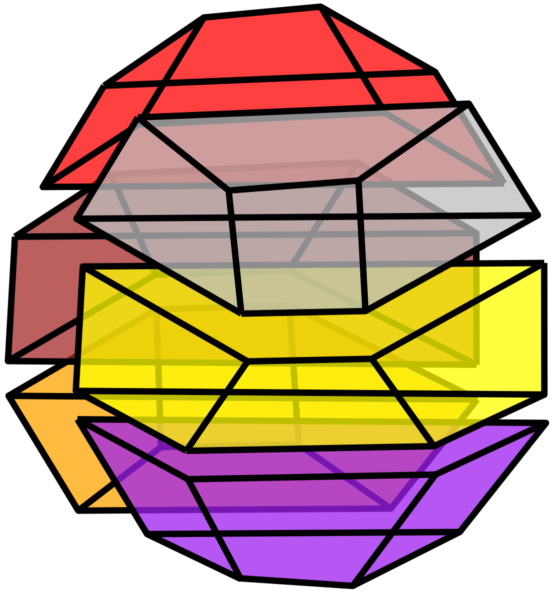
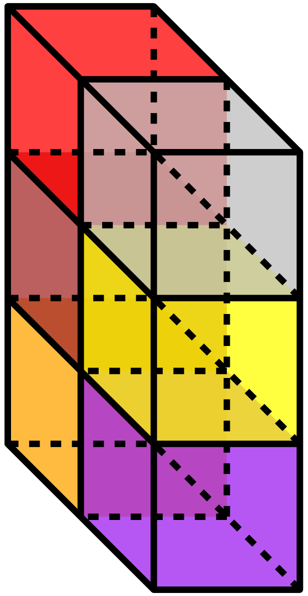
Algebraic combinatorics
around a problem in enumerative geometry





Philippe Nadeau

Soutenance HDR, ICJ, 5 mai 2025

Outline

I. Quasisymmetric polynomials revisited

We first recall the connection between symmetric and Schubert polynomials, via divided differences and permutations.

We present an analogue theory for quasisymmetric polynomials.

We have new trimming operators T_i. The combinatorics is given by certain forests, and we get a dual basis of forest polynomials.

II. The original problem in enumerative geometry

Let w be a permutation in S_n.

$a_w =$ number of points in the intersection of the permutahedral variety Perm_n with a generic Schubert variety X^w.

Manifestly positive rules for the coefficients a_w

III. Extra combinatorics (parking procedures)

I. Quasisymmetric polynomials revisited

Symmetric polynomials and divided differences

Polynomials Fix $n \geq 1$, and define $\mathrm{Pol}_n := \mathbb{Q}[x_1, \dots, x_n]$.

Let also $\mathrm{Pol} := \mathbb{Q}[x_1, x_2, \dots]$ which we see as $\mathrm{Pol} = \cup_n \mathrm{Pol}_n$.

Let S_n be the group of permutations of $\{1, \dots, n\}$, generated by $s_i = (i, i+1)$ for $i < n$.

Let also $S_\infty = \bigcup_n S_n$

$$= \{ \text{ Permutations } w \text{ of } \mathbb{Z}_{>0} = \{1, 2, \dots\} \text{ such that } w(i) = i \text{ for } i >> 0.\}.$$

S_n acts on Pol_n, and $f \in \mathrm{Pol}_n$ is called symmetric ($f \in \mathrm{Sym}_n$) if fixed by the action.

Example for $n = 3$, elementary symmetric polynomials

$e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3$.

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Operators Define the *divided difference* $\partial_i = \dfrac{\mathrm{id} - s_i}{x_i - x_{i+1}}$ on Pol_n and Pol.

Then $f \in \mathrm{Pol}_n$ is symmetric if and only if $\partial_i f = 0$ for $i = 1, \ldots, n-1$.

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Monoid We now look at the monoid of all composites $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k}$ in the ring of endomorphisms of Pol.

It is isomorphic to the *nilCoxeter monoid*, with underlying set S_∞ and multiplication $w \cdot w' = ww'$ if $\ell(w) + \ell(w') = \ell(ww')$, and 0 otherwise.

$$\Rightarrow \text{Composite operators } \partial_w := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{k-1}} \partial_{i_k} \text{ when } s_{i_1} \cdot s_{i_2} \cdots s_{i_{k-1}} \cdot s_{i_k} = w.$$

Schubert polynomials

Definition-Theorem. The Schubert polynomials $\mathfrak{S}_w$ for $w \in S_\infty$, are the unique family of homogeneous polynomials in Pol such that $\mathfrak{S}_{\text{id}} = 1$ and

(*i* is a descent)

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } w(i) > w(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof Sketch: Pick n such that $w \in S_n$, define $\mathfrak{S}_w = \partial_{w^{-1}w_o}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1)$, and check that this does not depend on n. $\square$

Ex ($w \in S_3$)

$$\mathfrak{S}_{123} = 1 \qquad \mathfrak{S}_{213} = x_1 \qquad \mathfrak{S}_{132} = x_1 + x_2 \qquad \mathfrak{S}_{321} = x_1^2 x_2 \qquad \mathfrak{S}_{312} = x_1^2 \qquad \mathfrak{S}_{231} = x_1 x_2$$

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Corollary (Duality). For any $w, w' \in S_\infty$,

$$\text{Constant term of } \partial_w(\mathfrak{S}_{w'}) = \begin{cases} 1 & \text{if } w = w' \\ 0 & \text{otherwise.} \end{cases}$$

Origin: $\mathfrak{S}_w$ encodes the cohomology class of the Schubert subvariety $X_{w_o w}$ inside the full flag variety. [Lascoux-Schützenberger '82]

Schubert polynomials

$\to$ The $\mathfrak{S}_w$ form **nice bases of various spaces:**

- $\mathfrak{S}_w$ is symmetric in $x_1, \dots, x_n \Leftrightarrow w$ has a unique descent at $i = n$.

Proposition. In that case $\mathfrak{S}_w = s_\lambda(x_1, \dots, x_n)$ (a Schur polynomial).

- The $\mathfrak{S}_w$ with $w \in S_\infty$ form an integral basis of Pol.

- Let $\mathrm{Sym}_n^+ \subset \mathrm{Pol}_n$ be the ideal generated by the $f \in \mathrm{Sym}_n$ with $f(0) = 0$.

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$\rightarrow$ **Positivity questions**

- From their definition, not clear that $\mathfrak{S}_w$ has positive coefficients.
  Needs extra work $\Rightarrow$ Combinatorial interpretation as pipe dreams.

