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EHUD DE SHALIT, CHARLOTTE HARDOUIN, AND JULIEN ROQUES

Abstract. The differential nature of solutions of linear difference equations over the projective line was recently elucidated. In contrast, little is known about the differential nature of solutions of linear difference equations over elliptic curves. In the present paper, we study power series $f(z)$ with complex coefficients satisfying a linear difference equation over a field of elliptic functions K, with respect to the difference operator $\phi f(z) = f(qz)$, $2 \leq q \in \mathbb{Z}$, arising from an endomorphism of the elliptic curve. Our main theorem says that such an f satisfies, in addition, a polynomial differential equation with coefficients from K, if and only if it belongs to the ring $S = K[z, z^{-1}, \zeta(z, \Lambda)]$ generated over K by z, z^{-1} and the Weierstrass ζ-function. This is the first elliptic extension of recent theorems of Adamczewski, Dreyfus and Hardouin concerning the differential transcendence of solutions of difference equations with coefficients in $\mathbb{C}(z)$, in which various difference operators were considered (shifts, q-difference operators or Mahler operators). While the general approach, of using parametrized Picard-Vessiot theory, is similar, many features, and in particular the emergence of monodromy considerations and the ring S, are unique to the elliptic case and are responsible for non-trivial difficulties. We emphasize that, among the intermediate results, we prove an integrability result for difference-differential systems over elliptic curves which is a genus one analogue of the integrability results obtained by Schäfke and Singer over the projective line.

1. Introduction

In 1886 Otto Hölder [Hol] proved that the Gamma function $\Gamma(z)$ is not only transcendental, but *hypertranscendental*: it does not satisfy any polynomial differential equation $P(f, f', , ..., f^{(r)}) = 0$ whose coefficients are polynomials in z. This lies in contrast to other well-known transcendental functions like $\exp(z)$, hypergeometric functions or theta functions, which all satisfy familiar (and important) polynomial differential equations over the field $\mathbb{C}(z)$ of rational functions.

It turns out that what prevents $\Gamma(z)$ from satisfying a polynomial differential equation over $\mathbb{C}(z)$ is the relation

$$\Gamma(z + 1) - z\Gamma(z) = 0,$$

in itself an instance of a linear *difference equation* in the difference operator $\phi f(z) = f(z+1)$. Indeed, as Adamczewski, Dreyfus and Hardouin showed recently [A-D-H], if F is a field of meromorphic functions on $\mathbb{C}$, invariant under ϕ, satisfying the following two properties

- $F^{\phi} = \{f \in F |\, \phi f = f\} = \mathbb{C}$,

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- $F \cap \mathbb{C}(z, \exp(az) | \, a \in \mathbb{C}) = \mathbb{C}(z)$,

and if $f \in F$ satisfies a linear ϕ-difference equation *and* a polynomial differential equation, both with coefficients from $\mathbb{C}(z)$, then $f \in \mathbb{C}(z)$. This includes Hölder's theorem as a special case. The above-cited paper considered several other theorems of the same nature, pertaining to difference operators that are shifts, q-difference operators or Mahler operators, all over the ground field $\mathbb{C}(z)$. This gives us a good understanding of the differential nature of solutions of difference equations on $\mathbb{P}^1(\mathbb{C})$.

By Hurwitz's automorphisms theorem, any automorphism of a compact Riemann surface of genus $g > 1$ is of finite order. Therefore, the algebro-differential study of solutions of difference equations over Riemann surfaces reduces to the genus zero or one case. Indeed, the structure of a difference equation associated with a finite order automorphism is not rich enough to capture the algebraic nature of its solutions. For genus one Riemann surfaces, that is, complex elliptic curves, there are essentially two kinds of endomorphisms: translations by a point of the curve and isogenies.

Our goal in the present work is to consider elliptic function fields as ground fields, and a difference operator arising from an isogeny of the elliptic curve[1]. In this framework, we prove an analogue of the main result of [A-D-H]. In our case, the dichotomy between rational solutions (that is, those belonging to the function field of the curve) and differentially transcendental solutions, observed in the genus zero case, is no longer true. Overcoming this difficulty leads to an explicit description of all the differentially algebraic solutions in terms of special functions. To describe our main result, let us fix some notation.

Let Λ_0 be a lattice in $\mathbb{C}$ and K the field of all meromorphic functions in the complex plane that are Λ-periodic with respect to some sublattice $\Lambda \subset \Lambda_0$. Fix $2 \leq q \in \mathbb{Z}$. The field K admits an *automorphism* ϕ and a *derivation* ∂ given by

$$\phi f(z) = f(qz), \quad \partial(f) = f', \quad \partial \circ \phi = q\phi \circ \partial.$$

Both ϕ and ∂ extend to M, the field of all meromorphic functions in the complex plane, and to the field of Laurent power series $F = \mathbb{C}((z))$. Clearly, $K \subset M \subset F$.

A power series $f \in F$ is said to satisfy a linear homogeneous *ϕ-difference equation* (called also a q-difference equation) over K if

$$\sum_{i=0}^{n} a_{n-i}\phi^i f = \sum_{i=0}^{n} a_{n-i}(z) f(q^i z) = 0 \tag{1.1}$$

for some n and $a_j \in K$. It is called *∂-algebraic* over K if there exists a nonzero polynomial $P \in K[X_0, ..., X_r]$, for some $r \geq 0$, such that

$$P(f, \partial f, ..., \partial^r f) = 0.$$

A power series that is not ∂-algebraic is called *hypertranscendental (over K)*.

The main result of this paper is the following complete description of the ∂-algebraic solutions in F of ϕ-difference equations of the form (1.1).

Theorem (Theorem 38)**.** *Assume that $f \in F$ satisfies a non-trivial linear homogeneous ϕ-difference equation over K. Then, the following properties are equivalent:*

[1] The case of a translation on the elliptic curve will be addressed in a forthcoming paper by the last two authors.

(1) f is ∂-algebraic over K;
(2) f lies in the ring generated over K by $z^{\pm 1}$ and the Weierstrass zeta function $\zeta(z, \Lambda_0)$ of Λ_0, i.e.,

$$f \in S = K[z^{-1}, z, \zeta(z, \Lambda_0)].$$

The ring S seems ubiquitous when studying functional equations over elliptic curves. It appeared also in [dS21], in the classification of elliptic (p, q)-difference modules (for p and q relatively prime ≥ 2). Its emergence in that work was attributed to the appearance of certain non-trivial vector bundles (Atiyah's bundles) over the elliptic curve $\mathbb{C}/\Lambda_0$. In the present work, the same ring S, or rather its subring $S_0 = K[z, \zeta(z, \Lambda_0)]$, arises, in the context of *linear* differential equations over K, from *monodromy considerations* (see Theorem 36). Once we pass to arbitrary *polynomial* differential equations over K, the ring S arises as the *Picard-Vessiot ring* of a certain fundamental ∂-integrable ϕ-module. Some properties of the ring S that are needed in later proofs are explored in §2 and Appendix A.

As an intermediate step toward the proof of the above theorem, we prove the following integrability result describing the elements in F that satisfy both a linear homogeneous ϕ-difference equation, and a *linear homogeneous ordinary differential equation* over K. It is the first elliptic analogue of several integrability results obtained for difference equations over the projective line, see [Ramis92, Bez93, Bez94, Sch-Sin].

Theorem (Corollary 37)**.** *Assume that $f \in F$ satisfies a linear homogeneous ϕ-difference equation over K. Then, the following properties are equivalent:*
(1) f satisfies a linear homogeneous ordinary differential equation over K;
(2) f lies in the ring S_0 generated over K by z and the Weierstrass zeta function $\zeta(z, \Lambda_0)$ of Λ_0, i.e.,

$$f \in S_0 = K[z, \zeta(z, \Lambda_0)].$$

As implied above, the proof of this result uses monodromy considerations for modules with a connection over K, which admit, in addition, a ϕ-structure (termed *ϕ-isomonodromic*). Dually, these modules can be regarded as ∂-integrable ϕ-modules. Section 3 reviews the language of ϕ-, ∂- and (ϕ, ∂)-modules, its relation to linear systems of ϕ-difference and differential equations, and the important concepts of ϕ-isomonodromy and ∂-integrability. Sections 6,7 and 8 lead the way to Theorem 36 and Corollary 37, but the results obtained on the way, like the relation between integrability and solvability (Corollary 35) are of independent interest, and are used again in the proof of the Main Theorem.

Once we consider arbitrary polynomial differential equations over K, we are forced to use the machinery of δ-parametrized Picard Vessiot (PPV) theory of ϕ-difference equations, and δ-parametrized ϕ-difference Galois theory. This theory, expounded in [H-S08], but less familiar than classical (non-parametrized) Picard Vessiot theory, is used in our work in roughly the same manner as in [A-D-H], where the ground field is $\mathbb{C}(z)$. The details pertaining to the ground field are, of course, different. In particular, since PPV theory only works under the assumption that ϕ and δ commute, we must replace the derivation ∂ by $\delta = z\partial$. This derivation, however, exists only over the field $K' = K(z)$, forcing us to base-change from K to K', and then use descent arguments. Classical and δ-parametrized Picard

Vessiot theory are surveyed in Section 4. Section 5 connects the PPV theory to the framework of ϕ-modules, providing a Galoisian criterion for δ-integrability (a variant of results already to be found in the literature).

The proof of the Main Theorem occupies Sections 9-12. Similarly to the program carried out in [A-D-H] over $\mathbb{C}(z)$, it starts with the rank 1 case, where again, some elliptic function theory is needed. We then consider an extreme case, in which the difference Galois group G associated with our ϕ-module is *simple.* A deep theorem of Cassidy (Theorem 19 of [Cas89], see also Theorem 22 below) allows us to deduce that in this case, the fundamental matrix of solutions has a maximal δ-transcendence degree, and in particular any particular solution is hypertranscendental. This also settles the Main Theorem in the *irreducible case, i.e.*, when the ϕ-module associated to the difference equation satisfied by f, is irreducible, or, equivalently, when the standard representation of G is irreducible.

To treat the general case, we use an inductive approach, similar to the one used in [A-D-H]. The rank 1 case, proved right at the beginning, becomes instrumental. The last stage of the proof may be described as "Galois acrobatics". At the very final step, Proposition 7, proved by a technical tour de force and unique to the elliptic set-up, plays a crucial role.

We end with a word for the experts, regarding our use of δ-parametrized Picard Vessiot theory, and linear differential algebraic groups (LDAG's) in general. We work in the classical language of Weil and Kolchin, over a "universal" differentially closed field of constants $\mathbb{C} \subset \widetilde{C}$, requiring descent arguments to come down to K and S. While a scheme-theoretic or Tannakian approach has been developed by various authors to some extent, not all the results we need are in the literature, and estabilishing them would have taken us beyond the scope of this work.

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2. Elliptic functions and related rings

2.1. The ground field K, the automorphism ϕ and the derivation ∂.

In this paper, we use standard notation of difference and differential algebra which can be found in [Kol], [Cohn]. Algebraic attributes (e.g. Noetherian) are understood to apply to the underlying ring. Attributes that apply to the difference (resp. differential) structure are usually prefixed with ϕ (resp. ∂). For instance, a ϕ-ring is a ring with an endomorphism ϕ, a ϕ-ideal is an ideal of a ϕ-ring that is set-wise invariant by ϕ etc.

