Berry Phases of Boundary Gravitons

Blagoje Oblak

(ETH Zurich)

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1710.06883 (J. Geom. Phys.)
1711.05753 (CQG)

MOTIVATION

Gravity has rich **asymptotic symmetries**

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- For definiteness : 3D gravity
- Generalize Thomas precession [Thomas 1926]

Plan of the talk

1. **Berry phases in group reps**

2. Virasoro Berry phases

3. Berry phases & asymptotic symmetries

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A. Berry phases

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A. Berry phases

B. Application to unitary reps

Berry phases

System with parameters $p_1, ..., p_n$

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- Coordinates on manifold $\mathcal{M}$

BERRY PHASES

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Berry phases & groups

Group G

BERRY PHASES & GROUPS

Group G, algebra $\mathfrak{g}$

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Note : Berry phase vanishes for $f(t)$ in stabilizer of $|\phi\rangle$

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- Parameter space is G/G_ϕ

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Example : $G = \mathrm{SU}(2)$, highest-weight state $|\, j\,\rangle$

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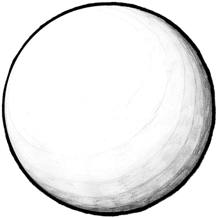

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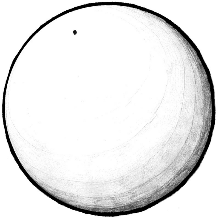

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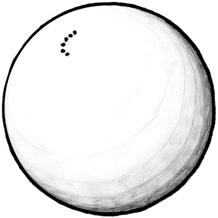

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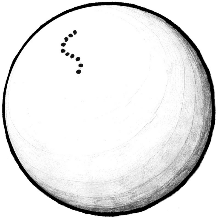

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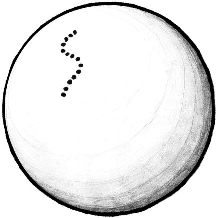

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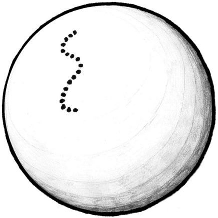

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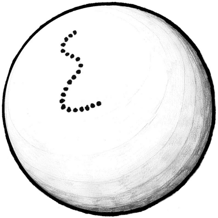

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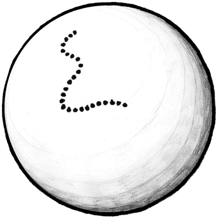

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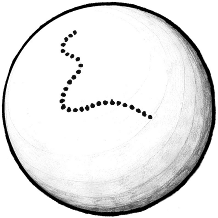

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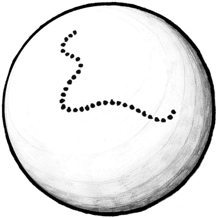

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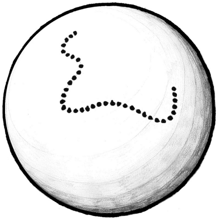

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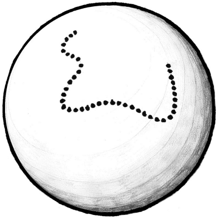

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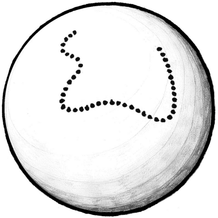

Berry phases & groups

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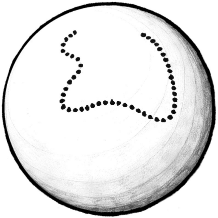

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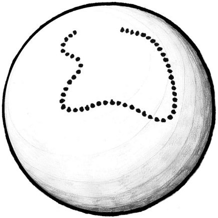

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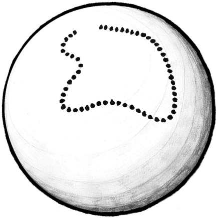

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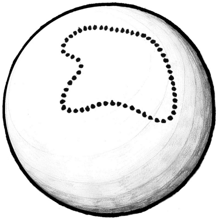

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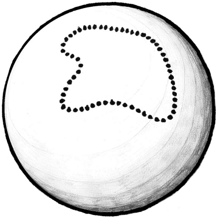

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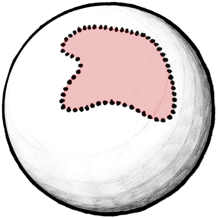

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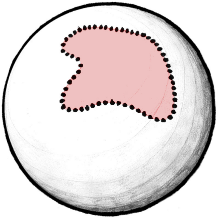

How does this work for **Virasoro** ?