- This approach says little about the known positivity of the coefficients c_{uv}^w:

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w.$$

Quasisymmetric polynomials

Definition. Let $f \in \mathrm{Pol}_n$. Then f is quasisymmetric if for all $a_1, \dots, a_k > 0$, for all $i_1, \dots, i_k$ such that $1 \le i_1 < \dots < i_k \le n$, Coeff of $x_1^{a_1} \cdots x_k^{a_k}$ = Coeff of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ in f.

$$\text{For } n = 3,\ f = \underbrace{4x_1^2x_2 + 4x_1^2x_3 + 4x_2^2x_3}_{a_1, a_2 = 2, 1} \underbrace{- 3x_1 - 3x_2 - 3x_3}_{a_1 = 1} + \underbrace{2x_1x_2x_3^2}_{a_1, a_2, a_3 = 1, 1, 2}.$$

We let QSym_n be the space of these polynomials, graded by degree.

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Motivation(s)

- Introduced in Stanley's thesis (1970), explicitly identified by Gessel (1984). They are the natural setting for certain generating functions for posets.
- Relation to symmetric polynomials: create bases that refine symmetric bases, expand (quasi)symmetric polynomials in these bases,...
- Terminal object in the category of combinatorial Hopf algebras.

(More precisely this holds for quasisymmetric *functions*, which are the power series limits of these polynomials)

Trimming operators

We define operators that "detect quasisymmetry".

Definition. For $f \in \mathrm{Pol}_n$ and $i < n$, define

$$\mathrm{R}_i(f(x_1, \dots, x_n)) := f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})$$

This is an *algebra morphism* $\mathrm{Pol}_n \to \mathrm{Pol}_{n-1}$.

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Lemma. $f \in \mathrm{QSym}_n$ if and only if $\mathrm{R}_1(f) = \mathrm{R}_2(f) = \cdots = \mathrm{R}_n(f)$.

This characterization is related to (Hivert, 2000).

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$\to$ “Trimming” operators T_i.

Definition. For $f \in \mathrm{Pol}_n$ and $i < n$,

$$\mathrm{T}_i := \frac{\mathrm{R}_{i+1} - \mathrm{R}_i}{x_i}.$$

$\Rightarrow f \in \mathrm{QSym}_n$ if and only if $\mathrm{T}_1 f = \mathrm{T}_2 f = \cdots = \mathrm{T}_{n-1} f = 0$.

Trimming operators

Explicitly,

$$\mathrm{T}_i(f) = \frac{f(x_1, \dots, x_{i-1}, x_i, 0, x_{i+1}, \dots, x_{n-1}) - f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})}{x_i}$$

Note that T_i(monomial of degree d) $= \pm$ a monomial of degree $d - 1$ (or zero).

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Let $n \to \infty$ and consider the T_i as operators on Pol.

The T_i satisfy the relations of the Thompson monoid.

(The group given by this presentation is Thompson group F)

$$\mathrm{T}_i \mathrm{T}_j = \mathrm{T}_j \mathrm{T}_{i+1} \text{ if } i > j.$$

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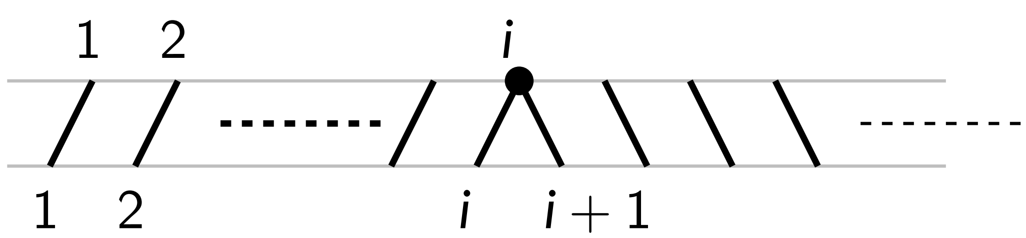
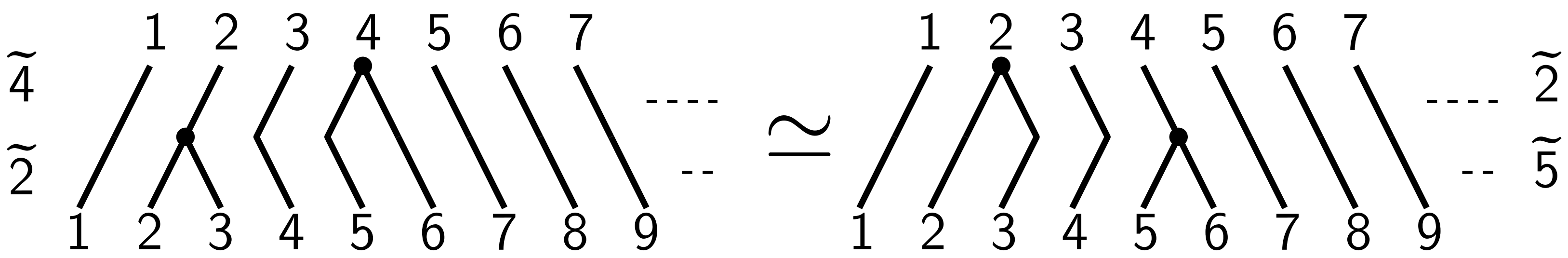
To study the combinatorics, associate to T_i the elementary diagram $\tilde{i}$:

