2+x_1^3x_3+x_1^3x_4$
$(2,2,0,0,0)$	$x_1^2x_2^2$
$(2,1,1,0,0)$	$x_1^2x_2x_3$
$(2,1,0,1,0)$	$x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_2x_4$
$(2,0,2,0,0)$	$x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_3^2$
$(2,0,1,1,0)$	$x_1^2x_2x_3+x_1^2x_2x_4+x_1^2x_3x_4$
$(1,3,0,0,0)$	$x_1x_2^3$
$(1,2,1,0,0)$	$x_1x_2^2x_3$
$(1,2,0,1,0)$	$x_1x_2^2x_3+x_1x_2^2x_4$
$(1,1,2,0,0)$	$x_1x_2x_3^2$
$(1,1,1,1,0)$	$x_1x_2x_3x_4$

Table A.1. Table of forest polynomials P_F.

	QSym_n	Sym_n
Divided differences	T_i	∂_i
Indexing combinatorics	$F \in$ Forest	$w \in S_\infty$
	Fully supported forests Forest$_n$	S_n
	Forest code $\mathrm{c}(F)$	Lehmer code $\mathrm{lcode}(w)$
	Left terminal set $\mathrm{LTer}(F)$	Descent set $\mathrm{Des}(w)$
	F/i for $i \in \mathrm{LTer}(F)$	ws_i for $i \in \mathrm{Des}(w)$
	Trimming sequences $\mathrm{Trim}(F)$	Reduced words w
	Zigzag forests $Z \in$ ZigZag$_n$	Grassmannian permutations λ
Monoid	Thompson monoid	nilCoxeter monoid
Pol-**basis**	Forest polynomials P_F	Schuberts $\mathfrak{S}_w$
Composites	$\mathsf{T}_F = \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_k}$ for $\mathbf{i} \in \mathrm{Trim}(F)$	$\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ for $\mathbf{i} \in w$
Pol_n-**basis**	$\{\mathsf{P}_F \mid \mathrm{LTer}(F) \subset [n]\}$	$\{\mathfrak{S}_w \mid \mathrm{Des}(w) \subset [n]\}$
Duality	$\mathrm{ev}_0\, \mathsf{T}_F \mathsf{P}_G = \delta_{F,G}$	$\mathrm{ev}_0\, \partial_w \mathfrak{S}_{w'} = \delta_{w,w'}$
Positive expansions	$\mathsf{P}_F \mathsf{P}_H = \sum c_{F,H}^G \mathsf{P}_G,\ c_{F,H}^G \geq 0$	$\mathfrak{S}_u \mathfrak{S}_w = \sum c_{u,w}^v \mathfrak{S}_v,\ c_{u,w}^v \geq 0$
Invariant basis	Fundamentals $F_{m,S}(\mathbf{x}_n)$	Schur polynomials $s_\lambda(\mathbf{x}_n)$
Coinvariant basis	$\{\mathsf{P}_F \mid F \in \text{Forest}_n\}$	$\{\mathfrak{S}_w \mid w \in S_n\}$

Table A.2. Comparing the symmetric and quasisymmetric stories; here $\mathrm{ev}_0(f) = f(0, 0, \ldots)$

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Combinatoire algébrique autour d'un problème de géométrie énumérative

Résumé: Les travaux présentés dans ce mémoire d'HDR s'inscrivent dans le domaine de la combinatoire algébrique et consistent en plusieurs contributions originales. Ils ont été motivés par une question initiale de géométrie énumérative, à savoir calculer explicitement le nombre de points d'intersection a_w entre la variété permutaédrale et une sous-variété de Schubert X_w de la variété des drapeaux.

La première partie est consacrée à la présentation du problème initial ainsi qu'à la description des techniques classiques de cohomologie permettant d'énoncer deux formules distinctes pour a_w. La première formule, sur les travaux de Klyachko, permet d'établir la stricte positivité des nombres a_w et de mettre en évidence diverses propriétés structurelles de ces derniers. La seconde formule, obtenue au terme d'un développement algébrique et combinatoire substantiel, fournit une interprétation combinatoire explicite de ces mêmes nombres.

La deuxième partie rassemble plusieurs contributions développées au cours de l'étude du problème original. On y introduit notamment des q-analogues des quantités apparaissant dans la première formule, les polynômes eulériens mixtes, qui constituent une famille riche englobant de nombreuses suites combinatoires classiques. Les propriétés combinatoires et algébriques de cette q-déformation font l'objet d'une étude approfondie. Par ailleurs, une variante de la théorie des P-partitions est développée, en lien avec la seconde formule, ainsi qu'une théorie des fonctions de parking bilatères, qui permet d'éclairer les aspects combinatoires de cette dernière.

La troisième et dernière partie est consacrée aux polynômes quasisymétriques, une famille qui joue un rôle central dans la démonstration de la seconde formule pour les nombres a_w. Une nouvelle approche fondée sur des opérateurs est proposée, permettant de simplifier et d'étendre les méthodes classiques d'étude de ces polynômes. Cette construction est directement inspirée de celle des polynômes symétriques, les polynômes forêts jouant ici un rôle analogue à celui des polynômes de Schubert.

Algebraic combinatorics around a problem in enumerative geometry

Abstract: The research presented in this habilitation thesis falls within the field of algebraic combinatorics, and consists of several original contributions. It was initially motivated by an enumerative geometry problem: namely, the explicit computation of the number of intersection points a_w between the permutahedral variety and a Schubert subvariety X_w within the flag variety.

The first part introduces the original problem and describes classical cohomological techniques that allow for the derivation of two distinct formulas for a_w. The first formula, based on Klyachko's work, establishes the strict positivity of the numbers a_w and highlights several of their structural properties. The second formula, obtained through substantial algebraic and combinatorial developments, provides an explicit combinatorial interpretation of these numbers.

The second part gathers various contributions that emerged during the resolution of the initial problem. In particular, q-analogues of the quantities appearing in the first formula, named remixed Eulerian numbers, are introduced and studied. This rich family encompasses many classical combinatorial sequences, and the combinatorial and algebraic aspects of this q-deformation are studied in depth. Additionally, in connection with the second formula, a variant of the theory of P-partitions is developed, along with a theory of bilateral parking functions which sheds light on certain combinatorial aspects of the formula.

The third and final part focuses on quasisymmetric polynomials, an important family playing a central role in the derivation of the second formula for a_w. A new operator-based approach is introduced, simplifying and extending classical methods for studying these polynomials. This construction is directly inspired by the theory of symmetric polynomials, with forest polynomials playing a role analogous to that of Schubert polynomials.

Image en couverture : Une dissection du permutaèdre en cubes.

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