2. Virasoro Berry phases

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A. Maurer-Cartan form of Diff S^1

2. Virasoro Berry phases

A. Maurer-Cartan form of Diff S^1

B. Maurer-Cartan form of Virasoro

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A. Maurer-Cartan form of Diff S^1

B. Maurer-Cartan form of Virasoro

C. Berry phases

Diff S^1

Diff S^1 = group of **circle diffeos**

DIFF S^1

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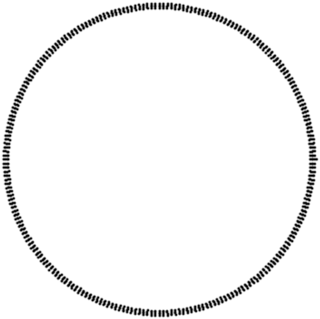

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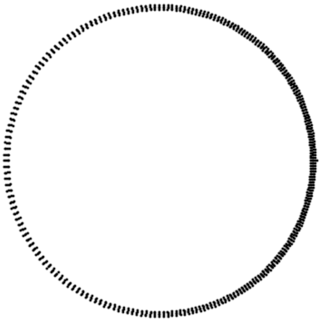

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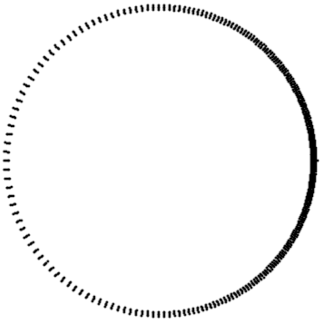

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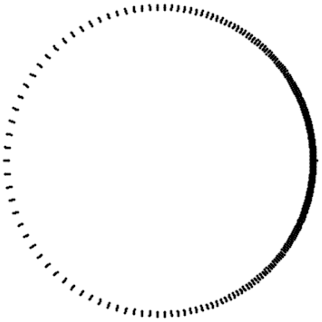

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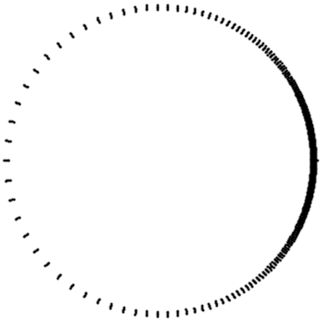

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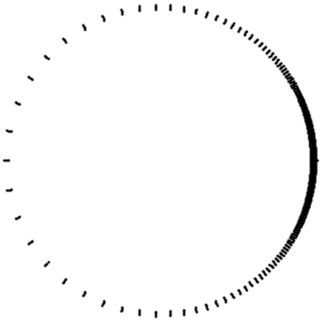

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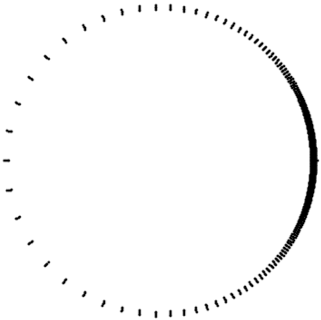

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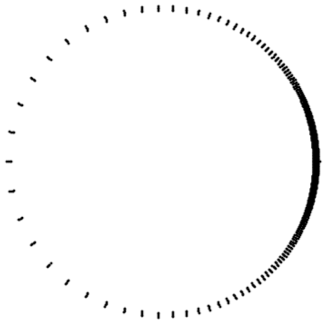

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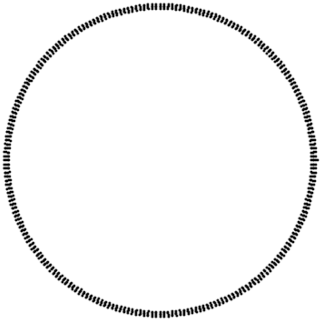

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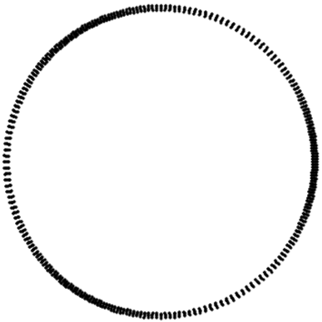

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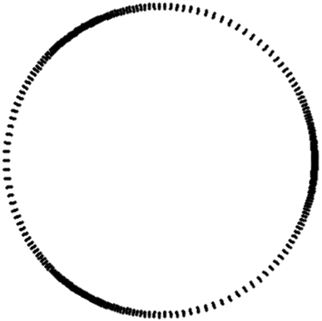

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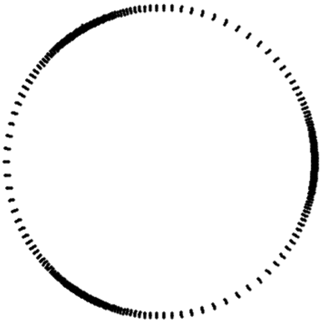