$\tilde{i} =$ 

Monoid elements as forests



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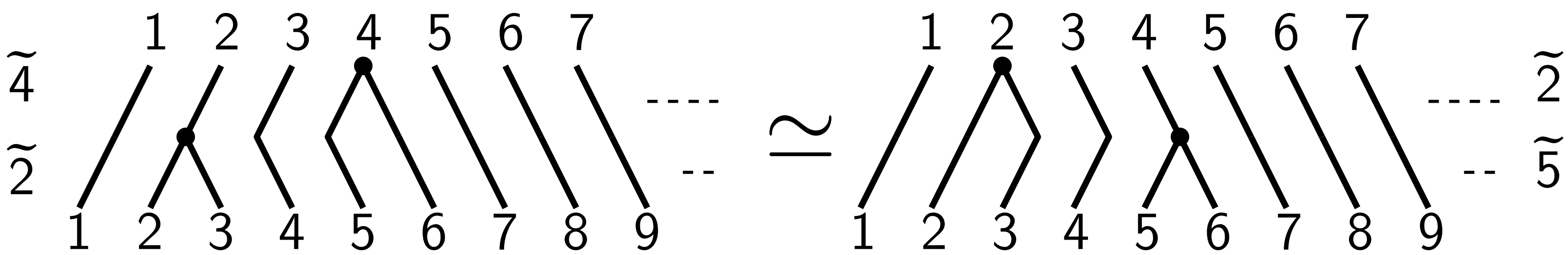
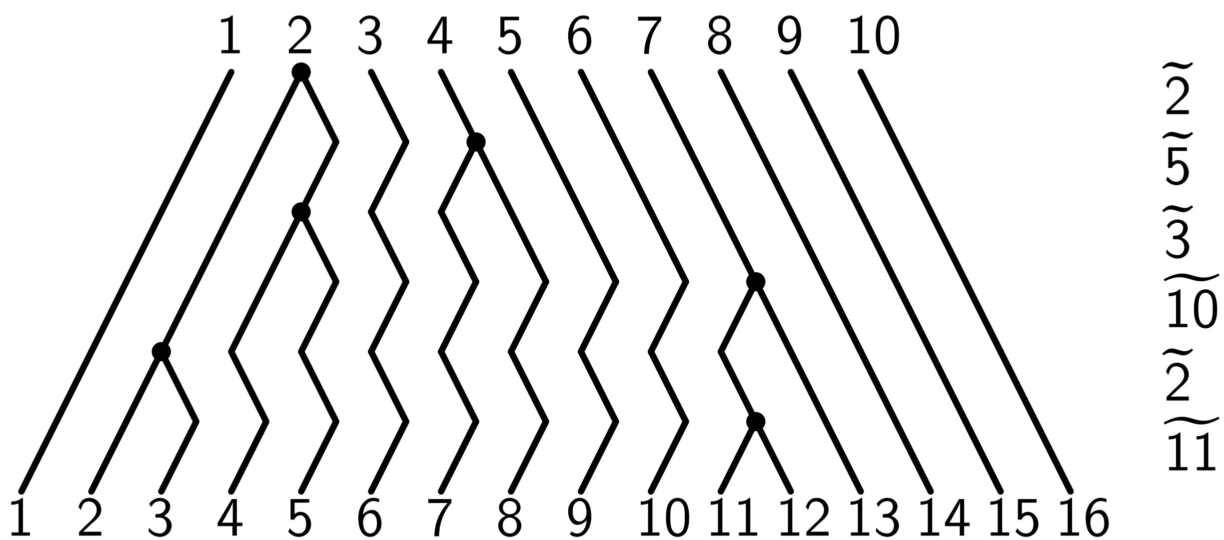
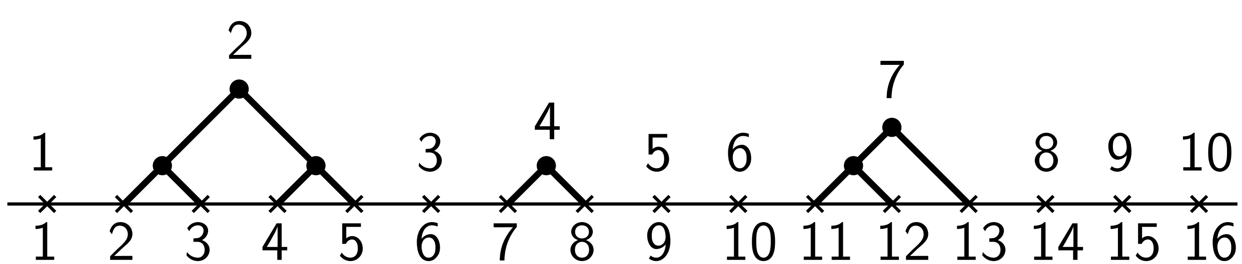


The equivalence class of

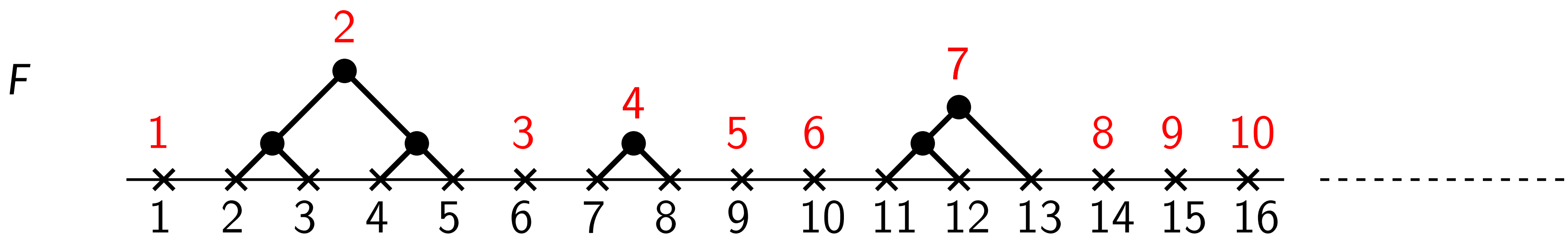


can be represented by



Combinatorics

Definition. An indexed forest F is a sequence of plane binary trees, eventually trivial.



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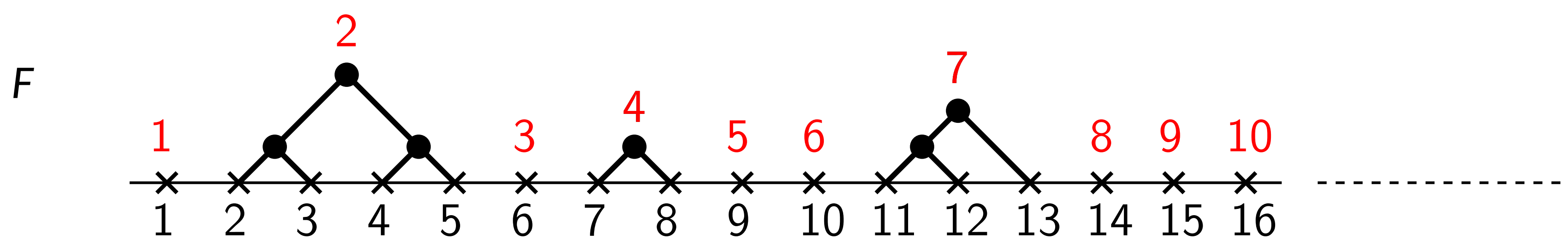

Let For be the set of indexed forests.