2.1.1. *The field K.* For a lattice $\Lambda \subset \mathbb{C}$ we denote by K_Λ the field of Λ-periodic meromorphic functions (Λ-elliptic functions). It is well known that

$$K_\Lambda = \mathbb{C}(\wp(z,\Lambda), \wp'(z,\Lambda))$$

is generated over $\mathbb{C}$ by the Weierstrass $\wp$-function

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{0\neq\omega\in\Lambda}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

and its derivative. If $\Lambda' \subset \Lambda$ is a sublattice of Λ then $K_\Lambda \subset K_{\Lambda'}$. Two lattices Λ and Λ' are called *commensurable* if $\Lambda \cap \Lambda'$ is a lattice, necessarily of finite index in each of them. Equivalently, Λ and Λ' are commensurable if their $\mathbb{Q}$- spans coincide: $\mathbb{Q}\Lambda = \mathbb{Q}\Lambda'$. The notion of being commensurable is an equivalence relation on the set of lattices. Fix an equivalence class $\mathfrak{L}$ (called a commensurability class) and let

$$K = \bigcup_{\Lambda \in \mathfrak{L}} K_\Lambda.$$

It is readily seen that K is a field, indeed equal to the union of K_Λ for all sublattices $\Lambda \subset \Lambda_0$, if Λ_0 is any given member of the class $\mathfrak{L}$. If E_Λ is the complex elliptic curve whose associated Riemann surface is $\mathbb{C}/\Lambda$, then K_{Λ_0} is the field of rational functions on E_{Λ_0}, and K is the field of rational functions on its universal cover (in the algebraic sense) $\lim_{\leftarrow} E_\Lambda$, where the limit ranges over all the unramified coverings $E_\Lambda \to E_{\Lambda_0}$ $(\Lambda \subset \Lambda_0)$.

2.1.2. *The automorphism ϕ.* Let

$$\phi(z) = qz$$

for $q \in \mathbb{Z}_{\geq 2}$. As ϕ preserves every $\Lambda \in \mathfrak{L}$, it induces a non-trivial endomorphism (an isogeny) of E_Λ, and an endomorphism ϕ of K_Λ such that $(\phi f)(z) = f(qz)$ for any f in K_Λ. As $\phi(K_{q\Lambda}) = K_\Lambda \subset K_{q\Lambda}$, we see that ϕ induces an *automorphism* of the field K. In the language of difference algebra, ϕ is a *difference operator*, and the pair (K, ϕ) is a difference field.

Remark 1. If the lattices $\Lambda \in \mathfrak{L}$ admit complex multiplication in a quadratic imaginary field k, then q could be taken to be any non-zero non-unit element of the ring of integers $\mathcal{O}_k$. For simplicity, however, we assume that q is a rational integer.

The following result will be used much later.

Lemma 2. *The field K does not admit any finite field extension L/K to which ϕ extends as an automorphism.*

Proof. See Proposition 7(iii) in [dS23]. □

2.1.3. *The field of Laurent series F.* Associating to an elliptic function $f \in K$ its Taylor-Maclaurin expansion at 0, the field K embeds in the field of Laurent series

$$F = \mathbb{C}((z)).$$

Clearly ϕ induces an automorphism of F as well, compatible with the embedding $K \hookrightarrow F$. As the *field of ϕ-constants*

$$F^\phi := \{f \in F \mid \phi f = f\} = \mathbb{C},$$

a-fortiori $K^\phi := \{f \in K \mid \phi f = f\} = \mathbb{C}$.

2.1.4. *The derivation ∂.* The fields K and F are equipped with the derivation

$$\partial = \frac{d}{dz}.$$

It satisfies the *commutation relation*

$$\partial \circ \phi = q\phi \circ \partial$$

with the automorphism ϕ.

2.2. **The Weierstrass zeta function $\zeta(z, \Lambda)$ and the rings S_0 and S.**

2.2.1. *The two rings.* From now on all the lattices Λ will belong to the commensurability class $\mathfrak{L}$ used to define K. The *Weierstrass zeta function* of Λ

$$\zeta(z,\Lambda)=\frac{1}{z}+\sum_{0\neq\omega\in\Lambda}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right)$$

is everywhere meromorphic on $\mathbb{C}$, has simple poles with residue 1 at the points of Λ and only there, and is characterized, up to an additive constant, by the relation $\zeta'(z,\Lambda)=-\wp(z,\Lambda)$. Its periods along $\omega\in\Lambda$,

$$\eta(\omega,\Lambda)=\zeta(z_0+\omega,\Lambda)-\zeta(z_0,\Lambda)=-\int_{z_0}^{z_0+\omega}\wp(z,\Lambda)dz$$

(independent of z_0), are given by the Legendre η-function. If (ω_1,ω_2) is an oriented basis of Λ (meaning that $\mathrm{Im}(\omega_1/\omega_2)>0$) and $\eta_i=\eta(\omega_i,\Lambda)$, then the *Legendre period relation*

$$\omega_1\eta_2-\omega_2\eta_1=2\pi i \tag{2.1}$$

holds.

Lemma 3. *The space $\mathbb{C}z+\mathbb{C}\zeta(z,\Lambda)$ realizes every homomorphism $\chi:\Lambda\to\mathbb{C}$,* i.e.*, for any such χ we have a unique function $h_\chi=az+b\zeta(z,\Lambda)$ with $a,b\in\mathbb{C}$ such that the ω-period*

$$\int_{z_0}^{z_0+\omega}dh_\chi=h_\chi(z_0+\omega)-h_\chi(z_0)$$

is $\chi(\omega)$ for any $\omega\in\Lambda$.

Proof. The ω_1- and ω_2-periods of $h_\chi=az+b\zeta(z,\Lambda)$ are given by $a\omega_1+b\eta_1$ and $a\omega_2+b\eta_2$ respectively. In order to prove the lemma, we thus have to prove that, for any $(\chi_1,\chi_2)\in\mathbb{C}^2$, there exist unique $(a,b)\in\mathbb{C}^2$ such that

$$\chi_1=a\omega_1+b\eta_1 \text{ and } \chi_2=a\omega_2+b\eta_2.$$

That this is true follows immediately from the fact that the matrix

$$\begin{pmatrix}\omega_1 & \eta_1\\ \omega_2 & \eta_2\end{pmatrix}$$

is invertible because it has nonzero determinant in virtue of (2.1). □

Lemma 4. *(i) If $\Lambda'\subset\Lambda$ is a sublattice then*

$$\zeta(z,\Lambda)-[\Lambda:\Lambda']\zeta(z,\Lambda')\in K+\mathbb{C}z+\mathbb{C}\subset K[z].$$

(ii) The rings of meromorphic functions

$$S_0=K[z,\zeta(z,\Lambda)]$$

and

$$S=K[z,z^{-1},\zeta(z,\Lambda)]$$

do not depend on the choice of the lattice Λ in $\mathfrak{L}$.

Proof. (i) The meromorphic function

$$\wp(z,\Lambda)-\sum_{\overline{\omega}\in\Lambda/\Lambda'}\wp(z+\omega,\Lambda')$$

is Λ-periodic and its poles are contained in Λ. But at 0 the poles of $\wp(z,\Lambda)$ and of $\wp(z,\Lambda')$ cancel each other, while the other terms have no pole. It follows that this Λ-periodic function has no poles, hence is a constant. Integrating, we find that

$$\zeta(z,\Lambda)-\sum_{\overline{\omega}\in\Lambda/\Lambda'}\zeta(z+\omega,\Lambda')=az+b$$

for some $a,b\in\mathbb{C}$. On the other hand $\zeta(z+\omega,\Lambda')-\zeta(z,\Lambda')\in K_{\Lambda'}\subset K$. It follows that

$$\zeta(z,\Lambda)-[\Lambda:\Lambda']\zeta(z,\Lambda')\in K+\mathbb{C}z+\mathbb{C}.$$

(ii) This follows easily from (i). □

2.2.2. *Algebraic independence of z and $\zeta(z,\Lambda)$ over K.*

Lemma 5. *The functions z and $\zeta(z,\Lambda)$ are algebraically independent over K.*

Proof. Since K is algebraic over K_Λ, it is enough to show that they are algebraically independent over K_Λ. The choice of the lattice Λ being clear from context, we abbreviate $\zeta(z,\Lambda)$ by ζ. Let (ω_1,ω_2) be an oriented basis of Λ. Lemma 3 ensures that there are linear combinations

$$u=az+b\zeta,\ \ v=cz+d\zeta$$

with coefficients $a,b,c,d\in\mathbb{C}$ such that u is ω_2-periodic but $u(z+\omega_1)-u(z)=1$, while v is ω_1-periodic, but $v(z+\omega_2)-v(z)=1$. Since $K_\Lambda(z,\zeta)=K_\Lambda(u,v)$, it is enough to show that u and v are algebraically independent over K_Λ. Suppose there were a nontrivial algebraic relation

$$\sum a_{ij}u^iv^j=0,$$

with $a_{ij}\in K_\Lambda$. Pick such a relation of lowest total degree. Without loss of generality u appears in it. The coefficient of the highest power of u, denoted $a_0(v)$, is a polynomial in $K_\Lambda[v]$ that does not vanish identically, as otherwise $a_0(v)=0$ would be a relation of lower total degree between u and v. Divide by it and rewrite the relation as

$$u^n+\alpha_1u^{n-1}+\cdots+\alpha_{n-1}u+\alpha_n=0$$

where $n\geq 1$, $\alpha_i\in K_\Lambda(v)$. Let z_0 be a point where $u(z_0)$ and all the $\alpha_i(z_0)$ are analytic, and $z_k=z_0+k\omega_1$. Using the periodicity of the α_i in ω_1 we get, after evaluating at z_k and using the fact that $u(z_k)=u(z_0)+k$,

$$(u(z_0)+k)^n+\alpha_1(z_0)(u(z_0)+k)^{n-1}+\cdots+\alpha_{n-1}(z_0)(u(z_0)+k)+\alpha_n(z_0)=0$$

Thus, the polynomial $X^n+\alpha_1(z_0)X^{n-1}+\cdots+\alpha_{n-1}(z_0)X+\alpha_n(z_0)$ has infinitely many roots and this yields a contradiction. □

2.2.3. *∂-simplicity of S_0 and S.* The derivation ∂ extends from K to S_0 and S. A ring with a derivation ∂ is called *∂-simple* if it does not admit a non-trivial ideal I invariant under ∂.

Lemma 6. *The rings S_0 and S are ∂-simple.*

Proof. Since S is a localization of S_0, it is enough to check the assertion for S_0. Let $\zeta=\zeta(z,\Lambda)$. The functions z,ζ are algebraically independent over K, hence every element of S_0 has a unique expression

$$f=\sum a_{i,j}z^i\zeta^j,\ \ a_{i,j}\in K.$$

Order the pairs (i,j) lexicographically (first by i, then by j). If $f\neq 0$, we denote the maximal (i,j) for which $a_{i,j}\neq 0$ by $d(f)$.