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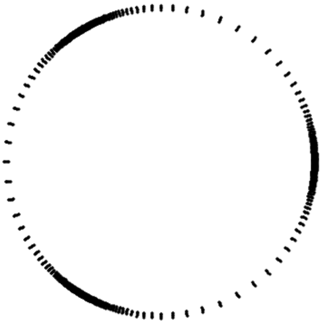

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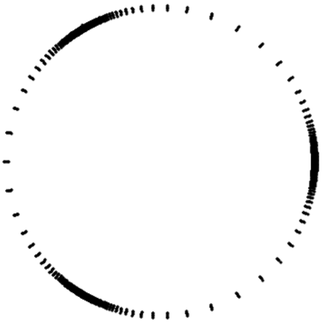

DIFF S^1

Maurer-Cartan form ?

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Let $f(t) =$ path in G

DIFF S^1

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For Diff S^1 : path $f(t,\varphi)$

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VIRASORO

Virasoro group = Central extension of Diff S^1

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$$\textcolor{red}{\textbf{Virasoro group}} = \text{Central extension of Diff}\, S^1 = \text{Diff}\, S^1 \times \mathbb{R}$$

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Virasoro group $=$ Central extension of $\text{Diff}\, S^1 = \text{Diff}\, S^1 \times \mathbb{R}$

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- Extra central entry in Maurer-Cartan form

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$$(f^{-1} \cdot \dot{f})_{\text{Virasoro}}$$

Virasoro

Virasoro group = Central extension of Diff S^1 = Diff $S^1 \times \mathbb{R}$

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$$(f^{-1} \cdot \dot{f})_{\text{Virasoro}} = \left(\frac{\dot{f}}{f'} \partial_\varphi \, , \right.$$

Virasoro

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$$(f^{-1}\cdot\dot{f})_{\text{Virasoro}} = \left(\frac{\dot{f}}{f'}\partial_\varphi\,,\;\frac{1}{48\pi}\int_0^{2\pi}d\varphi\,\frac{\dot{f}}{f'}\left(\frac{f''}{f'}\right)'\right)$$

[Alekseev-Shatashvili, Rai-Rodgers 1989]

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[Alekseev-Shatashvili, Rai-Rodgers 1989]

- We can compute Berry phases !

Berry phases

$$\text{Berry}: \quad B_\phi[f] \; = i \oint dt \, \langle \phi | \mathfrak{u}[f^{-1}\dot{f}] | \phi \rangle$$

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Apply this to Virasoro !

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- **Central charge** c

Berry phases

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Apply this to Virasoro !

- **Central charge** c
- **Highest weight** h :

$$\mathfrak{u}[\ell_0]|h\rangle = h|h\rangle \qquad \mathfrak{u}[\ell_n]|h\rangle = 0 \quad \text{if } n > 0$$

Berry phases

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Apply this to Virasoro !

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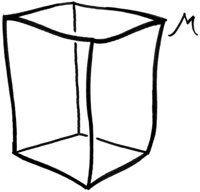

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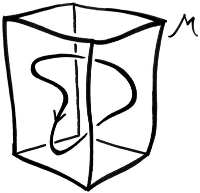

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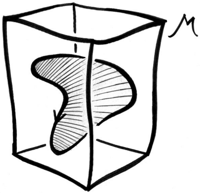

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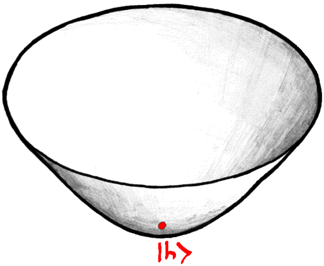

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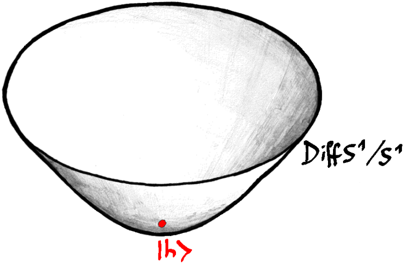

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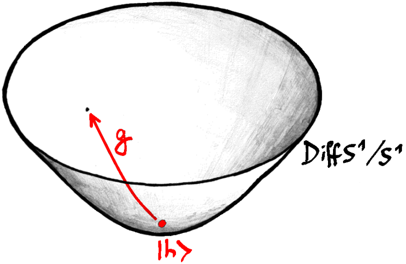