Proposition. Define $F \cdot G =$ the forest H obtained by identifying the leaves of F with the roots of G. Then For $\simeq$ Thompson monoid.

$\Rightarrow$ We can define $\mathrm{T}_F = T_{i_1} \cdots T_{i_k}$ by taking any decomposition $F = \tilde{i_1} \cdots \tilde{i_k}$.

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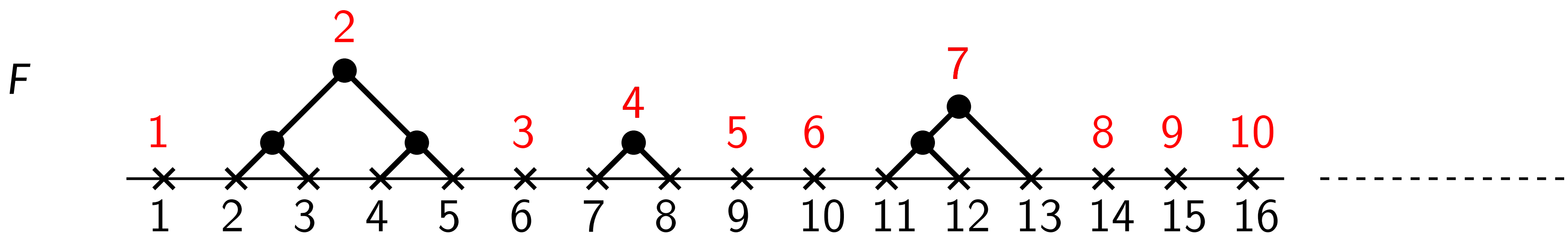

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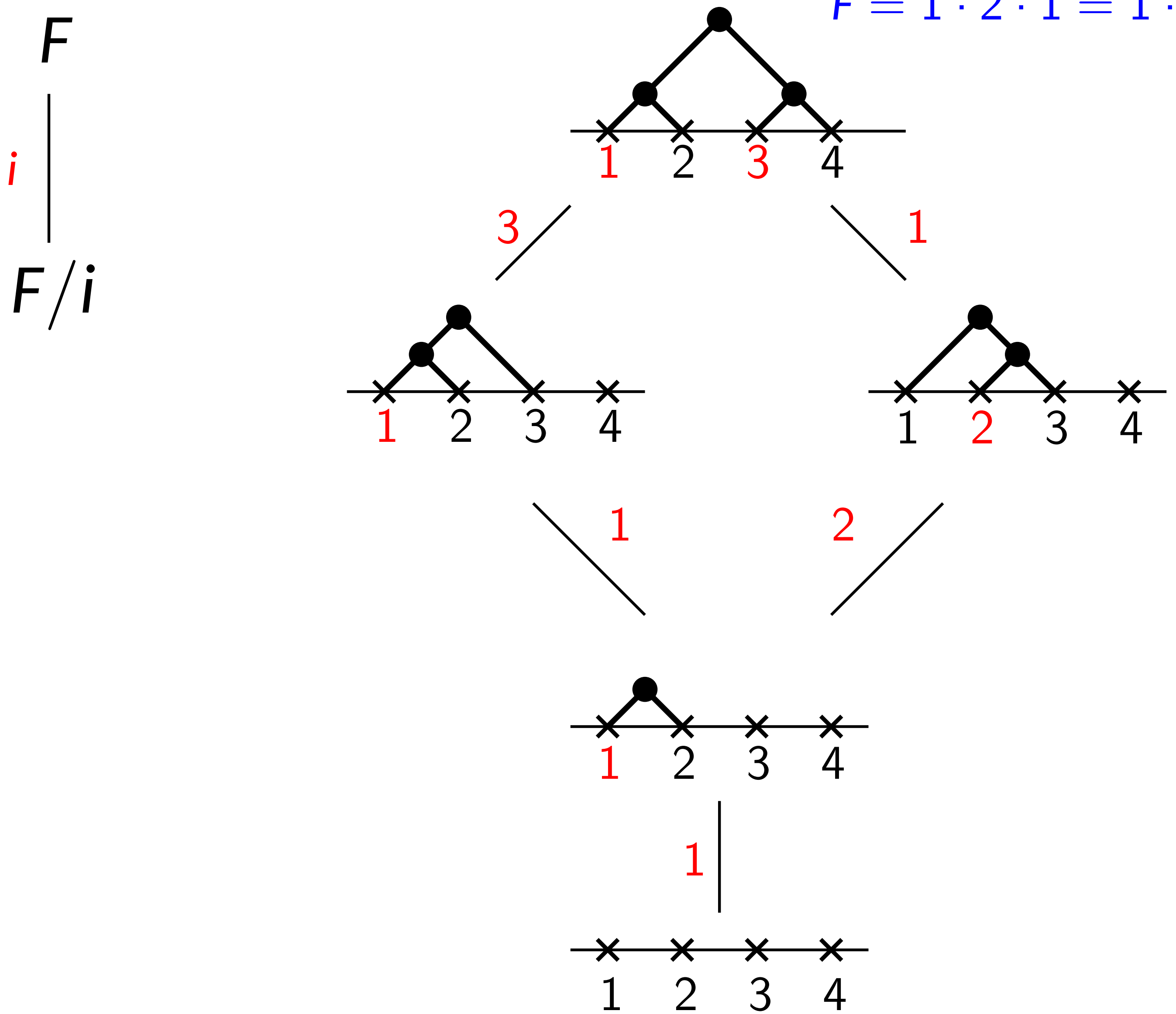
• Let $\mathrm{LTer}(F)$ be the set of left leaves i of a terminal node of F.

Example $\mathrm{LTer}(F) = \{2, 4, 7, 11\}$ above

• F/i is defined when $i \in \mathrm{LTer}(F)$ by removing said terminal node.