Let I be a non-zero proper ∂-ideal, and consider $0\neq f\in I$ of minimal $d(f)$. Since I is proper, $d(f)=(i_0,j_0)>(0,0)$. We may also assume that $a_{i_0,j_0}=1$. Since $\partial(z)=1$ and $\partial(\zeta)\in K$,

$$d(\partial(f))<d(f).$$

By our assumption, $\partial(f)\in I$, hence by the minimality of $d(f)$, we must have $\partial(f)=0$. This forces f to be constant, a contradiction. □

The ring S (but not S_0) is also ϕ-simple, see Lemma 18 below.

2.3. **A technical result on the ring S_0.** The following Proposition is a crucial ingredient in the proof of the main theorem of the paper. However, it will be needed only at the very end, and its proof, which is lengthy, is deferred to Appendix A.

Besides ∂, the rings S_0 and S (but not K) carry also the derivation

$$\delta=z\partial=z\frac{d}{dz}.$$

The advantage of δ over ∂ is that it commutes with ϕ: $\delta\circ\phi=\phi\circ\delta$.

Proposition 7. *Let $f,g\in S_\Lambda=K_\Lambda[z,\zeta(z,\Lambda)]$, $a,c\in\mathbb{C}$ and $p\in\mathbb{C}[z]$ be such that*

$$(\delta-c)(g)=(\phi-a)(f)+p.$$

Then $g=(\phi-a)(f_1)+p_1$ for some $f_1\in S_\Lambda$ and $p_1\in\mathbb{C}[z]$. Furthermore, if $a=q^r$ for some $r\geq 0$ we can take $p_1=dz^r$, $d\in\mathbb{C}$, and otherwise we can take $p_1=0$.

Corollary 8. *Let $\mathscr{L}\in\mathbb{C}[\delta]$ be a monic polynomial in δ, $a\in\mathbb{C}$ and $f,b\in S_\Lambda$ such that*

$$\mathscr{L}(b)=(\phi-a)(f).$$

Then $b=(\phi-a)(h)+p$ for some $h\in S_\Lambda$ and $p\in\mathbb{C}[z]$. Furthermore, if $a=q^r$ for some $r\geq 0$ we can take $p=dz^r$, $d\in\mathbb{C}$, and otherwise we can take $p=0$.

Proof. (of corollary) Factor $\mathscr{L}=(\delta-c_k)\circ\cdots\circ(\delta-c_1)$ with $c_i\in\mathbb{C}$, and use descending induction on $0\leq j\leq k-1$ to find $f_j\in S_\Lambda$ and $p_j=d_jz^r\in\mathbb{C}[z]$ (with $d_j=0$ if $a\notin\{1,q,q^2,...\}$) such that

$$(\delta-c_j)\circ\cdots\circ(\delta-c_1)(b)=(\phi-a)(f_j)+p_j.$$

At the end of the induction, set $h=f_0$ and $p=p_0$. □

3. Linear difference and differential equations, and their associated modules

The purpose of this section is to review some standard results and set up notation.

3.1. **Systems of linear difference equations and difference modules.** References for the results mentioned below may be found in [S-vdP97, Section 1.4].

3.1.1. *Systems and modules.* Let (K, ϕ) be a field of characteristic 0, equipped with an automorphism ϕ, called a *difference operator.*

To avoid trivialities we assume that ϕ is of infinite order: no positive power of ϕ is the identity. We denote by

$$C = K^{\phi} = \{f \in K \mid \phi(f) = f\}$$

the *field of ϕ-constants*, and assume that it is algebraically closed, although for most of what we do, at least in the beginning, this is not essential.

A system of linear equations

$$\phi(y) = Ay, \tag{3.1}$$

where $A \in \mathrm{GL}_n(K)$, is called a linear system of ϕ-difference equations. One seeks solutions $y \in R^n$ where (R, ϕ) is a ϕ-ring extension of (K, ϕ), *i.e.*, a ring extension together with a compatible extension of ϕ.

A *difference module* over K (called also a *ϕ-module*) is a pair (W, Φ) where W is a finite dimensional K-vector space and Φ a bijective ϕ-semilinear endomorphism of W, *i*.e., $\Phi : W \to W$ is a bijective map such that, for all $a \in K$ and $w, w' \in W$,

$$\Phi(aw + w') = \phi(a)\Phi(w) + \Phi(w').$$

To the system (3.1) we attach the difference module (K^n, Φ) with $\Phi(y) = A^{-1}\phi(y)$. Conversely, if (W, Φ) is a difference module, $e_1, ..., e_n$ is a basis of W and

$$\Phi(e_j) = \sum_{i=1}^{n} a_{ij} e_i,$$

we attach to (W, Φ) the system (3.1) with $A^{-1} = (a_{ij})$. This process is not canonical, as it depends on the chosen basis $e_1, ..., e_n$ of W. If $\widetilde{e}_1, ..., \widetilde{e}_n$ is another basis of the same module and

$$e_j = \sum_{i=1}^{n} p_{ij} \widetilde{e}_i,$$

so that $P = (p_{ij})$ is the change-of-basis matrix, then we get the system $\phi(y) = \widetilde{A}y$ where

$$\widetilde{A} = \phi(P)AP^{-1}.$$

Two matrices A and $\widetilde{A}$ related by such a relation with $P \in \mathrm{GL}_n(K)$ are said to be *gauge equivalent* over K. The above procedure establishes a bijection between systems (3.1), up to gauge equivalence over K, and difference modules (W, Φ), up to isomorphism over K.

A difference module isomorphic to (K^n, ϕ) is called *trivial.* The module associated to (3.1) is trivial if and only if A is gauge equivalent to the identity matrix.

3.1.2. *Base change and solutions set.* If (R, ϕ) is a ϕ-ring extension of (K, ϕ) and (W, Φ) is a difference module over K, then we may consider its base change

$$(W_R, \Phi_R) = (R \otimes_K W, \phi \otimes \Phi).$$

Note that W_R is free of rank $n = \dim_K W$ over R. Let $C_R = R^{\phi}$ be the ring of ϕ-constants of R. If R is a *ϕ-simple* ϕ-ring (does not have any non-trivial ϕ-ideal), and in particular if it is a field, then C_R is a field. Assume from now on that R is ϕ-simple. We continue to denote Φ_R simply by Φ. The set W_R^{Φ} of Φ-fixed vectors of W_R is a vector space over C_R. The Casoratian Lemma asserts that

$$R \otimes_{C_R} W_R^{\Phi} \hookrightarrow R \otimes_K W \tag{3.2}$$

is injective, and in particular $\dim_{C_R} W_R^\Phi \le \dim_K W$, as can be seen by calculating the R-ranks of the two sides.

If $(W, \Phi) = (K^n, A^{-1}\phi)$ is the difference module attached to the system (3.1), then the fixed vectors $\mathcal{U}_R = W_R^\Phi$ are the solutions of the system in R. For this reason we sometimes call it the *solution set*. We say that the system (3.1) attains a *full set of solutions* over a ϕ-simple ϕ-ring R if $\dim_{C_R} \mathcal{U}_R = n$. This is equivalent to the existence of a matrix $U \in \mathrm{GL}_n(R)$ satisfying $\phi(U) = AU$. Such a matrix is called a *fundamental matrix* over R, and its columns span $\mathcal{U}_R = UC_R^n$. If U' is another fundamental matrix over R then

$$U' = UT$$

with $T \in \mathrm{GL}_n(C_R)$. Yet another way to say that (3.1) attains a full set of solutions over R is that (W_R, Φ_R) is trivial: the embedding (3.2) is an isomorphism.

Assume that W attains a full set of solutions over the ϕ-simple ϕ-ring R and R' is a ϕ-ring extension of R (not necessarily ϕ-simple) such that $C_{R'} = C_R$. If $U \in \mathrm{GL}_n(R)$ is a fundamental matrix and $v \in W_{R'}^\Phi$, then $U^{-1}v$ is fixed by ϕ, hence $v \in UC_R^n = \mathcal{U}_R$. Thus the space of solutions does not grow when R grows, as long as the ring of ϕ-constants stays the same.

3.1.3. *An irreducibility criterion.* We shall need the following lemma. Compare Lemma 4.6 of [A-D-H-W], where it is deduced from the cyclic vector lemma. We give a more direct proof here. A ϕ-module W over K is called irreducible if it does not admit any ϕ-submodule other than 0 and W itself.

Lemma 9. *Let $K' = K(z)$ be a transcendental extension of K and W a ϕ-module over K. Extend ϕ to K' by the rule $\phi(z) = qz$, where $q \in C^\times$, $C = K^\phi$. Then W is irreducible if and only if $W_{K'}$ is irreducible.*

Proof. It is clear that if W is reducible, so is $W_{K'}$. To prove the converse, let $k = K((z))$ and $R = K[[z]]$, and extend the action of ϕ to them, so that $K \subset K' \subset k$ is a tower of ϕ-fields. If $W_{K'}$ is reducible, so is W_k, so it is enough to prove that if W_k is reducible, so is W.

Let $V \subset W_k$ be a ϕ-submodule over k, with $0 < \dim_k V < \dim_K W$. Then $V_0 = V \cap W_R$ is a ϕ-submodule of W_R over R. Since W_R is a free R-module and since R is a principal ideal domain, V_0 is a free R-module as well. Using the fact that k is the quotient ring of R, it is easily seen that the maximal number of R-linearly independent elements of V_0 is equal to the maximal number of k-linearly independent elements of V, so $\mathrm{rk}_R V_0 = \dim_k V$.

The inclusion $V_0 \subset W_R$ of ϕ-modules over R induces the monomorphism

$$\overline{V_0} := V_0/(V_0 \cap zW_R) \hookrightarrow \overline{W_R} := W_R/zW_R$$

of ϕ-modules over K. Note that $V_0 \cap zW_R = V \cap zW_R = z(V \cap W_R) = zV_0$, so $\overline{V_0} = V_0/zV_0$ and, hence, $\dim_K \overline{V_0} = \mathrm{rk}_R V_0 = \dim_k V$. Moreover, $\overline{W_R}$ and W are clearly isomorphic as ϕ-modules over K. Therefore, W has a ϕ-submodule over K of dimension $\dim_k V$ and, hence, is reducible.

□

Remark 10. With the notations of the previous proof, we stress that, in general, the base change of $\overline{V_0}$ to k need not be V, and the submodule V need not descend to K.

3.2. **Systems of linear differential equations and modules with connections.** This analogue of the difference equation set-up is even more classical, see [S-vdP03]. We give it only to fix notation.

3.2.1. *Systems and modules.* Let (K, ∂) be a field of characteristic 0, equipped with a non-trivial derivation. We denote by $C = K^\partial$ the kernel of ∂, and call it the field of *∂-constants.* Although not essential for many of the arguments, we assume that C is algebraically closed.

Assume we are given a *linear system of differential equations*

$$\partial(y) = By, \tag{3.3}$$

where $B \in \mathfrak{gl}_n(K)$. One seeks solutions in ∂-ring extensions (R, ∂) of K.