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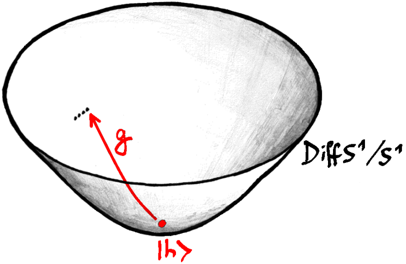

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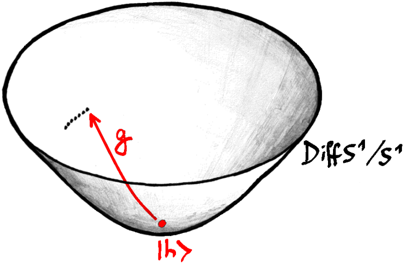

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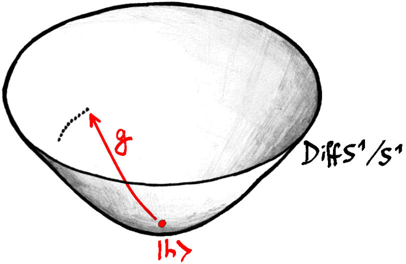

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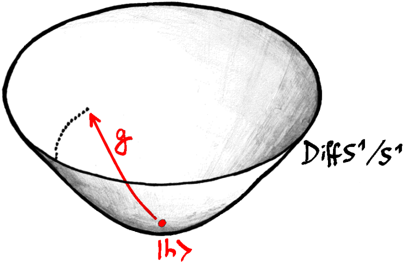

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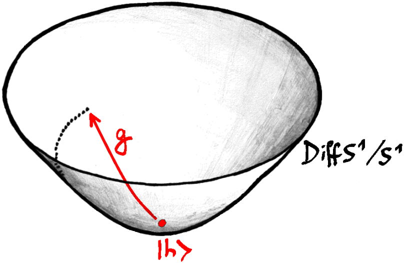

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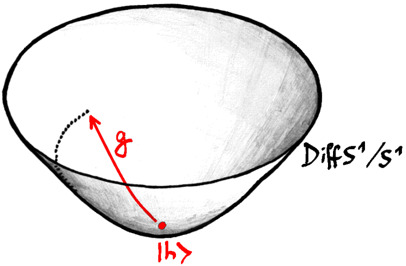

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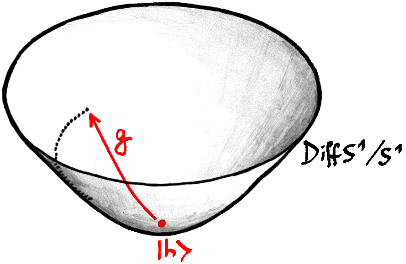

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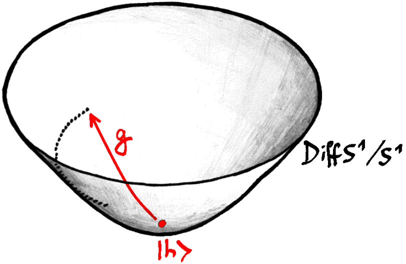

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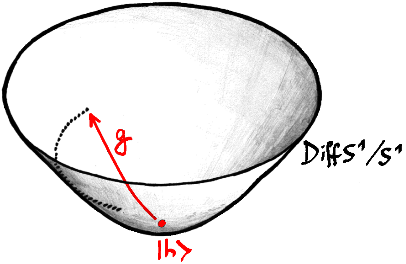

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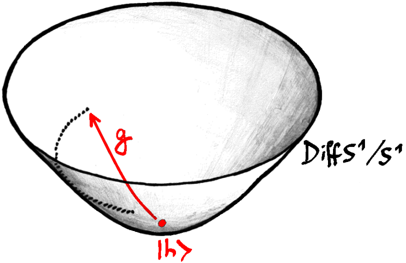

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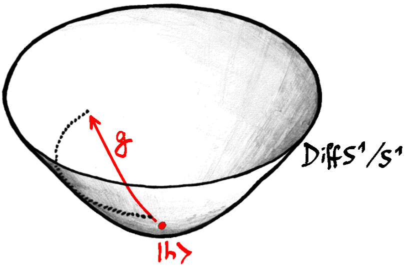

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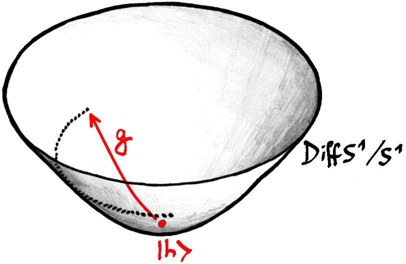