Example

$F = \tilde{1} \cdot \tilde{2} \cdot \tilde{1} = \tilde{1} \cdot \tilde{1} \cdot \tilde{3}.$



Forest polynomials

Definition-Theorem[N.-Spink-Tewari '24] The forest polynomials $\mathfrak{P}_F$ for $F \in \mathrm{For}$ are the unique family of homogeneous polynomials such that $\mathfrak{P}_\emptyset = 1$ and

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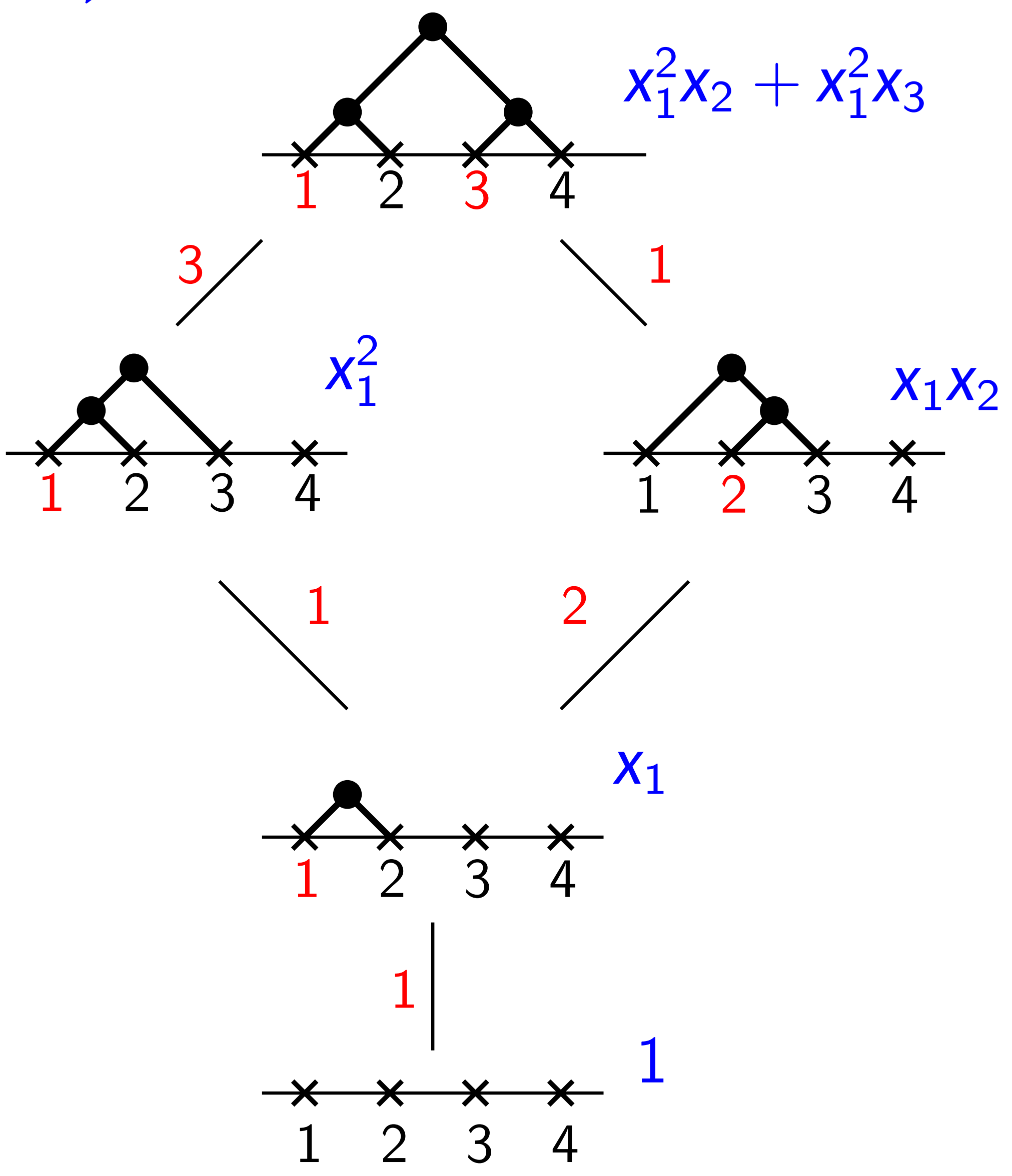
By iteration one gets:

Corollary. (Duality) For $F, G \in \mathrm{For}$, we have

$$\text{Constant term of } \mathrm{T}_F(\mathfrak{P}_G) = \begin{cases} 1 & \text{if } G = F, \\ 0 & \text{otherwise.} \end{cases}$$

Back to Example

Some polynomials $\mathfrak{P}_F$



Forest polynomials

$\rightarrow$ **Nice bases of various spaces:**

- $\mathfrak{P}_F$ is quasisymmetric in $x_1, \ldots, x_n$ if and only F has a unique terminal node at $i = n$.

Proposition. If so, $\mathfrak{P}_F$ is a fundamental quasisymmetric polynomial $F_\alpha(x_1, \ldots, x_n)$.

- $(\mathfrak{P}_F)_F$ is an integral basis of Pol.

- Let $\mathrm{QSym}_n^+ \subset \mathrm{Pol}_n$ be the ideal generated by the $f \in \mathrm{QSym}_n$ with $f(0) = 0$.

Proposition. $\mathfrak{P}_F$ for $F \in \mathrm{For}_n$ project to a basis of the coinvariant space $\mathrm{Pol}_n/\mathrm{QSym}_n^+$.

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→ **Positivity results**

- By their combinatorial definition, the $\mathfrak{P}_F$ have positive coefficients.

- The structure constants $\mathfrak{P}_F\mathfrak{P}_G = \sum_H d_{FG}^H \mathfrak{P}_H$ are positive.
  This can be proved combinatorially.

(**Key**: Leibniz rule $\mathrm{T}_i(fg) = \mathrm{T}_i(f)\mathrm{R}_{i+1}(g) + \mathrm{R}_i(f)\mathrm{T}_i(g)$.)

Positivity of Schubert polynomials

A direct check shows:

$$\mathrm{T}_i = \mathrm{R}_i \partial_i$$

Now for $f \in \mathrm{Pol}$ with $f(0) = 0$,

$$f = \sum_{i=1}^{\infty} (\mathrm{R}_{i+1}(f) - \mathrm{R}_i(f)) + \mathrm{R}_1(f)$$

$$= \sum_{i=1}^{\infty} x_i \mathrm{T}_i(f) + \mathrm{R}_1(f) = \sum_{i=1}^{\infty} x_i \mathrm{R}_i \partial_i(f) + \mathrm{R}_1(f)$$

Choose $f = \mathfrak{S}_w$ with $w \neq \mathrm{id}$

$$\mathfrak{S}_w = \sum_{i \in \mathrm{Des}(w)} x_i \mathrm{R}_i(\mathfrak{S}_{ws_i}) + \mathrm{R}_1(\mathfrak{S}_w).$$

- This is a **new recurrence**.
- Probably the simplest proof that $\mathfrak{S}_w$ **has positive coefficients.**
- Can be interpreted combinatorially on pipe dreams.