A *module with a connection* (W, ∇) over K (called also a ∂-module) is a finite dimensional vector space over K equipped with a ∂-connection, *i.e.*, an additive map $\nabla : M \to M$ such that $\nabla(aw) = \partial(a)w + a\nabla(w)$ for all a in K and w in W. To the system (3.3) one attaches the module

$$W = K^n, \ \ \nabla(y) = \partial(y) - By.$$

Conversely, if (W, ∇) is a module with a connection and $e_1, ..., e_n$ is a basis of W, then

$$\nabla(e_j) = \sum_{i=1}^n b_{ij} e_i,$$

and we associate to it the system (3.3) with $B = -(b_{ij})$. If $\widetilde{e}_1, ..., \widetilde{e}_n$ is another basis of the same module and

$$e_j = \sum_{i=1}^n p_{ij} \widetilde{e}_i,$$

then the matrix giving ∇ in the new basis is $-\widetilde{B}$ where

$$\widetilde{B} = PBP^{-1} + \partial P \cdot P^{-1}.$$

We call a pair of matrices $B, \widetilde{B}$ related as above *gauge equivalent*. The above procedure establishes a bijection between systems (3.3), up to gauge equivalence, and modules with a connection (W, ∇), up to isomorphism.

A *trivial* ∂-module is a module isomorphic to (K^n, ∂). The module associated to (3.3) is trivial if and only if B is gauge equivalent to the zero matrix.

3.2.2. *Base change and solutions set.* If (R, ∂) is a ∂-ring extension of (K, ∂) and (W, ∇) is a ∂-module over K, then we may consider its base change

$$(W_R, \nabla_R) = (R \otimes_K W, \partial \otimes 1 + 1 \otimes \nabla).$$

Note that W_R is free of rank $n = \dim_K W$ over R. Let $C_R = R^\partial$ be the ring of ∂-constants of R. If R is a *∂-simple* ∂-ring (does not have any non-trivial ∂-ideal), and in particular if it is a field, then C_R is a field. Assume from now on that R is ∂-simple, and recall that ∂-simple rings are domains (have no zero divisors).

The kernel of ∇ in W_R, denoted W_R^∇, is a vector space over C_R. The Wronskian Lemma asserts that

$$R \otimes_{C_R} W_R^\nabla \hookrightarrow R \otimes_K W \tag{3.4}$$

is injective, and in particular $\dim_{C_R} W_R^\nabla \leq \dim_K W$, as can be seen by calculating the R-ranks of the two sides.

If $(W,\nabla)=(K^n,\partial-B)$ is the ∂-module attached to the system (3.3), then $\mathcal{U}_R=W_R^\nabla$ are the solutions of the system over R. For this reason we sometimes call it the *solution set*. We say that the system (3.3) attains a *full set of solutions* over a ∂-simple ∂-ring R if $\dim_{C_R}\mathcal{U}_R=n$. This is equivalent to the existence of a matrix $U\in \mathrm{GL}_n(R)$ satisfying $\partial(U)=BU$. Such a matrix is called a *fundamental matrix* over R, and its columns span $\mathcal{U}_R=UC_R^n$. If U' is another fundamental matrix over R then

$$U'=UT$$

with $T\in\mathrm{GL}_n(C_R)$. Yet another way to say that (3.3) attains a full set of solutions over R is that (W_R,∇_R) is trivial: the embedding (3.4) is an isomorphism.

Assume that W attains a full set of solutions over the ∂-simple ∂-ring R and R' is a ∂-ring extension of R (not necessarily ∂-simple) such that $C_{R'}=C_R$. If $U\in\mathrm{GL}_n(R)$ is a fundamental matrix and $v\in W_{R'}^\nabla$, then $U^{-1}v$ is killed by ∂, hence $v\in UC_R^n=\mathcal{U}_R$. Thus the space of solutions does not grow when R grows, as long as the ring of ∂-constants stays the same.

3.3. **Systems of linear (ϕ,∂)-equations and (ϕ,∂)-modules.**

3.3.1. *(ϕ,∂)-fields.* Consider a triplet (K,ϕ,∂) where K is a field of characteristic zero, ϕ is an automorphism, and ∂ a derivation satisfying

$$\partial\circ\phi=q\phi\circ\partial.$$

We assume that $q\in K^\phi\cap K^\partial$. Such a triplet will be called a (ϕ,∂)-field. The main examples considered in this work are:

- K,ϕ and ∂ are as in § 2.1.
- The same example, where K is replaced by the field M of meromorphic functions on $\mathbb{C}$, or by the field $F=\mathbb{C}((z))$.
- K is replaced by $K'=K(z)\subset M$ or by F, ϕ is the same, but ∂ is replaced by the derivation $\delta=z\partial$. In this example $\delta\circ\phi=\phi\circ\delta$, so the q appearing in the commutation relation is not the q defining ϕ, but 1.

We let C be the field of ϕ-constants in K. In the examples mentioned above $C=\mathbb{C}$. It coincides with the field $K^\partial=\{f\,|\,\partial(f)=0\}$ of ∂-constants, so we simply call it the field of constants. However, we shall have to consider later on an extension $(\widetilde{K}',\phi,\delta)$ of $(K(z),\phi,\delta)$, in which the δ-field$(\widetilde{K}')^\phi=\widetilde{C}$ is a *differential closure* of $\mathbb{C}$. The differential closure $\widetilde{C}$ of $\mathbb{C}$ is a δ-field extension of $\mathbb{C}$, unique up to δ-isomorphism, characterized by the following two properties:

- $\widetilde{C}$ is *differentially closed*: Every finite system of polynomial differential equations in several variables in the operator δ, with coefficients from $\widetilde{C}$, which has a solution in some δ-field extension of $\widetilde{C}$, already has a solution in $\widetilde{C}$.
- If $(\mathscr{C},\delta)$ is another differentially closed field containing $\mathbb{C}$ then there exists a δ-embedding of $\widetilde{C}$ in $\mathscr{C}$ over $\mathbb{C}$.

See [M-MTDF] for a survey, including proofs of the existence and uniqueness of $\widetilde{C}$. The existence is due to Kolchin and (with a simpler characterization than the one given here) to Blum. The uniqueness is due to Shelah. Blum and Shelah's approaches are proved by model theoretic means; differentially closed fields do not show up "naturally" in algebra, and are considered "huge". Despite the analogy with the algebraic closure of a field, caution must be exercised. For example, $\widetilde{C}$ fails to satisfy *minimality*: it has proper subfields δ-isomorphic to it over $\mathbb{C}$.

It is proven in Proposition 2.11 of loc.cit. that the δ-constants of a δ-closure $\widetilde{C}$ of $\mathbb{C}$ are just $\mathbb{C}$. Proposition 1.1 of [M-RDCF] proves that the group $\mathrm{Aut}_\delta(\widetilde{C}/\mathbb{C})$ of δ-automorphisms of $\widetilde{C}/\mathbb{C}$ is rich: the only elements of $\widetilde{C}$ fixed by $\mathrm{Aut}_\delta(\widetilde{C}/\mathbb{C})$ are the elements of $\mathbb{C}$. Both these facts will play a role later on.

3.3.2. *Systems and modules.* Given a (ϕ,δ)-field K, we consider the double system of equations

$$\begin{cases} \phi(y) = Ay \\ \partial(y) = By \end{cases}$$

where $A \in \mathrm{GL}_n(K)$ and $B \in \mathfrak{gl}_n(K)$. One seeks solutions in (ϕ,∂)-ring extensions R of K. It is readily checked that a necessary condition for the existence of a $U \in \mathrm{GL}_n(R)$, for some extension R, satisfying both sets of equations, is the *consistency relation*

$$q\phi(B) = ABA^{-1} + \partial A \cdot A^{-1}. \tag{3.5}$$

If this relation holds we call the above system of equations *consistent*.

A (ϕ,∂)-*module* over K is a triple (W,Φ,∇) such that (W,Φ) is a ϕ-module, (W,∇) is a ∂-module and

$$\nabla \circ \Phi = q\Phi \circ \nabla.$$

The (ϕ,∂)-module associated to a consistent system as above is (K^n,Φ,∇) where $\Phi(y) = A^{-1}\phi(y)$ and $\nabla(y) = \partial(y) - By$. The consistency relation (3.5) between A and B is equivalent to the commutation relation between Φ and ∇. Conversely, if (W,Φ,∇) is a (ϕ,∂)-module, $e_1,...,e_n$ is a basis of W over K and A and B are constructed as in the previous sections, then we get a consistent system of equations.

If we change the basis as before, the new pair of matrices that we obtain, with respect to the new basis, is

$$(\widetilde{A},\widetilde{B}) = (\phi(P)AP^{-1}, PBP^{-1} + \partial P \cdot P^{-1}).$$

We call two pairs $(\widetilde{A},\widetilde{B})$ and (A,B) related in this way *gauge equivalent*. As before, this establishes a bijection between consistent systems as above, up to gauge equivalence, and isomorphism classes of (ϕ,∂)-modules.

The module (W,Φ,∇) is called *trivial* if it is isomorphic to the module (K^n,ϕ,∂). In terms of the associated systems, this means that (A,B) is gauge equivalent to $(I,0)$. It may happen that (W,Φ) or (W,∇) are trivial but (W,Φ,∇) is not. In such a case we use the terminology ϕ-*trivial* or ∂-*trivial*.

A particular case of gauge equivalence occurs when we take the transition matrix between the bases to be $P = A$. In this case we get, thanks to the consistency relation, the pair

$$(\widetilde{A},\widetilde{B}) = (\phi(A), q\phi(B)).$$

3.3.3. *Base change and solution sets.* It is tempting to look for a (ϕ,∂)-ring extension R over which the (ϕ,∂)-system above attains a full set of solutions. While the definitions can be immitated, at this point we take a different path. We shall be primarily interested in the *difference* Picard Vessiot theory, *i.e.*, the construction of a "minimal" ring over which (W,Φ) alone is trivialized. We shall show that *if* W carries a consistent ∂-structure, *i.e.*, can be extended to become a (ϕ,∂)-module, this imposes serious restrictions on the *difference Galois group* of the system $\phi(y)=Ay$. Similarly, we might be interested in a ∂-module (W,∇), its *differential* Picard Vessiot theory, and the restrictions that a consistent ϕ-structure imposes on its *differential Galois group.* This leads us to the notions of ∂-integrability and ϕ-isomonodromy[2] discussed below.

3.4. **∂-integrable ϕ-modules.** Let K be a (ϕ,∂)-field. Let (W,Φ) be a ϕ-module over K and $\phi(y)=Ay$ the associated system of difference equations, in some basis of W. Concretely, we may take $(W,\Phi)=(K^n,A^{-1}\phi)$.

Let $K[\varepsilon]$ be the ring of dual numbers over K ($\varepsilon^2=0$). Extend the automorphism ϕ to $K[\varepsilon]$ as $\phi(a+b\varepsilon)=\phi(a)+q\phi(b)\varepsilon$, and define a ring homomorphism $h_\partial: K\to K[\varepsilon]$ by $h_\partial(a)=a+\partial(a)\varepsilon$. Both the inclusion $K\hookrightarrow K[\varepsilon]$ and h_∂ commute with ϕ. Consider the two ϕ-modules

$$W_\varepsilon = K[\varepsilon]\otimes_K W,\ \ \Phi_\varepsilon=\phi\otimes\Phi$$

and

$$W_\varepsilon^{(\partial)} = K[\varepsilon]\otimes_{h_\partial,K} W,\ \ \Phi_\varepsilon^{(\partial)}=\phi\otimes\Phi.$$

One can thereby construct an endofunctor of the category of ϕ-modules over K which attaches to any module W the module $W_\varepsilon^{(\partial)}$. Such a functor is called the *prolongation functor* and has been introduced by Ovchinnikov [Ov] and Kamensky [Kam] when defining differential Tannakian categories. The ϕ-module $W_\varepsilon^{(\partial)}$ is an extension of W by itself in the category of ϕ-modules over K. Moreover, if $\phi(y)=Ay$ is a system associated to W then $\phi(y)=\begin{pmatrix}A & \partial(A)\\ 0 & A\end{pmatrix}y$ is a system associated to $W_\varepsilon^{(\partial)}$.