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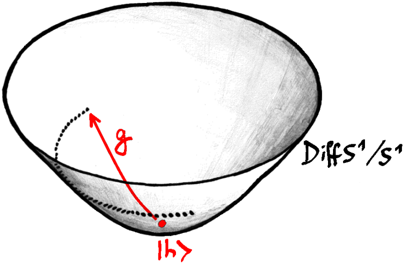

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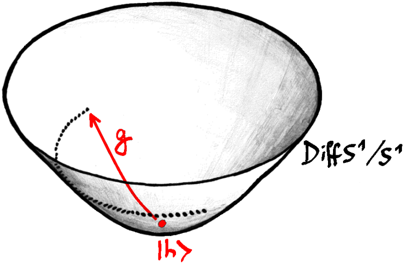

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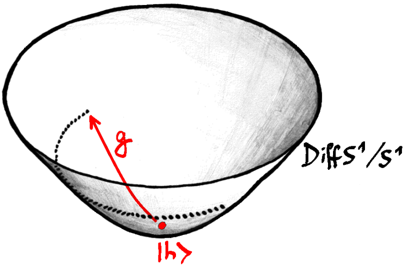

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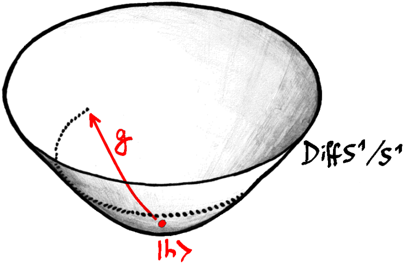

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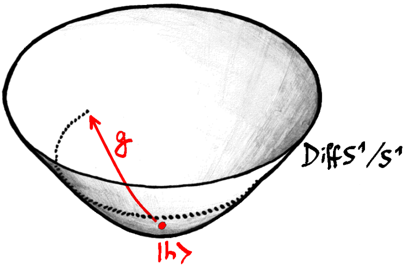

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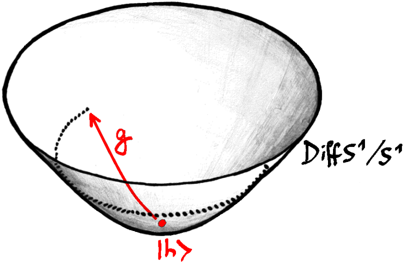

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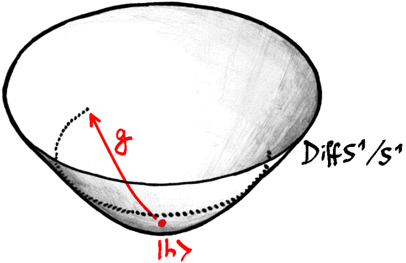

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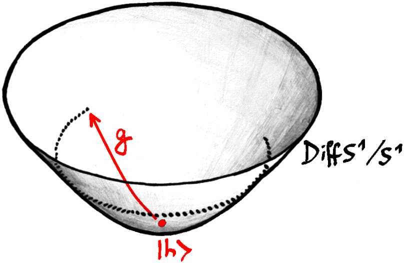

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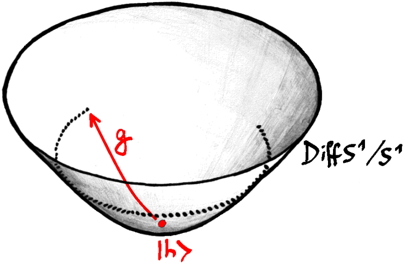

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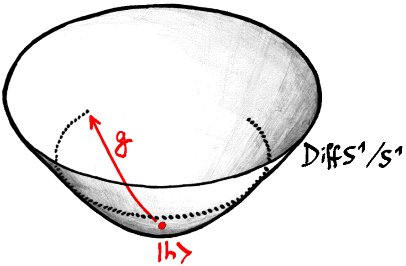

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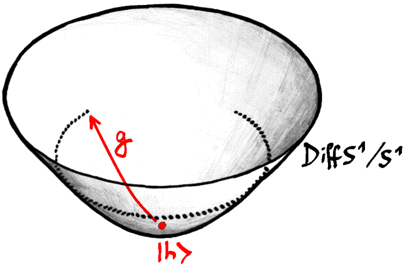

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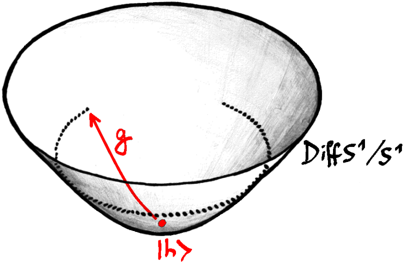