General framework

Summary In both cases, we have

- Operators $X = (X_i)$ of degree -1 which generate a certain monoid M.
- A theorem stating the existence and uniqueness of homogenous dual polynomials S_m for $m \in M$ (i.e. $S_1 = 1$ and $X_i S_m = S_{m/i}$ if m/i exists, 0 otherwise).

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We give a solution based on the existence of creation operators Y_i: these must satisfy for $f \in \mathrm{Pol}$ with $f(0) = 0$,

$$\sum_{i=1}^{\infty} Y_i X_i(f) = f.$$

Theorem[N.-Spink-Tewari '24] *Under certain conditions*, if Y_i are creation operators, then the dual family $(S_m)_{m \in M}$ is unique, forms a basis of Pol, and is given by

$$S_m = \sum_{(i_1,\ldots,i_k) \in \mathrm{Fact}(m)} Y_{i_k} \cdots Y_{i_1}(1).$$

If the creation operators preserve coefficient positivity of polynomials, we get immediately that the S_m have positive coefficients.

Related work

$\rightarrow$ Double Schubert polynomials are a generalization of Schubert polynomials, related to equivariant algebraic geometry.
We can also define double versions of forest polynomials, using an operator approach.
These have surprising connections with noncrossing partitions.

This is joint work with N. Bergeron, L. Gagnon, H. Spink and V. Tewari: see *Equivariant quasisymmetry and noncrossing partitions*, arXiv:2504.15234.

$\rightarrow$ There are non-homogenous versions of Schubert polynomials called Grothendieck polynomials, related to K-theory.
We can also define non-homogeneous versions of forest polynomials, called grove polynomials, using an operator approach.

This is work in progress with H. Spink and V. Tewari.

II. A problem in enumerative geometry

The flag variety and its cohomology

$\rightarrow$ The flag variety $\mathrm{Fl}(n)$ is the set of complete flags

$$V_\bullet = (V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n).$$

It admits a natural structure of a smooth projective variety of dimension $\binom{n}{2}$.

Fix $V_\bullet^{ref} \in \mathrm{Fl}(n)$. The Schubert varieties $X_w(V_\bullet^{ref}) \subset \mathrm{Fl}(n)$ are defined for $w \in S_n$.

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For any irreducible subvariety $Y \subset \mathrm{Fl}(n)$ of dimension d, we have a *fundamental class* $[Y] \in H^{n(n-1)-2d}(\mathrm{Fl}(n))$. In particular $\sigma_w := [X_{w_0 w}(V_\bullet^{ref})] \in H^{2\ell(w)}$.

These form a linear basis: $H^*(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q}\sigma_w$.

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The numbers $b_w(Y)$ are nonnegative integers: they count the number of intersection points of Y with generic Schubert subvarieties of codimension d.

> In particular when $Y = X_u \cap X_v$, finding a manifestly positive rule for $c_{uv}^w = b_w(Y)$ is a major open problem.

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→ Connection with Schubert polynomials: There exists a surjective morphism $j_n : \mathrm{Pol}_n \to H^*(\mathrm{Fl}(n))$ with kernel Sym_n^+. It satisfies $j_n(\mathfrak{S}_w) = \sigma_w$ for $w \in S_n$.

The numbers a_w

The permutahedral variety $\mathrm{Perm}(n)$ is the generic orbit closure of the maximal torus acting in $\mathrm{Fl}(n)$.

(It can also be described as a regular semisimple Hessenberg variety, or abstractly as the toric variety associated with the braid arrangement.)

It is a smooth variety of dimension $n-1$.

Write S_n' for the set of permutations in S_n of length $n-1$.

Definition Let $w \in S_n'$ and $V_\bullet^{ref}$ generic.

$$a_w \text{ is the number of points in } \mathrm{Perm}(n) \cap X_{w_o w}(V_\bullet^{ref}).$$

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Note that if $j_n(f) = [\mathrm{Perm}(n)]$, then one transforms the computation of a_w into finding the coefficients of a polynomial in its Schubert basis expansion.

The first two formulas are based on this approach; the third one is based on computations in the cohomology ring of $\mathrm{Perm}(n)$.

A (non manifestly positive) formula

We introduce the operator of divided symmetrization $\langle \cdot \rangle_n$. This is the linear operator on Pol_n defined by:

$$\langle f(x_1, \ldots, x_n) \rangle_n := \sum_{w \in S_n} w \cdot \left(\frac{f(x_1, \ldots, x_n)}{\prod_{1 \le i \le n-1} (x_i - x_{i+1})} \right).$$

Proposition[N.-Tewari '21] For any $w \in S'_n$,

$$a_w = \langle \mathfrak{S}_w(x_1, \ldots, x_n) \rangle_n.$$

This follows from the fact that j_n sends

$$\prod_{\substack{1 \le i < j \le n \\ j > i+1}} (x_i - x_j)$$

to $[\mathrm{Perm}(n)]$, a special case of a result of Anderson and Tymoczko ['10].

Formula 1

Proposition [N.-Spink-Tewari '24] For any $f \in \mathrm{Pol}_n$ of degree $n-1$,

$$\langle f \rangle_n = \mathsf{T}_1(\mathsf{T}_1 + \mathsf{T}_2)\cdots(\mathsf{T}_1 + \cdots + \mathsf{T}_{n-1})f.$$

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One can show that $\mathrm{T}_i\mathfrak{S}_w$ is always a sum of Schubert polynomials.
$\Rightarrow$ this shows by induction that $a_w \in \mathbb{Z}_{\geq 0}$.

By a careful analysis, one can in fact show from this formula that $a_w \in \mathbb{Z}_{>0}$.

Formula 2

Here we start again with the expression

$$a_w = \left\langle \mathfrak{S}_w(x_1, \dots, x_n) \right\rangle_n.$$

This time we decompose $\mathfrak{S}_w = \sum b_w^F \mathfrak{P}_F$, via a certain combinatorial procedure.

Now forest polynomials $\mathfrak{P}_F$ behave well with $\left\langle \cdot \right\rangle_n$: they either vanish or give a simple combinatorial quantity.
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Formula 2 [N.-Tewari '24] Let $w \in S'_n$. Then a_w is the number of reduced words of w^{-1} that are also Ω-parking functions.

From this $a_w \in \mathbb{Z}_{\geq 0}$ (but it is not obvious that $a_w > 0$!).

Formula 3

This last formula relies on seminal work of Klyachko ['85].

