PARAMETERS ESTIMATION OF A THRESHOLD CIR PROCESS

Sara Mazzonetto[1], Benoît Nieto[2]

[1] Institut Elie Cartan, Nancy

[2] Institut Camille Jordan, École Centrale de Lyon

Introduction

We refer by threshold Cox-Ingersoll-Ross (TCIR) to a continuous-time threshold autoregressive process. It follows the CIR dynamics when above or below a fixed level, yet at this level (threshold) its coefficients can be discontinuous. Goal: We discuss maximum likelihood estimation of the drift parameters, both assuming continuous and discrete time observations. In the ergodic case, we derive consistency and speed of convergence of these estimators in long time and high frequency.

Let $(X_t)_{t\geq 0}$, the process solution of

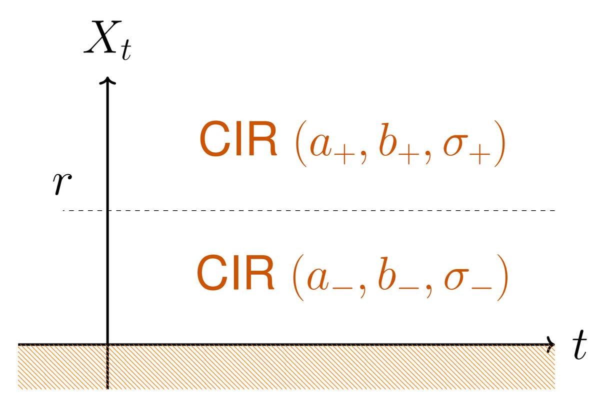
$$\begin{cases} dX_t = (a_r(X_t) - b_r(X_t)X_t)dt + \sqrt{2\sigma_r(X_t)X_t}dB_t, \quad t \geq 0, \\ X_0 = x_0. \end{cases} \tag{1}$$

$$a_r(x) = \begin{cases} a_+ \in \mathbb{R} & \text{si } x \geq r, \\ a_- \in \mathbb{R}_+ & \text{si } x < r, \end{cases} \quad b_r(x) = \begin{cases} b_+ \in \mathbb{R} \text{ si } x \geq r, \\ b_- \in \mathbb{R} \text{ si } x < r, \end{cases}$$

$$\sigma_r(x) = \begin{cases} \sigma_+ > 0 \text{ si } x \geq r, \\ \sigma_- > 0 \text{ si } x < r. \end{cases} \tag{2}$$

with $r > 0$ et $x_0 > 0$.

For all $t \geq 0$, $X_t \geq 0$.



Statistics of diffusion

Our goal is to estimate the parameter $\theta_0 = (a_{+,0}, b_{+,0}, a_{-,0}, b_{-,0})$ making use of the continuous time observation of $(X_t^{\theta_0})_{t\in[0,T]}$.

We denote $\theta = (a_+, b_+, a_-, b_-)$, the likelihood ratio evaluated at time T is given by

$$G_T(\theta) = \frac{d\mathbb{P}_{\theta,T}}{d\mathbb{P}_{\theta_0,T}} = \exp\left(\int_0^T \frac{a_r(X_s) - b_r(X_s)X_s}{2\sigma_r(X_s)X_s} dX_s - \frac{1}{4}\int_0^T \frac{(a_r(X_s) - b_r(X_s)X_s)^2}{\sigma_r(X_s)X_s} ds\right).$$

Remark: The estimators and the Likelihood is well defined iff $a_- > \sigma_-$ *i.e.*

$$\mathbb{P}_\theta\left(\int_0^T \frac{\mathbb{1}_{X_s \leq r} ds}{X_s} < \infty\right) = 1.$$

For the estimation problem, we suppose r and $\sigma_\pm$ to be known.

Process long time behavior depends on the parameters:

	Null recurrent	Positive Recurrent (Ergodic)	Transience
$\{0\}$ Absorbing point ($a_- = 0$)	$[b_+ = 0$ and $0 \leq a_+ < \sigma_+]$ and $[b_- \in \mathbb{R}$ and $a_- \leq 0]$	$\emptyset$	$[b_+ < 0$ or $a_+ \geq \sigma_+]$
$\{0\}$ Instantaneous reflecting point ($0 < a_- \leq \sigma_-$)	$[b_+ = 0$ and $0 \leq a_+ < \sigma_+]$ and $[b_- \in \mathbb{R}$ and $0 < a_- \leq \sigma_-]$.	$[[b_+ > 0$ and $a_+ \in \mathbb{R}]$ or $[b_+ = 0$ and $a_+ < 0]]$ and $[b_- \in \mathbb{R}$ and $0 < a_- \leq \sigma_-]$.	$[b_+ < 0$ or $a_+ \geq \sigma_+]$
$\{0\}$ is not reached ($a_- > \sigma_-$)	$[b_+ = 0$ and $0 \leq a_+ < \sigma_+]$ and $[b_- \in \mathbb{R}$ and $a_- > \sigma_-]$	$[[b_+ > 0$ and $a_+ \in \mathbb{R}]$ or $[b_+ = 0$ and $a_+ < 0]]$ and $[b_- \in \mathbb{R}$ and $a_- > \sigma_-]$.	$[b_+ < 0$ or $a_+ \geq \sigma_+]$