Definition 11. The ϕ-module W is called *∂-integrable* if there exists an isomorphism of ϕ-modules $(W_\varepsilon,\Phi_\varepsilon)\simeq(W_\varepsilon^{(\partial)},\Phi_\varepsilon^{(\partial)})$ over $K[\varepsilon]$, reducing to the identity modulo ε.

It is easily seen that a module W is *∂-integrable* if and only if the extension of W by itself corresponding to $W_\varepsilon^{(\partial)}$ splits. Moreover, we can interpret the notion of ∂-integrability in the following terms.

Lemma 12. *Let (W,Φ) be a ϕ-module over K. Then W is ∂-integrable if and only if it carries a structure of a (ϕ,∂)-module extending the given ϕ-module structure. Explicitly, this means that there exists a ∂-connection*

$$\nabla: W\to W$$

such that $q\Phi\circ\nabla=\nabla\circ\Phi$. In terms of the matrix A, the module W (or system (3.1)) is ∂-integrable if and only if there exists a matrix $B\in\mathfrak{gl}_n(K)$ such that

$$\partial(A)=q\phi(B)A-AB.$$

Proof. Let $\iota:(W_\varepsilon^{(\partial)},\Phi_\varepsilon^{(\partial)})\simeq(W_\varepsilon,\Phi_\varepsilon)$ be an isomorphism of ϕ-modules over $K[\varepsilon]$, reducing to the identity modulo ε. Write $W_\varepsilon^{(\partial)}=1\otimes W+\varepsilon\otimes W$. Note that only the second summand is a K-subspace, as

$$\lambda\otimes w = 1\otimes\lambda w-\partial(\lambda)\varepsilon\otimes w\ \ (\lambda\in K,\ w\in W)$$

[2]Some authors use ϕ-integrability, to stress the analogy with ∂-integrability. However, as there are no differential equations involved with ϕ, we prefer the term ϕ-isomonodromy.

need not lie in $1\otimes W$. Nevertheless, as ι is $K[\varepsilon]$-linear, it is determined by the values $\iota(1\otimes w)$. Since ι reduces to the identity modulo ε we can define ∇ by

$$\iota(1\otimes w)-1\otimes' w=\varepsilon\otimes'\nabla(w).$$

To avoid confusion we used $\otimes'$ for the tensor product in W_ε. We then have

$$\varepsilon\otimes'\nabla(\lambda w)=\iota(1\otimes\lambda w)-1\otimes'\lambda w=\iota(\lambda\otimes w+\partial(\lambda)\varepsilon\otimes w)-\lambda\otimes' w.$$

This equals $\lambda\varepsilon\otimes'\nabla(w)+\partial(\lambda)\varepsilon\otimes' w$. It follows that

$$\nabla(\lambda w)=\lambda\nabla(w)+\partial(\lambda)w,$$

so ∇ is indeed a ∂-connection. That it satisfies $q\Phi\circ\nabla=\nabla\circ\Phi$ follows from $\iota\circ\Phi_\varepsilon^{(\partial)}=\Phi_\varepsilon\circ\iota$. These arguments can be reversed, proving the Lemma. □

Note that any difference system $\phi(y)=Ay$ with $A\in\mathrm{GL}_n(K^\partial)$ is ∂-integrable (take $B=0$).

3.5. **ϕ-isomonodromic ∂-modules.** Still assuming that K is a (ϕ,∂)-field, let (W,∇) be a module with a connection, and $\partial(y)=By$ the associated system of differential equations, in some basis of W. Concretely, we may take $(W,\nabla)=(K^n,\partial-B)$.

The module $(W^{(\phi)},\nabla^{(\phi)})$ is defined as

$$W^{(\phi)}=K\otimes_{\phi,K}W,\ \ \nabla^{(\phi)}(a\otimes w)=\partial a\otimes w+qa\otimes\nabla w.$$

It is again a ∂-module.

Definition 13. A ∂-module W is called *ϕ-isomonodromic* (or *ϕ-integrable*) if $W\simeq W^{(\phi)}$.

Lemma 14. *Let (W,∇) be a ∂-module over K. Then W is ϕ-isomonodromic if and only if it carries a structure of a (ϕ,∂)-module extending the given ∂-module structure. In terms of the matrix B the module W (or the system (3.3)) is ϕ-isomonodromic if and only if there exists an $A\in\mathrm{GL}_n(K)$ such that*

$$q\phi(B)=ABA^{-1}+\partial A\cdot A^{-1},$$

i.e.*, if and only if B and $q\phi(B)$ are gauge equivalent.*

Proof. If $\iota:W^{(\phi)}\simeq W$ is an isomorphism of ∂-modules, put $\Phi(w)=\iota(1\otimes w)$. Then $\Phi(aw)=\phi(a)\cdot\Phi(w)$ and the relation $\nabla\circ\Phi=q\Phi\circ\nabla$ follows from $\iota\circ\nabla^{(\phi)}=\nabla\circ\iota$. Conversely, given Φ define $\iota(a\otimes w)=a\Phi(w)$. □

Example 15. Let K be as in § 2.1. If $f\in F=\mathbb{C}((z))$ satisfies a linear differential equation of degree n with coefficients from K, as well as a ϕ-difference equation of degree m over K, then $W=Span_K\{\phi^i\partial^j f\,|\,0\le i<m,\ 0\le j<n\}\subset F$ is a (ϕ,∂)-module over K, when we let $\nabla=\partial$ and $\Phi=\phi$.

If f only satisfied a differential equation and we take $N=Span_K\{\partial^j(f)\}$ then $(N,\nabla)=(N,\partial)$ is a ∂-module, and $\alpha:1\otimes g\mapsto\phi(g)$ yields $\alpha:(N^{(\phi)},\nabla^{(\phi)})\simeq(\phi(N),\partial)$ (as a subspace of F). Indeed, for $g\in N$

$$\alpha\circ\nabla^{(\phi)}(1\otimes g)=\alpha(q\otimes\partial g)=q\phi(\partial(g))=\partial(\phi(g))=\partial\circ\alpha(1\otimes g).$$

If $N=\phi(N)$ then clearly $N\simeq\phi(N)$ and $N=W$ has the structure of a (ϕ,∂)-module.

4. Picard-Vessiot and parametrized Picard-Vessiot theory

4.1. **Picard Vessiot theory of difference equations.** A classical exposition of Picard-Vessiot theory for difference equations is [S-vdP97] whereas parametrized Picard-Vessiot theory is introduced in [H-S08]. The articles [O-W], [dV-H] and [Wi12] contain some schematic approaches.

4.1.1. *The Picard Vessiot extension and the Picard Vessiot ring.* Let (K, ϕ) be a ϕ-field of characteristic 0, with an algebraically closed field of constants $C = K^{\phi}$. Consider the system of equations (3.1). One is interested in constructing a minimal ϕ-extension (L, ϕ) in which it attains a full set of solutions, much like the construction of a splitting field of a polynomial in Galois theory. Easy examples show that L can not always be a field. The next best option works.

Definition 16. (i) A *ϕ-pseudofield* L is a direct product of finitely many fields

$$L = L_1 \times \cdots \times L_r,$$

together with an automorphism ϕ, permuting the L_i cyclically: $\phi(L_i) = L_{i+1 \mod r}$.

(ii) A *Picard Vessiot (PV) extension* of K for (3.1) is a ϕ-pseudofield (L, ϕ) containing (K, ϕ), satisfying:

- There exists a fundamental matrix $U \in \mathrm{GL}_n(L)$ for (3.1), and $L = K(U)$, *i.e.*, L is generated, as a pseudofield, by the entries of U.
- $L^{\phi} = C$.

Picard Vessiot extensions exist, and are unique up to K-ϕ-isomorphism [O-W, Prop.2.14 and Theorem 2.16]. If L is a PV extension for (3.1) as above then L_1 is a PV extension of the system

$$\phi^r(y) = A^{(r)}y, \;\; A^{(r)} = \phi^{r-1}(A) \cdots \phi(A)A,$$

over the difference field (K, ϕ^r). The projection of U to L_1 is a fundamental matrix for this new system. In many cases, depending on the desired conclusion, it is harmless to replace the original system by this new one, ϕ by ϕ^r, reducing the situation to the case where L is a field.

Since the fundamental matrix is unique up to multiplication on the right by a matrix from $\mathrm{GL}_n(C)$, the subring

$$R = K[U, \det(U)^{-1}]$$

of L is canonical, and is invariant under ϕ as well. It turns out to be ϕ-simple. A ϕ-simple K-algebra generated by a fundamental matrix U and the inverse of its determinant (thus making $U \in \mathrm{GL}_n(R)$) is called a PV *ring* for (3.1). It is again unique up to isomorphism, and its total ring of fractions (its localization in the set of non zero divisors) is a PV extension [O-W, Proposition 2.14].

If (L, ϕ) is a PV extension of (K, ϕ) for the system (3.1) and (K', ϕ) is an intermediate ϕ-extension, $K \subset K' \subset L$, then (L, ϕ) is a PV extension of the same system over K' as well.

Remark 17. Let (F, ϕ) be a ϕ-field extension of (K, ϕ) containing the coordinates of a solution $0 \neq u \in F^n$ of $\phi(u) = Au$. Assume $F^{\phi} = C$. Then we may assume, without loss of generality, that the PV extension L contains the subfield $K(u)$ of F. Indeed, replace (K, ϕ) by $(K(u), \phi)$ and look for a PV extension L for the original system over $K(u)$. Then L will satisfy the requirements to become a PV extension of the same system also over K.

4.1.2. *The ring S as a Picard-Vessiot ring.* In the notation of § 2, the automorphism ϕ also extends to S_0 and S. The following holds.

Lemma 18. *The ring S is a Picard-Vessiot ring over K and thereby is ϕ-simple.*

Proof. One can give a proof along the lines of Lemma 6. However, we give a slicker proof, that will be useful later. Consider the system of ϕ-difference equations

$$\phi(y) = \begin{pmatrix} q & f_\zeta(z,\Lambda) \\ 0 & 1 \end{pmatrix} y$$

where $f_\zeta(z,\Lambda) = \zeta(qz,\Lambda) - q\zeta(z,\Lambda) \in K_\Lambda$. The matrix

$$U = \begin{pmatrix} z & \zeta(z,\Lambda) \\ 0 & 1 \end{pmatrix}$$

is a fundamental matrix, and its determinant is z. The field $E = K(z, \zeta(z,\Lambda))$ is a Picard-Vessiot extension for the system over K because it is generated by the entries of U, and its field of ϕ-constants is $\mathbb{C}$. The last point is most easily seen if we embed E in $F = \mathbb{C}((z))$ and use the fact that $F^\phi = \mathbb{C}$. The subring $S \subset E$ is generated by the entries of U and $\det(U)^{-1}$, hence is the Picard-Vessiot *ring* for the system. But Picard-Vessiot rings are known to be ϕ-simple [O-W, Prop. 2.14] by general principles. □

The ring S_0 is not ϕ-simple. The proper ideal (z) is ϕ-invariant. The ϕ-simplicity of S has the following useful consequence.