He essentially computed the action of the natural map $H^*(\mathrm{Fl}(n)) \mapsto H^*(\mathrm{Perm}(n))$ on Schubert classes.

One also needs to introduce Postnikov's mixed Eulerian numbers $A_{\mathbf{i}}$ ['09]: these are defined as "mixed volumes of hypersimplices" $\Rightarrow$ The $A_{\mathbf{i}}$ are in $\mathbb{Z}_{>0}$.

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Formula 3 [N.-Tewari '23] For any $w \in S'_n$,

$$a_w = \sum_{\mathbf{i} \in \mathrm{Red}(w)} \frac{A_{\mathbf{i}}}{(n-1)!}.$$

This formula implies that $a_w > 0$ (but not that it is an integer!).

Because of the properties of mixed Eulerian numbers, it also follows that $a_w = a_{w^{-1}}$, a property that is more mysterious from the other formulas.

Perspectives

$\rightarrow$ The link between the various formulas can be explained algebraically. It would be interesting to also match the combinatorics of the various expressions.

$\rightarrow$ One should also try to find manifestly positive rules to compute the “Schubert coefficients” of the other regular semimple Hessenberg varieties.

This is an interesting family of size the nth Catalan number, whose cohomology is linked to the Shareshian–Wachs conjecture about certain chromatic (quasi)symmetric functions.

Extra Combinatorics
(Parking procedures)

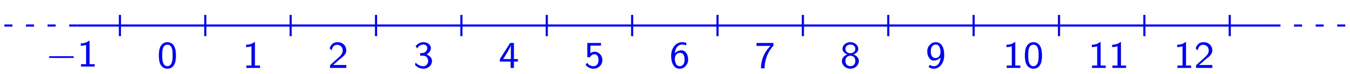
Classical Parking Functions

Classical parking procedure ([Konheim-Weiss '66]).

- r cars want to park on $\mathbb{Z}$.
- The ith car has a preferred spot a_i.
- If the spot is available, it parks there.
- If not, it parks in the *nearest available spot on the right*.

Example.

$$a_1a_2a_3a_4a_5a_6a_7 = 3525895$$



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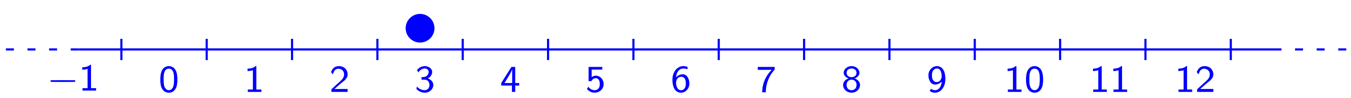

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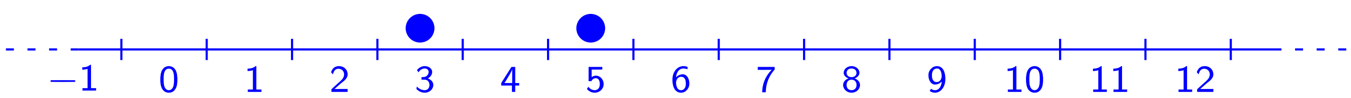

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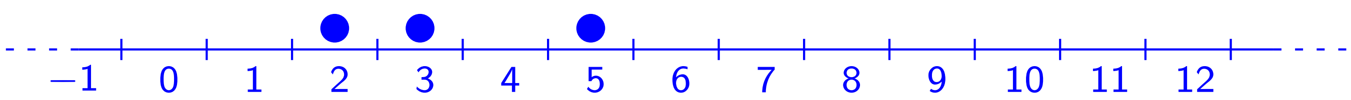

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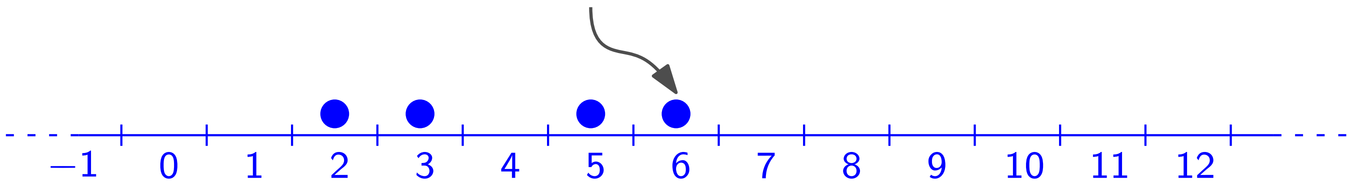

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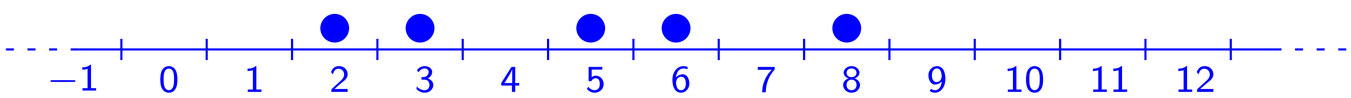

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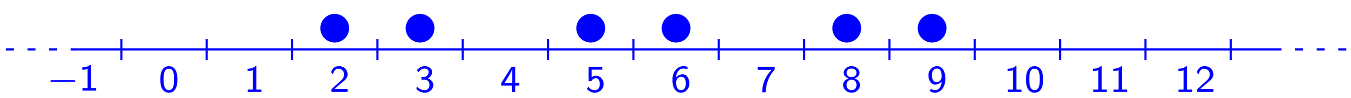

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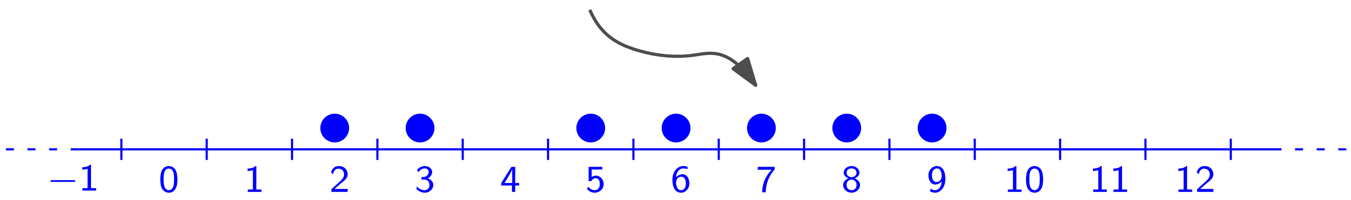

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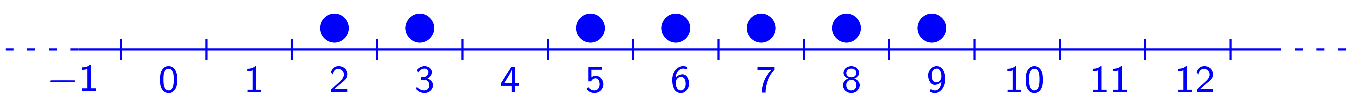

Definition. A word $a_1a_2 \ldots a_r$ is called a parking function if the parking procedure ends up with all spots $1, \ldots, r$ occupied.