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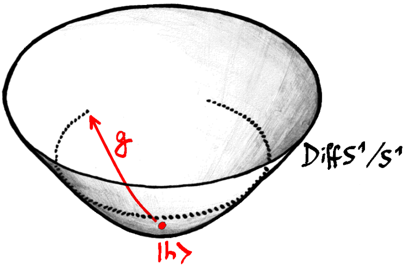

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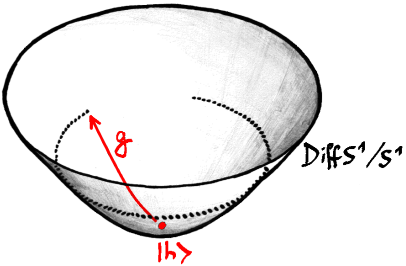

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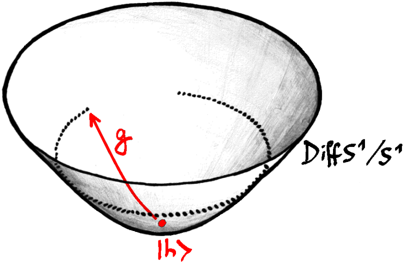

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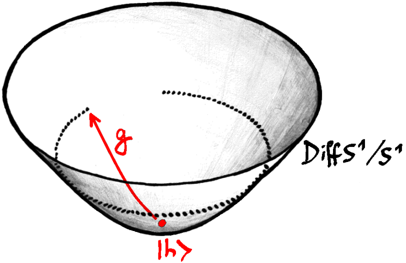

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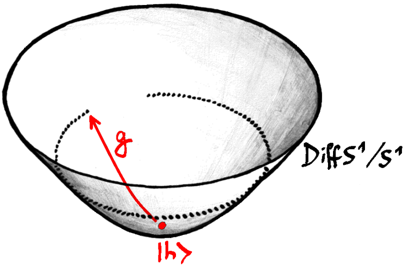

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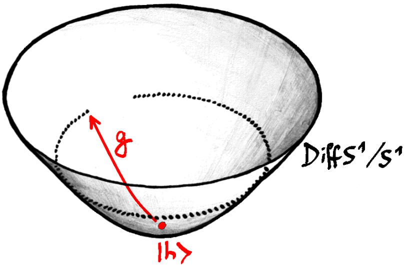

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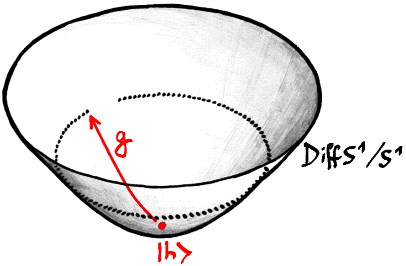

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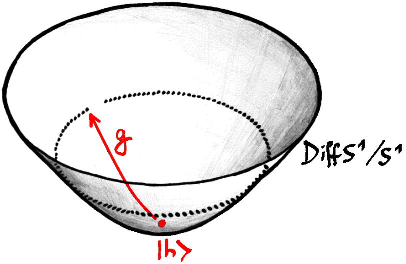

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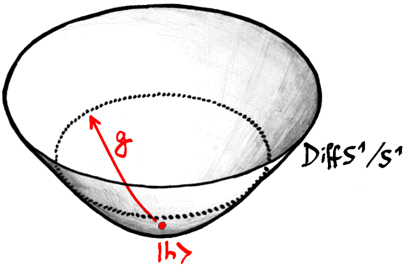

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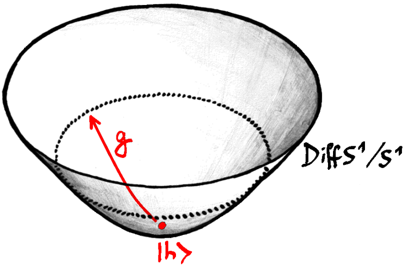

3. Berry phases & asymptotic symmetries

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A. Berry phases in AdS_3

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A. Berry phases in AdS_3

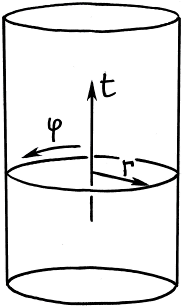
B. Berry phases in flat space

Berry phases in AdS_3

AdS_3 space-time

BERRY PHASES IN AdS_3

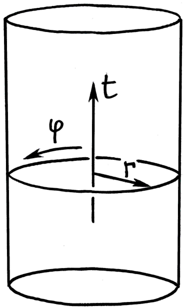
AdS_3 space-time



Berry phases in AdS_3

AdS_3 space-time

- Infinity = time-like cylinder



Berry phases in AdS$_3$

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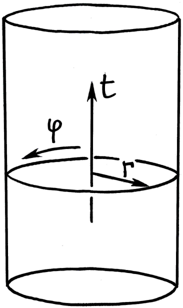