Corollary 19. *Let E be the field of fractions of S. The following properties relative to an $f \in E$ are equivalent:*

- *$f \in S$;*
- *the K-vector space*

$$M_f = \mathrm{Span}_K\{\phi^i(f) |\, i \geq 0\} \subset E$$

is finite dimensional.

Proof. Assume that $\dim_K M_f < \infty$. We first claim that $\phi(M_f) = M_f$. Indeed, since $\phi(M_f) = \mathrm{Span}_K\{\phi^i(f) |\, i \geq 1\}$, it is sufficient to prove that f belongs to $\phi(M_f)$. Let

$$\sum_{i=0}^{m} a_i \phi^i(f) = 0$$

($a_i \in K$) be a nontrivial linear relation between the $\phi^i(f)$ with smallest possible m. Then $a_0 \neq 0$, for otherwise applying ϕ^{-1} to the relation would give a similar one with $m-1$ replacing m. It follows that $a_0 f$, and hence f, belongs to $\mathrm{Span}_K\{\phi^i(f) |\, i \geq 1\} = \phi(M_f)$. This concludes the proof of our claim.

Let

$$I = \{a \in S |\, aM_f \subset S\}.$$

Since M_f is spanned by $\phi^i(f)$ for $0 \leq i \leq m-1$, a common denominator of these m generators belongs to I, so $I \neq 0$. The set I is clearly an ideal in S, and it is ϕ-invariant because if $a \in I$ then $\phi(a)M_f = \phi(a)\phi(M_f) = \phi(aM_f) \subset S$, so $\phi(a) \in I$. By the ϕ-simplicity of S we must have $1 \in I$, and in particular $f \in S$.

For the converse, note that the set of f's for which $\dim_K M_f$ is finite is a K-algebra. As it contains z, z^{-1} and $\zeta(z,\Lambda)$, it contains the ring S. □

4.1.3. *The difference Galois group and its standard representation.* Let (L, ϕ) be a PV extension of (3.1).

Definition 20. The group

$$G = \mathrm{Aut}_\phi(L/K)$$

of ϕ-automorphisms of L leaving K pointwise fixed is called the *(difference) Galois group* of (3.1).

Fix a fundamental matrix U of (3.1) so that $L = K(U)$ and every $\sigma \in G$ is determined by its effect on U. Since σ and ϕ commute, $\sigma(U)$ is another fundamental matrix, so

$$\sigma(U) = UV_\sigma$$

for $V_\sigma \in \mathrm{GL}_n(C)$. [S-vdP97, Theorem 1.13] shows that $\sigma \mapsto V_\sigma$ is an embedding of G as a *Zariski closed* subgroup of $\mathrm{GL}_n(C)$. A different choice of U changes the embedding by conjugation.

Another look at G is given by the ϕ-module $(W, \Phi) = (K^n, A^{-1}\phi)$ associated with (3.1), and the solution set

$$\mathcal{U} = W_L^\Phi = UC^n.$$

The G-action on $W_L = W \otimes_K L$ (σ acting like $1 \otimes \sigma$) commutes with Φ, so induces an action on $\mathcal{U}$. If $\sigma \in G$ and Uv ($v \in C^n$) is a vector in $\mathcal{U}$, then $\sigma(Uv) = UV_\sigma v$. It follows that the representation $\sigma \mapsto V_\sigma$ is nothing but the matrix representation afforded by $\mathcal{U}$, in the basis consisting of the columns of U. We call $\mathcal{U}$ the *standard representation* of G.

4.1.4. *The Galois correspondence theorem.* Since the characteristic is 0, the algebraic group G must be reduced, but need not be connected. [S-vdP97, Lemma 1.28] proves that

$$L^G = K.$$

The last fact is the basis for the Galois correspondence between ϕ-pseudofields $K \subset E \subset L$ and Zariski closed subgroups $\{e\} \subset H \subset G$. With E we associate

$$H(E) = \mathrm{Aut}_\phi(L/E).$$

With H we associate its fixed field

$$E(H) = L^H.$$

The Galois correspondence theorem asserts that these two assignments are inverse to each other, and set the family of all intermediate ϕ-pseudofields and the family of all Zariski closed subgroups of G in an order-reversing bijection [S-vdP97, Theorem 1.29].

An intermediate ϕ-pseudofield E is called *normal* if it is the PV extension of a system $\phi(y) = A_1 y$ for some $A_1 \in \mathrm{GL}_m(K)$ over K. [An, Théorème 3.5.2.2] yields that an intermediate ϕ-pseudofield E is normal if and only if the corresponding group $H = H(E)$ is normal in G, and in this case

$$\mathrm{Aut}_\phi(E/K) = G/H.$$

As in classical Galois theory, when it comes to showing that normal subextensions of a Galois extension are splitting fields of suitable polynomials, the system $\phi(y) = A_1 y$ corresponding to the fixed field of a normal subgroup of G is neither unique, nor related in any canonical way to the original system used to define L.

An important normal subgroup of G is its connected component G^0, the smallest Zariski closed subgroup H of finite index in G. Its fixed field $E(G^0)$ is therefore the largest finite ϕ-extension of K in L. If L is a field, this is the algebraic closure of K in L. Thus, if L happens to be a field, G is connected if and only if K is algebraically closed in L. In this case, if K does not admit any finite field extension to which ϕ extends as an automorphism, G will be connected. The K of § 2.1 is such an example, see Lemma 2.

4.1.5. *A special case of the Tannakian correspondence.* As C is algebraically closed of characteristic 0, the algebraic group G may be identified, as we did, with its C-points $G(C)$. It can be defined also as a functor on the category of C-algebras by

$$G(D) = \mathrm{Aut}_\phi(D \otimes_C R / D \otimes_C K)$$

where ϕ is extended to $D \otimes_C R$ as $1 \otimes \phi$. One proves then that this functor is representable by a linear algebraic group over C, and the standard Yoneda Lemma shows that it determines G uniquely. A more sophisticated approach is to consider the abelian category of ϕ-modules, the tensor subcategory $\{W\}$ generated by the ϕ-module W attached to our system, and use the Tannakian formalism to obtain G as its Tannakian fundamental group (see [An, Théorème 3.4.2.3] for a discussion on the notions of fiber functors on $\{W\}$ and Picard-Vessiot extensions for W).

We shall need the following special case of the Tannakian correspondence, which can be proven directly. We assume that C is algebraically closed.

Proposition 21. *Let the notation be as above. The applications*

$$\mathcal{V} \subset \mathcal{U} \longmapsto V := (\mathcal{V} \otimes_C L)^G \subset W \text{ and } V \subset W \longmapsto \mathcal{V} := (V \otimes_K L)^\Phi \subset \mathcal{U}$$

are bijections between G-submodules $\mathcal{V}$ of $\mathcal{U}$, and ϕ-submodules V of W, which are inverse to each other. We have $\dim_K V = \dim_C \mathcal{V}$. In particular W is an irreducible ϕ-module if and only if $\mathcal{U}$ is an irreducible representation of G.

Here G acts on $\mathcal{V} \otimes_C L$ diagonally, and the operator Φ on V is derived from $1 \otimes \phi$. Similarly, Φ acts on $V \otimes_K L$ diagonally, and the G-action on $\mathcal{V}$ is derived from its action on L.

In the next section we review the δ-parametrized Picard Vessiot theory. There too, the three approaches (Picard-Vessiot, schematic and Tannakian) coexist. The last two, however, have not been fully developed in the literature[3], so we adopt the Picard-Vessiot approach. As we shall explain, the δ-parametrized Galois group is a *linear differential algebraic group*, which is not determined by its points in C, but rather in a *differential closure* $\widetilde{C}$ of C. This forces us to extend scalars from C to $\widetilde{C}$, and then use descent arguments to go back.

4.2. δ-parametrized Picard Vessiot theory of difference equations.

4.2.1. *LDAG's.* Linear differential algebraic groups (LDAG's) have been defined and studied by Kolchin [Kol] and Cassidy [Cas72]. For a quick introduction see, for example, the summary in §2 of [Mi-Ov].

Let C be an algebraically closed field of characteristic 0, equipped with a derivation δ, which may be trivial. We fix a differential closure $(\widetilde{C}, \delta)$ of C. Let

$$\widetilde{C}\{X_{ij}, \det(X)^{-1}\}_\delta$$

[3]Despite partial results of Buium, Kamensky, Kovacic and Ovchinnikov.

denote the ring of differential polynomials in the variables X_{ij} $(1 \leq i, j \leq n)$, with $\det(X)$ inverted. A LDAG $\widetilde{\mathcal{G}}$ is a subgroup of $\mathrm{GL}_n(\widetilde{C})$ that is the zero set of some radical δ-ideal $\widetilde{\mathcal{I}}$ of $\widetilde{C}\{X_{ij}, \det(X)^{-1}\}_\delta$. We call $\widetilde{\mathcal{I}}$ the *ideal of definition of* $\widetilde{\mathcal{G}}$. It is a radical Hopf δ-ideal and by the Ritt-Raudenbusch theorem, is the radical of a finitely generated δ-ideal.

If the generators of $\widetilde{\mathcal{I}}$ as a δ-ideal can be taken to be differential polynomials with coefficients from C, we say that $\widetilde{\mathcal{G}}$ is *defined over* C. For such a LDAG, we attach a *differential group scheme* $\mathcal{G}$ over C as follows. We define the ideal of definition of $\mathcal{G}$ as $\mathcal{I} = \widetilde{\mathcal{I}} \cap C\{X_{ij}, \det(X)^{-1}\}_\delta$ and we let $C\{\mathcal{G}\} = C\{X_{ij}, \det(X)^{-1}\}_\delta/\mathcal{I}$ be the *differential coordinate ring of* $\widetilde{\mathcal{G}}$ *over* C. Then, we define the points of $\mathcal{G}$ in any δ-ring extension D of C as

$$\mathcal{G}(D) = \mathrm{Hom}_\delta(C\{\mathcal{G}\}, D).$$

As $\widetilde{\mathcal{I}}$ is δ-generated by $\mathcal{I}$, we see that $\mathcal{G}(\widetilde{C}) = \widetilde{\mathcal{G}}$, $\mathcal{I}$ is a radical Hopf δ-ideal in $C\{X_{ij}, \det(X)^{-1}\}_\delta$, and $\mathcal{G}(D)$ is a *subgroup* of $\mathrm{GL}_n(D)$.