We take Θ such that

$\Theta := \{(a_\pm, b_\pm)$ s.t. $[[b_+ > 0$ and $a_+ \in \mathbb{R}]$ or $[b_+ = 0$ and $a_+ < 0]]$ and $[b_- \in \mathbb{R}$ and $a_- > \sigma_-]\}$.

Maximum Likelihood Estimators

Imagine that we observe $(X_t)_{t\in[0,T]}$ in continuous time. For $T \in (0, \infty)$ and $m = -1, 0, 1$, we define

$$\mathcal{Q}_T^{\pm,m} = \int_0^T X_s^m \mathbb{1}_{\{\pm(X_s - r)\geq 0\}} ds \quad \text{and} \quad \mathcal{M}_T^{\pm,m} = \int_0^T X_s^m \mathbb{1}_{\{\pm(X_s - r)\geq 0\}} dX_s.$$

Let $\theta_0 = (a_0^\pm, b_0^\pm)$, the parameter to be estimated and $\theta \in \Theta \subset \mathbb{R}^4$. We denote

$$(\alpha_T^\pm, \beta_T^\pm) = \underset{\theta\in\Theta}{\text{Argmax}}\, G_T(\theta).$$

ML-Estimators:

For every $T \in (0, \infty)$ the MLE are given by

$$\alpha_T^\pm = \frac{\mathcal{M}_T^{\pm,-1}\mathcal{Q}_T^{\pm,1} - \mathcal{M}_T^{\pm,0}\mathcal{Q}_T^{\pm,0}}{\mathcal{Q}_T^{\pm,1}\mathcal{Q}_T^{\pm,-1} - (\mathcal{Q}_T^{\pm,0})^2} \quad \text{and} \quad \beta_T^\pm = \frac{\mathcal{M}_T^{\pm,-1}\mathcal{Q}_T^{\pm,0} - \mathcal{Q}_T^{\pm,-1}\mathcal{M}_T^{\pm,0}}{\mathcal{Q}_T^{\pm,1}\mathcal{Q}_T^{\pm,-1} - (\mathcal{Q}_T^{\pm,0})^2}.$$

Theorem MLE: Long-time behavior for continuous time observations

- $\frac{1}{T}\mathcal{Q}_T^{\pm,m} \xrightarrow[N\to\infty]{\text{a.s.}} \mathcal{Q}_\infty^{\pm,m} \in \mathbb{R}_+^\star$.
- The estimator is strongly consistent:
$$(\alpha_T^\pm - a_\pm, \beta_T^\pm - b_\pm) \xrightarrow[N\to\infty]{\text{a.s.}} (0, 0),$$
- and asymptotically normal:
$$\sqrt{T}\left(\alpha_T^\pm - a_\pm,\, \beta_T^\pm - b_\pm\right) \xrightarrow[N\to\infty]{\text{stably}} N^\pm := \left(N^{\pm,\alpha},\, N^{\pm,\beta}\right),$$
where N^+, N^- are two mutually independent, independent of $(X_t)_{t\geq 0}$, two-dimensional Gaussian r.v. with covariance matrices given by
$$2\sigma_\pm\Gamma_\pm^{-1} \text{ with } \Gamma_\pm = \begin{pmatrix} \mathcal{Q}_\infty^{\pm,-1} & -\mathcal{Q}_\infty^{\pm,0} \\ -\mathcal{Q}_\infty^{\pm,0} & \mathcal{Q}_\infty^{\pm,1} \end{pmatrix}.$$

Outlook

- Our results can be extended to the multi-threshold CIR.
- Some work is necessary to prove the strong existence and unicity of the SDE (1): extension of Le Gall 1985.
- These results can be extended to the case where $0 < a_- \leq \sigma_-$ making use of the Quasi- Likelihood.
- A statistical test can be performed to verify if the process hit zero or not.
- A second statistical test can be performed to verify the existence of a threshold.
- A quadratic estimator is proposed to estimate $\sigma_\pm$.

High Frequency and Long time case

We assume in this section to observe the process on the discrete time grid $0 = t_0 < t_1 \cdots < t_{N-1} < t_N = T$, for $N \in \mathbb{N}$, $T \in (0, \infty)$, and set $\Delta_N = \max_{k=1,\cdots,N}\{t_k - t_{k-1}\}$. We define $X_i := X_{t_i}$, with $i = 0, \cdots, N$. The discretized likelihood provides an estimator $(\alpha_{T,N}^\pm, \beta_{T,N}^\pm)$. Which is the discretized version of the one from continuous time observations.