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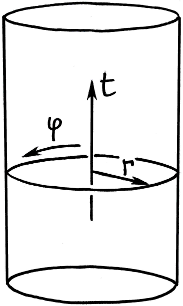

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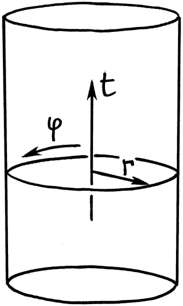

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[Brown-Henneaux 1986]

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  [Brown-Henneaux 1986]

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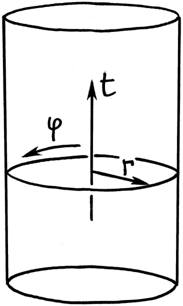

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$$ds^2 = \frac{\ell^2}{r^2}dr^2 - \left(rdx^+ - \frac{4G\ell}{r}\bar{T}(x^-)dx^-\right)\left(rdx^- - \frac{4G\ell}{r}T(x^+)dx^+\right)$$

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BERRY PHASES IN AdS_3

UIRREP of $\text{Diff}\, S^1 \times \text{Diff}\, S^1$

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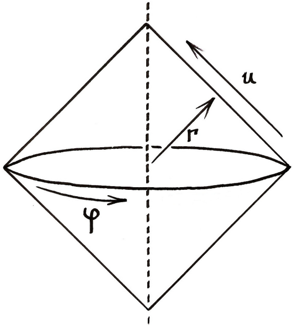
- Let's turn to Minkowskian gravity !

Berry phases in flat space

3D **Minkowski** space-time

Berry phases in flat space

3D **Minkowski** space-time



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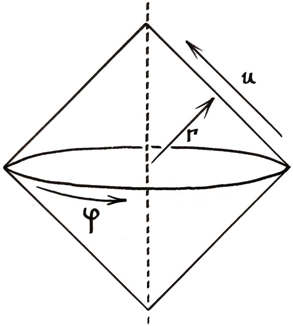
- Future null infinity



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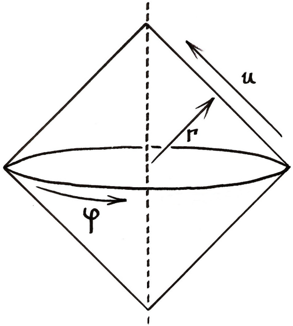

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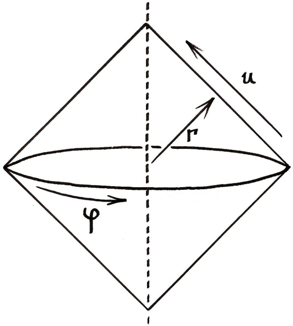
$$\mathbf{BMS}_3 = \text{Diff}\, S^1 \ltimes \text{Vect}\, S^1$$



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- $\varphi \mapsto f(\varphi)$
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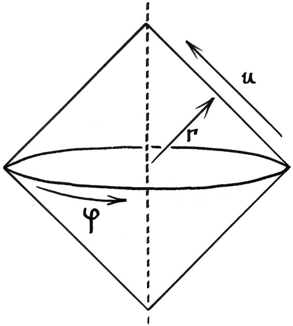

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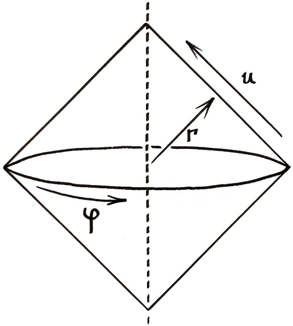

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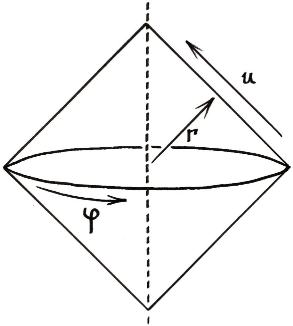

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BMS_3 UIRREPS = dressed particles in flat space

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$\downarrow$

dual to supertranslations

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$\swarrow$ $\searrow$

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Path $f(t)$ without supertranslations

Berry phases in flat space

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Circular path $f(t, \varphi) = g(\varphi) + \omega t$

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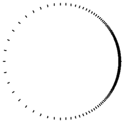

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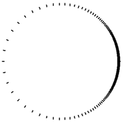

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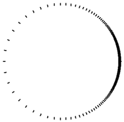

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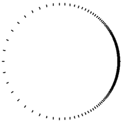

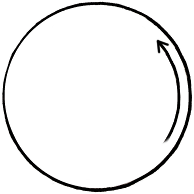

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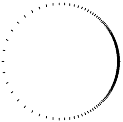