Though $\mathcal{G}(\widetilde{C}) = \widetilde{\mathcal{G}}$, we caution that the group of C-points $\mathcal{G}(C)$ tells us little about the nature of $\mathcal{G}$. For example, the single equation

$$\delta(\frac{\delta X}{X}) = 0$$

is easily seen to define a differential subgroup $\mathcal{G}$ of $\mathrm{GL}_1(\widetilde{C})$. If C is the field of complex numbers, equipped with the trivial derivation, then $\mathcal{G}(\mathbb{C}) = \mathbb{C}^\times$. If δ is extended to $\mathbb{C}(z)$ so that $\delta(z) = 1$ we still have $\mathcal{G}(\mathbb{C}(z)) = \mathbb{C}^\times$. However, over the field of meromorphic functions on $\mathbb{C}$ (with $\delta = d/dz$) we find new points, namely the solutions $X = e^{az}$ for any $a \in \mathbb{C}$.

The *Zariski closure* $\widetilde{G}$ of $\widetilde{\mathcal{G}}$ is a linear algebraic group defined over $\widetilde{C}$. Its ideal of definition as a linear algebraic group over $\widetilde{C}$ is

$$\widetilde{I} = \widetilde{\mathcal{I}} \cap \widetilde{C}[X_{ij}, \det(X)^{-1}].$$

Every linear algebraic group $\widetilde{G} \subset \mathrm{GL}_n(\widetilde{C})$ may be considered also a LDAG, which we denote $[\delta]_*\widetilde{G}$. If G is a linear algebraic group defined over C, then $G(\widetilde{C})$ is a LDAG defined over C. Abusing notation, we denote by $[\delta]_*G$ the differential group scheme over C associated with $G(\widetilde{C})$. The Zariski closure $\widetilde{G}$ of $\widetilde{\mathcal{G}}$ is characterized by the property that $\widetilde{\mathcal{G}} \subset [\delta]_*\widetilde{G}$, and for no proper Zariski closed subgroup $\widetilde{H} \subsetneq \widetilde{G}$ do we have $\widetilde{\mathcal{G}} \subset [\delta]_*\widetilde{H}$.

If $\mathcal{G}$ is a connected differential group scheme over C, *i.e.*, $C\{\mathcal{G}\}$ is a domain, then its field of fractions $C\langle\mathcal{G}\rangle$ is a δ-field. Its δ-*transcendence degree* over C, denoted by $\delta\mathrm{tr.deg.}(C\langle\mathcal{G}\rangle/C)$, is the maximal number of elements of $C\langle\mathcal{G}\rangle$ which are δ-algebraically independent over C, *i.e.*, do not satisfy any differential polynomial with coefficients from C. It coincides with the δ-*dimension* of $\mathcal{G}$, as defined by Kolchin ([Kol], IV.3, p.148):

$$\delta \dim \mathcal{G} = \delta\mathrm{tr.deg.}(C\langle\mathcal{G}\rangle/C).$$

In general $\delta \dim \mathcal{G}$ is equal to the δ-dimension of its connected component (in the Kolchin topology). For example, the δ-dimension of the example given above is 0.

A LDAG $\widetilde{\mathcal{G}}$ is called δ-*constant* if its ideal of definition $\widetilde{\mathcal{I}}$ contains $\delta(X_{ij})$, or equivalently $\widetilde{\mathcal{G}} \subset \mathrm{GL}_n(\widetilde{C}^\delta)$. The following theorem of Cassidy ([Cas89], Theorem 19) is instrumental to our work.

Theorem 22. *Suppose that* $\widetilde{\mathcal{G}}$ *is a LDAG, Zariski dense in a* simple *linear algebraic group* $\widetilde{G} \subset \mathrm{GL}_n(\widetilde{C})$. *If* $\widetilde{\mathcal{G}} \subsetneq [\delta]_* \widetilde{G}$, *then* $\widetilde{\mathcal{G}}$ *is conjugate, in* $\mathrm{GL}_n(\widetilde{C})$, *to a* δ*-constant LDAG.*

4.2.2. *The* δ*-parametrized Picard Vessiot extension and Picard Vessiot ring.* Let (K, ϕ, δ) be a (ϕ, δ)-field of characteristic 0, where ϕ is an automorphism and δ a derivation commuting with ϕ:

$$\phi \circ \delta = \delta \circ \phi.$$

We assume that the field $C = K^\phi$ of ϕ-constants is algebraically closed. Since δ and ϕ commute, it is a δ-field, and we denote by $\widetilde{C}$, as before, a differential closure, on which we let ϕ act trivially.

Let (3.1) be a linear system of difference equations over K.

Definition 23. A δ*-parametrized Picard Vessiot (PPV) extension* of K is a (ϕ, δ)-pseudofield $\mathcal{L}$, *i.e.*,

$$\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_r,$$

where the $\mathcal{L}_i$ are δ-field extensions of K permuted cyclically by ϕ, satisfying:

- $\mathcal{L} = K \langle U \rangle_\delta$ is δ-generated as a pseudofield by the entries of a fundamental matrix U for the system.
- $\mathcal{L}^\phi = C$.

By [Wi12], Corollary 10, a δ-parametrized Picard Vessiot extension exists. By Proposition 6.16 of [H-S08], if $C = \widetilde{C}$, it is also unique up to K-(ϕ, δ)-isomorphism.

The subring $\mathcal{R} = K\{U, \det(U)^{-1}\}_\delta$ of $\mathcal{L}$ that is δ-generated over K (as a δ-ring) by U and the inverse of its determinant, does not depend on the choice of U and is ϕ-simple [O-W, Prop.2.14]. Such a ring is called a δ*-parametrized Picard Vessiot ring* for the system[4]. The subring $R = K[U, \det(U)^{-1}]$ of $\mathcal{R}$ is an (ordinary) PV ring then, and its total ring of fractions $L \subset \mathcal{L}$ an (ordinary) PV extension.

4.2.3. *The* δ*-parametrized Galois group.* Let (K, ϕ, δ) be a (ϕ, δ)-field of characteristic zero with algebraically closed field of ϕ-constants C.

For $\mathcal{L}$ a PPV extension of (3.1) over K and $\mathcal{R} \subset \mathcal{L}$ the PPV-ring, we define the δ*-parametrized Galois group scheme of* $\mathcal{L}$ *over* K as the functor which associates to any δ-ring extension D of C, the group

$$\mathcal{G}(D) = \mathrm{Aut}_{\phi,\delta}(D \otimes_C \mathcal{R} / D \otimes_C K).$$

This functor is indeed representable by a differential group scheme over C, which is unique up to isomorphism by the Yoneda Lemma.

If $\widetilde{C}$ is a differential closure of C, one can consider $\widetilde{K} = Frac(\widetilde{C} \otimes_C K)$, the base-changed ground field ($\widetilde{C} \otimes_C K$ is a domain because $\widetilde{C}$ is C-regular), and $\mathfrak{S} = \widetilde{C} \otimes_C K - \{0\}$ (a multiplicative set). Let $\widetilde{\mathcal{R}} = \widetilde{C} \otimes_C \mathcal{R}[\mathfrak{S}^{-1}] = \widetilde{K} \otimes_K \mathcal{R}$. We claim that $\widetilde{\mathcal{R}}$ is a PPV ring over $\widetilde{K}$. For that we only have to show that it is ϕ-simple, and as it is obtained by localization from $\widetilde{C} \otimes_C \mathcal{R}$ it is enough to check that the latter is ϕ-simple. But this is clear, since the action of ϕ on $\widetilde{C}$ is trivial and $\mathcal{R}$ is ϕ-simple [S-vdP97, Lemma 1.11]. Let $\widetilde{\mathcal{L}}$ be the total ring of fractions of $\widetilde{\mathcal{R}}$. It is a a PPV extension over $\widetilde{K}$ with $\widetilde{\mathcal{L}}^\phi = \widetilde{C}$ [H-S08, Cor. 6.15].

[4]In [H-S08] one only asks that $\mathcal{R}$ be (ϕ, δ)-simple, a weaker condition, but it is shown in Corollary 6.22 that if C is differentially closed, $\mathcal{R}$ is then actually ϕ-simple. See the discussion in [Wi12] why, when C is only algebraically closed, it makes better sense to impose the stronger condition of being ϕ-simple.

Since any (ϕ, δ)-automorphism σ of $\widetilde{\mathcal{L}}$ over $\widetilde{K}$ is determined by its effect on U, and $\sigma(U) = UV_\sigma$ where V_σ has entries in $\widetilde{\mathcal{L}}^\phi = \widetilde{C}$, such a σ actually induces an automorphism of $\widetilde{C} \otimes_C \mathcal{R}$ over $\widetilde{C} \otimes_C K$. The converse is equally clear. We define the *δ-parametrized Galois group* of $\widetilde{\mathcal{L}}$ over $\widetilde{K}$ as $\mathrm{Aut}_{\phi,\delta}(\widetilde{\mathcal{L}}/\widetilde{K})$. Since $\widetilde{C}$ is δ-closed, this group can be embedded as a LDAG $\widetilde{\mathcal{G}}$ in $\mathrm{GL}_n(\widetilde{C})$ via its action on a fundamental matrix in $\mathrm{GL}_n(\widetilde{R})$ and it is independent, up to conjugation in $\mathrm{GL}_n(\widetilde{C})$, of the choice of the PPV extension over $\widetilde{K}$ or of U [H-S08, Proposition 6.18]. Finally, [dV-H, Proposition 1.20] yields

$$\widetilde{\mathcal{G}} = \mathrm{Aut}_{\phi,\delta}(\widetilde{\mathcal{L}}/\widetilde{K}) = \mathcal{G}(\widetilde{C}).$$

Let $\widetilde{G} = \mathrm{Aut}_\phi(\widetilde{L}/\widetilde{K})$ be the (ordinary) difference Galois group of (3.1) over $\widetilde{K}$. Then the group $\widetilde{\mathcal{G}}$ is Zariski dense in $\widetilde{G}$, see [H-S08], Proposition 6.21.

Finally, the torsor theorem yields the following equality

$$\delta \dim \mathcal{G} = \delta \mathrm{tr.deg.}(\mathcal{L}/K),$$

where $\delta\mathrm{trdeg}(\mathcal{L}/K)$ is the differential transcendence degree of $\mathcal{L}_1$ over K in the notation of §4.2.2. This result was proved for the δ-parametrized Galois group $\widetilde{\mathcal{G}}$ in [H-S08, Proposition 6.26]. For the δ-parametrized Galois group scheme $\mathcal{G}$, the proof of the above equation is entirely analogous to the proof of [DVHW, Lemma 2.7] in the symmetric context of differential equations with a difference parameter.

4.2.4. *The δ-parametrized Galois correspondence.* We shall need the following result. The proof of Lemma 6.19 in [H-S08], although set in a different context, applies here as well, without any change.

Proposition 24. *Let the notation be as in the previous section. For every $x \in \widetilde{\mathcal{L}} - \widetilde{K}$ there exists a $\sigma \in \widetilde{\mathcal{G}}$ with $\sigma(x) \neq x$,* i.e.,

$$\widetilde{\mathcal{L}}^{\widetilde{\mathcal{G}}} = \widetilde{K}.$$

This is the key to the δ-parametrized Galois correspondence, analogous to what we described in the non-parametrized framework. See [H-S08] and note that since we do not work schematically, we must extend scalars to $\widetilde{C}$.