Example.

$(r = 1)$ 1
$(r = 2)$ 11, 12, 21
$(r = 3)$ 111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213, 221, 231, 311, 312, 321

Bilateral Parking Procedures

Bilateral parking procedure ([Nadeau '22]).

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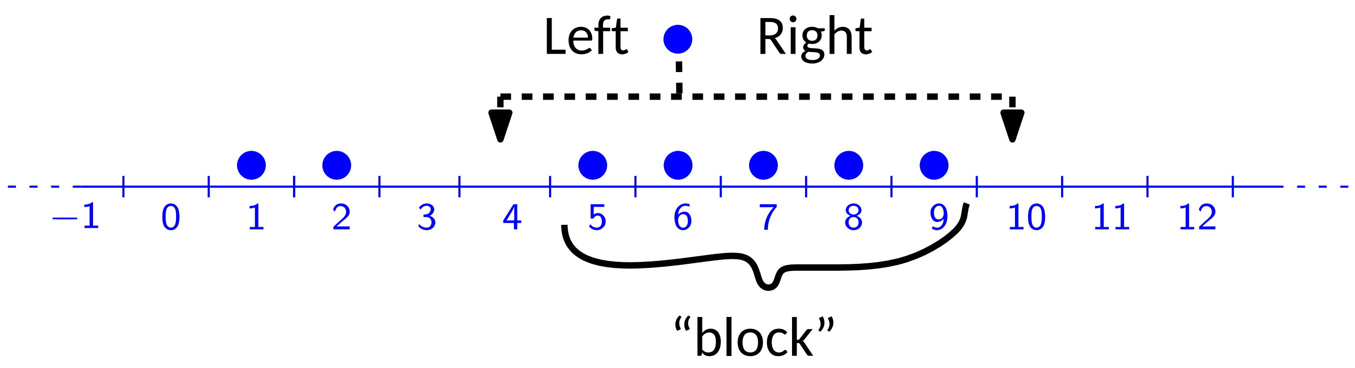

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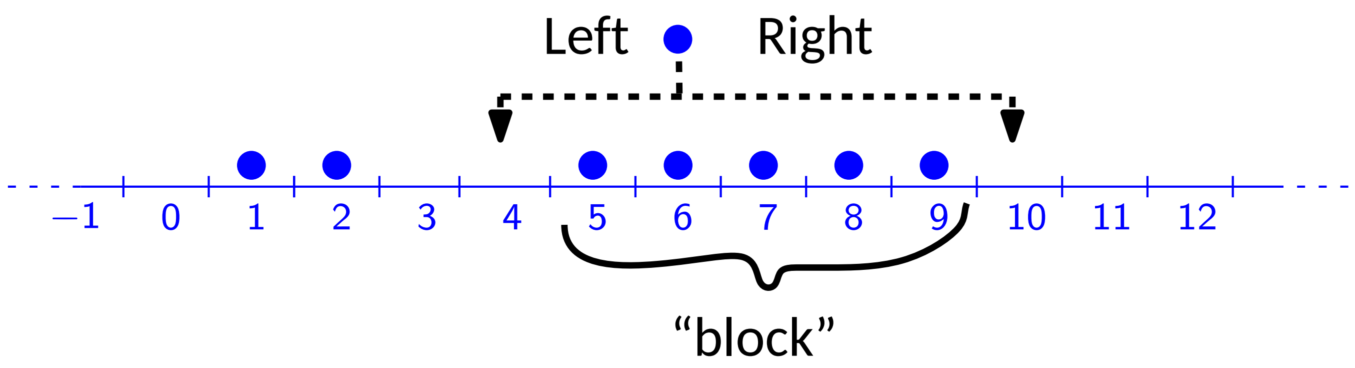

Example. ($\mathcal{P}_{prime}$) Count the number of cars on the block.
If this is a *prime number*, go right, otherwise go left.

Example. ($\mathcal{P}_{closest}$) Count the number of cars on the block to your left (n_L) and to your right (n_R). If $n_L \geq n_R$, go right, otherwise go left.

Example. (Ω) If the desired spot of the last car that parked on the block is to your left, go right, otherwise go left.

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• A $\mathcal{P}$-parking function is a word $a_1 \cdots a_r$ such that all cars park in the spots $\{1, \ldots, r\}$.

• One can define the notion of a local procedure $\mathcal{P}$: roughly speaking, this means the the rule is "shift-invariant" and depends only on the block where one wants to park.

All of our previous examples are local.

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$\rightarrow$ We have the following "discrete universality result":

Theorem ([N. '22+]). Let $\mathcal{P}$ be a local parking procedure.
Then the number of $\mathcal{P}$-parking functions of size r is $(r+1)^{r-1}$.

The proof relies on a generalization of Pollak's "cyclic lemma" argument.

A probabilistic parking procedure

$\rightarrow$ Fix a real number $q \geq 0$, and consider the following procedure $\mathcal{P}^q$: when the desired spot is occupied, go one spot to the left with probability $q/(1+q)$, and to the right with probability $1/(1+q)$.
Continue until you find an empty parking spot.

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Let $p_{\mathbf{v_1}\cdots\mathbf{v_r}}(q)$ be the probability that all cars are parked in $\{1, \ldots, r\}$ starting from the list of desired spots $v_1, \cdots, v_r$.

$\rightarrow$ Given a word $\mathbf{v}$, define the remixed Eulerian number ([N.-Tewari '21])

$$A_{\mathbf{v}}(q) := (r)_q! p_{\mathbf{v}}(q).$$

Here $(r)_q! = (r)_q(r-1)_q \cdots (1)_q$ where $(i)_q = (1-q^i)/(1-q)$.

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$\rightarrow$ More generally these polynomials comprise several well-known families of standard "q-analogues" of classical numbers. Furthermore,

Theorem [N.-Tewari '23] $A_{\mathbf{v}}(q)$ is a polynomial in q with positive integral coefficients. It is symmetric and unimodal.

FIN