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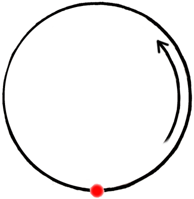

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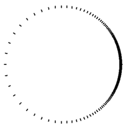

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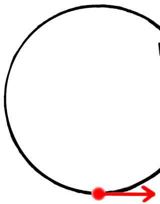

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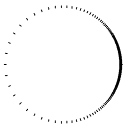

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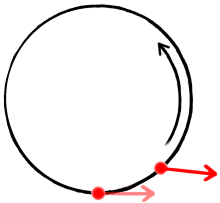

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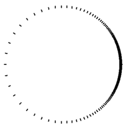

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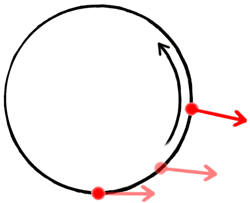

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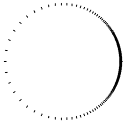

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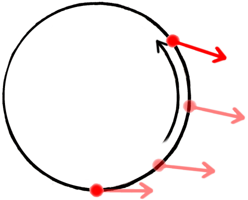

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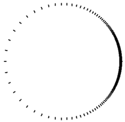

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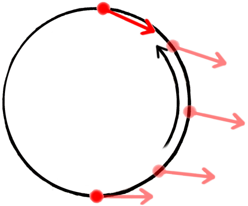

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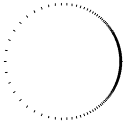

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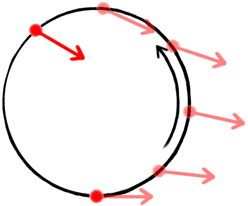

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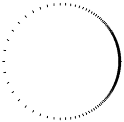

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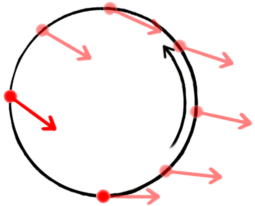

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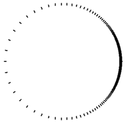

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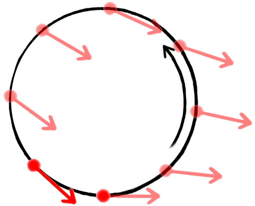

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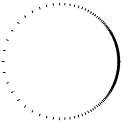

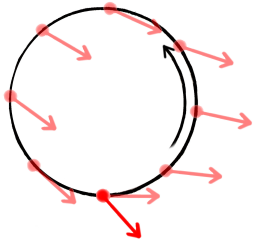

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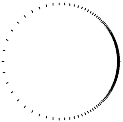

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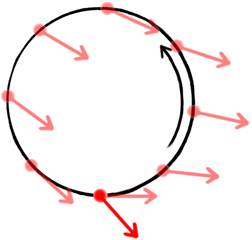

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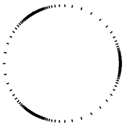

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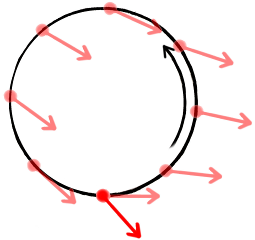

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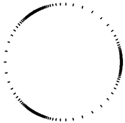

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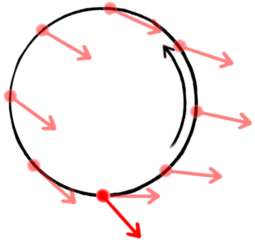

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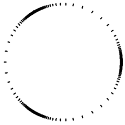

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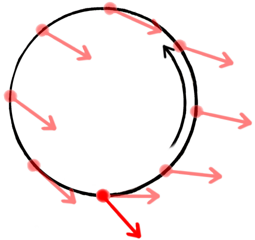

Berry phases in flat space

Paths with supertranslations ?

Berry phases in flat space

Paths with supertranslations :

- $$\boxed{B_{\text{scalar}} = -\int \frac{dt\,d\varphi}{2\pi}\frac{\dot{\alpha}\circ f}{f'}\left[M - \frac{1}{8G} + \frac{1}{4G}\{f;\varphi\}\right]}$$ [B.O. 2017]

Berry phases in flat space

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BERRY PHASES IN FLAT SPACE

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- $$B_{\text{scalar}} = -\int \frac{dt\,d\varphi}{2\pi} \frac{\dot{\alpha} \circ f}{f'} \left[M - \frac{1}{8G} + \frac{1}{4G} \{f;\varphi\} \right]$$ [B.O. 2017]

- Involves Planck mass

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- Observable in quantum **Hall effect** ?

[with T. Neupert & F. Schindler]

Thank you for listening !