Remark: Consistency and asymptotic normality holds for $(\alpha_{T_N,N}^\pm, \beta_{T_N,N}^\pm)$. Where $\lim_{N\to+\infty} T_N = +\infty$ and $\lim_{N\to+\infty} \Delta_N = 0$.

Numerical application

We simulate the threshold OU process using the Euler scheme (time scale 10^{-3}).

a_+	a_-	b_+	b_-	σ_+	σ_-	r	X_0
5	3	2.5	3	1	1.5	1	0.6

Table 1: Parameter choice for simulation.

parameter	a_-	a_+	b_-	b_+
bias	0.086	0.112	-0.009	0.012
MSE	$4,12 \times 10^{-6}$	2.67×10^{-6}	1.05×10^{-5}	$9,02 \times 10^{-7}$

Table 2: Mean, mean squared error (MSE) of the MLE estimators with parameters as in Table 1, on $n = 10^3$ trajectories, with $T = 10^3$ and $N = 10^6$ observations on each trajectory.

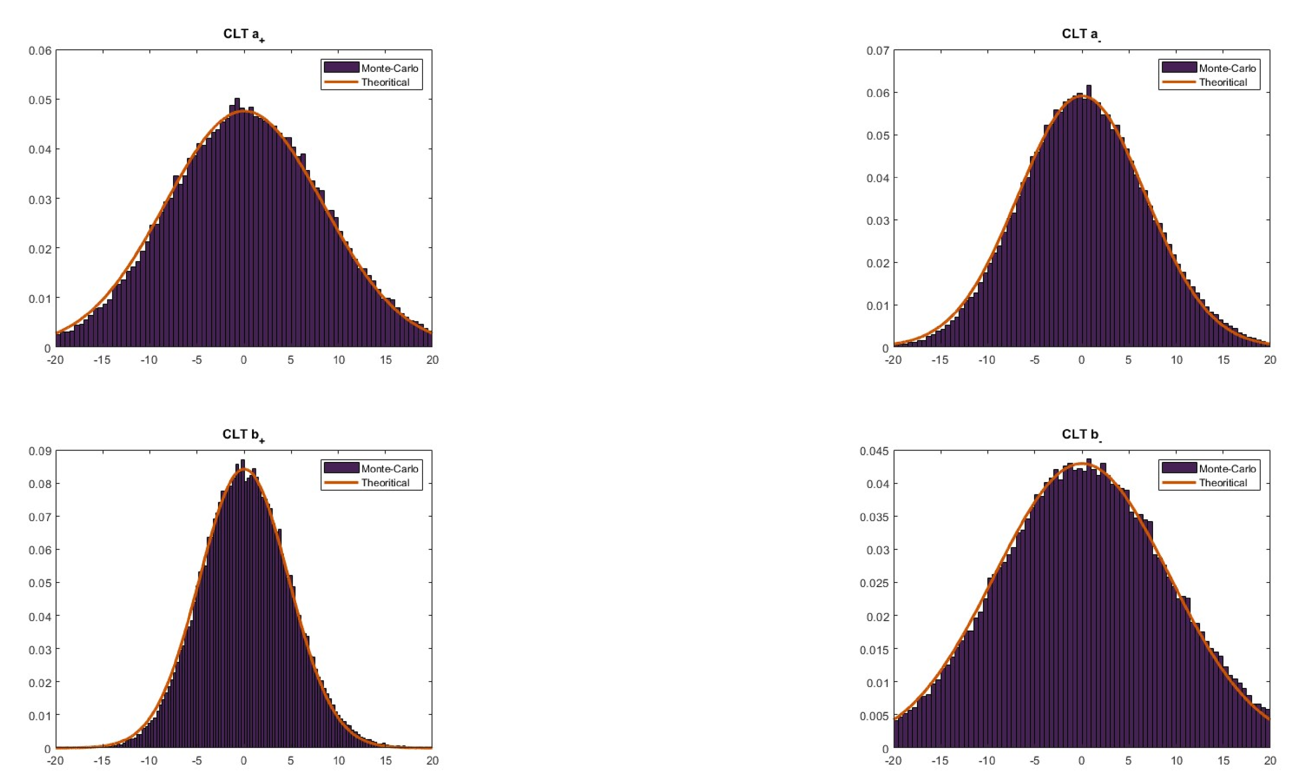


Figure 2: CLT in Theorem MLE, with parameters as in Table 1. We plot the theoretical distribution of the estimation error and compare with the distribution of the error on $n = 10^3$ trajectories, with $T = 10^3$ and $N = 10^6$ observations on each trajectory.

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