4.3. **δ-algebraic solutions.** Notation as in section 4.2.3, let $(W, \Phi) = (K^n, A^{-1}\phi)$ be the ϕ-module associated with the linear system $\phi(y) = Ay$, L a PV extension over K and $\mathcal{L}$ a δ-parametrized PV extension over K containing L. Let

$$\mathcal{U} = W_L^\Phi = W_{\mathcal{L}}^\Phi = UC^n$$

be the solution space, where $U \in \mathrm{GL}_n(L)$ is a fundamental matrix. As above, we denote by $\widetilde{C}$ a differential closure of C, and add a tilde to denote the same objects over $\widetilde{K}$. Since $\widetilde{\mathcal{L}}^\phi = \widetilde{C}$, we have $\widetilde{\mathcal{U}} = \widetilde{C} \otimes_C \mathcal{U}$.

Let $\mathcal{L}_a \subset \mathcal{L}$ be the set of elements that are δ-algebraic over K. Since $x \in \mathcal{L}_a$ if and only if

$$\mathrm{tr.deg.} K(x, \delta x, \delta^2 x, ...)/K < \infty,$$

it is clear that $\mathcal{L}_a$ is a *δ-invariant subfield* of $\mathcal{L}$. Since ϕ and δ commute, it is also *ϕ-invariant.* Let

$$\mathcal{U}_a = \mathcal{U} \cap \mathcal{L}_a^n$$

be the C-subspace of $\mathcal{U}$ consisting of solutions all of whose coordinates are δ-algebraic. Similarly, define $\widetilde{\mathcal{L}}_a \subset \widetilde{\mathcal{L}}$ to be the field of elements that are δ-algebraic over $\widetilde{K}$, and $\widetilde{\mathcal{U}}_a = \widetilde{\mathcal{U}} \cap \widetilde{\mathcal{L}}_a^n$.

If $\sigma \in \widetilde{\mathcal{G}} = \mathcal{G}(\widetilde{C}) = \mathrm{Aut}_{\phi,\delta}(\widetilde{\mathcal{L}}/\widetilde{K})$ and $x \in \widetilde{\mathcal{L}}_a$, then $\sigma(x) \in \widetilde{\mathcal{L}}_a$ because σ commutes with δ. Thus $\widetilde{\mathcal{G}}$ preserves $\widetilde{\mathcal{L}}_a$, and in its standard representation on $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{U}}_a$ becomes an invariant subspace.

Lemma 25. *We have* $\widetilde{\mathcal{U}}_a = \widetilde{C} \otimes_C \mathcal{U}_a$.

Proof. Let $\mathcal{R}$ be the δ-parametrized PV ring in $\mathcal{L}$. As $\mathcal{U} \subset \mathcal{R}^n$, $\widetilde{\mathcal{U}} \subset \widetilde{C} \otimes_C \mathcal{R}^n$ and it is enough to prove that

$$(\widetilde{C} \otimes_C \mathcal{R})_a = \widetilde{C} \otimes_C \mathcal{R}_a. \tag{4.1}$$

The proof of (4.1) is done precisely as in [A-D-H-W], Lemma A.15, where the same statement is proved if the derivation δ is replaced by a difference operator (*i.e.* a field automorphism, denoted there σ). The only non-formal fact used in the proof of that Lemma would be, in our context, the statement that for any $x \in \widetilde{C} - C$ there exists a δ-automorphism of $\widetilde{C}$ over C not fixing x. For this, see the *proof* of Proposition 1.1 in [M-RDCF]. ☐

Corollary 26. *The C-subspace $\mathcal{U}_a \subset \mathcal{U}$ is G-invariant.*

Proof. As $\widetilde{\mathcal{U}}_a$ is $\widetilde{\mathcal{G}}$-invariant and $\widetilde{\mathcal{G}}$ is Zariski dense in $\widetilde{G}$, $\widetilde{\mathcal{U}}_a$ is $\widetilde{G}$-invariant. But the subspace $\mathcal{U}_a$ and the algebraic group $G \subset \mathrm{GL}(\mathcal{U})$ are defined over C, $\widetilde{\mathcal{U}}_a$ is nothing but the $\widetilde{C}$-points of $\mathcal{U}_a$ (by the last lemma) and $\widetilde{G} = G(\widetilde{C})$. Thus $\mathcal{U}_a$ is G-invariant. ☐

5. A Galoisian criterion for δ-integrability

Let (K, ϕ, δ) be as above (ϕ and δ commuting with each other, $C = K^\phi$ algebraically closed). Recall that (3.1) is called δ-integrable if the associated ϕ-module (W, Φ) carries a compatible connection ∇ making (W, Φ, ∇) a (ϕ, δ)-module over K. Equivalently, integrability means that there exists a matrix $B \in \mathfrak{gl}_n(K)$ such that

$$\delta(A) = \phi(B)A - AB. \tag{5.1}$$

Proposition 27. *The system (3.1) is δ-integrable if and only if there exists a matrix $D \in \mathfrak{gl}_n(C)$ such that*

$$\delta(V_\sigma) = V_\sigma D - D V_\sigma \tag{5.2}$$

for every $\sigma \in \mathcal{G}$.

To be precise, the meaning of the criterion is the following. Fix a fundamental solution matrix U with coefficients in $\mathcal{R}$. For any δ-ring extension $C \subset C'$, and for any $\sigma \in \mathcal{G}(C')$, the matrix $V_\sigma \in \mathrm{GL}_n(C')$ such that $\sigma(U) = UV_\sigma$ satisfies (5.2) in $\mathfrak{gl}_n(C')$. Alternatively, if $\mathcal{I} \subset C\{X_{ij}, \det(X)^{-1}\}_\delta$ is the ideal of definition of $\mathcal{G}$, then for any $1 \le i, j \le n$

$$\delta(X_{ij}) - \sum_\ell X_{i\ell} D_{\ell j} + \sum_\ell D_{i\ell} X_{\ell j} \in \mathcal{I}.$$

Proof. Suppose A and B satisfy (5.1). Let $\mathcal{R} \subset \mathcal{L}$ be the δ-parametrized PV ring and extension, and $U \in \mathrm{GL}_n(\mathcal{R})$ a fundamental matrix. Then

$$\phi(\delta(U) - BU) = \delta(AU) - \phi(B)AU = A(\delta(U) - BU),$$

so

$$D := U^{-1}(\delta(U) - BU) \in \mathfrak{gl}_n(C),$$

as it is fixed by ϕ and $\mathcal{L}^\phi = C$. Calculating $\delta(\sigma U) = \sigma(\delta U)$ for $\sigma \in \mathcal{G}(C')$, C' as above, we find

$$\delta(U)V_\sigma + U\delta(V_\sigma) = \delta(UV_\sigma) = \delta(\sigma U) = \sigma(\delta U) = \sigma(BU + UD) = BUV_\sigma + UV_\sigma D,$$

or

$$\delta(V_\sigma) = V_\sigma D - DV_\sigma.$$

Conversely, if $D \in \mathfrak{gl}_n(C)$ satisfies the last equation, define

$$B := \delta(U)U^{-1} - UDU^{-1} \in \mathfrak{gl}_n(\mathcal{R}) \subset \mathfrak{gl}_n(\mathcal{L}) \subset \mathfrak{gl}_n(\widetilde{\mathcal{L}}).$$

Then, for $\sigma \in \mathcal{G}(\widetilde{C})$

$$\sigma(B) = \delta(UV_\sigma)V_\sigma^{-1}U^{-1} - UV_\sigma DV_\sigma^{-1}U^{-1} =$$
$$= \delta(U)U^{-1} + U(V_\sigma D - DV_\sigma)V_\sigma^{-1}U^{-1} - UV_\sigma DV_\sigma^{-1}U^{-1} = B,$$

so $B \in \mathfrak{gl}_n(\widetilde{K})$ by the δ-parametrized Galois correspondence: $\widetilde{\mathcal{L}}^{\mathcal{G}} = \widetilde{K}$. But $\mathcal{R} \cap \widetilde{K} = K$ so we can descend the field of ϕ-scalars and deduce that $B \in \mathfrak{gl}_n(K)$. We compute

$$\phi(B)A - AB = \delta(AU)U^{-1} - AUDU^{-1} - AB$$
$$= \delta(A) + A\delta(U)U^{-1} + A(B - \delta(U)U^{-1}) - AB = \delta(A).$$

Thus A satisfies the condition for δ-integrability. □

Remark. i) The above Proposition does not require $K^\phi = C$ to be differentially closed.

ii) Compare [H-S08], Proposition 2.9. Assuming C is differentially closed, the relation $\delta(V_\sigma) = [V_\sigma, D]$ is *integrated* there to conclude that $\mathcal{G}$ is *conjugate* to a δ-constant group (see Section 4.2.1). If C is not differentially closed, such a conjugation exists only over a suitable non-trivial δ-extension of C, as we need to find an invertible matrix E solving $\delta E = -DE$, *i.e.*, a fundamental matrix for $\delta y = -Dy$. Having such an E at hand,

$$\delta(E^{-1}V_\sigma E) = -E^{-1}\delta(E)E^{-1}V_\sigma E + E^{-1}\delta(V_\sigma)E + E^{-1}V_\sigma\delta(E)$$
$$= E^{-1}(DV_\sigma - V_\sigma D + \delta(V_\sigma))E = 0.$$

6. ∂-modules over M and monodromy

6.1. Triviality of ∂-modules over M.

From now on:

- $K = \bigcup_{\Lambda \in \mathfrak{L}} K_\Lambda$, $\phi f(z) = f(qz)$ and $\partial f(z) = f'(z)$ are as in § 2.1.
- $M = \mathscr{M}(\mathbb{C})$ is the field of meromorphic functions on $\mathbb{C}$, with the same ∂, ϕ.
- $F = \mathbb{C}((z))$, $\mathcal{O}_F = \mathbb{C}[[z]]$, same ∂, ϕ.
- $K \subset M \subset F$, as (ϕ, ∂)-fields.

Let $B \in \mathfrak{gl}_n(M)$. For any $\zeta \in \mathbb{C}$ let M_ζ be the field of germs of meromorphic functions at ζ, and $\mathcal{O}_\zeta$ its valuation ring. Consider the system

$$\partial y = By \tag{6.1}$$

and the associated ∂-module $W = M^n$, $\nabla(y) = \partial y - By$.

Lemma 28. *The following are equivalent:*

i) The system (6.1) has a full set of solutions in M_ζ.
ii) The ∂-module $W_\zeta = M_\zeta \otimes_M W$ is trivial.
iii) B has an apparent singularity at ζ, i.e., *there exists a gauge transformation*

$$\widetilde{B} = \partial P \cdot P^{-1} + PBP^{-1}$$

with $P \in \mathrm{GL}_n(M_\zeta)$ such that $\widetilde{B}$ is regular (holomorphic) at ζ.

Proof. (i) is equivalent to $\dim_{\mathbb{C}} W_\zeta^\nabla = n$. If this is the case then by the Wronskian Lemma $W_\zeta = W_\zeta^\nabla \otimes_{\mathbb{C}} M_\zeta$. This proves that (i) implies (ii). If (ii) holds then there exists even a P with $\widetilde{B} = 0$, whence (iii). Note that in this case, since $\partial(P^{-1}) = -P^{-1}\partial(P)P^{-1}$ we have $\partial(P^{-1}) = BP^{-1}$ so P^{-1} is a fundamental matrix of solutions in M_ζ. Finally, if (iii) holds then to get (i) we may assume, by a change of coordinates, that B was regular at ζ to begin with. By the basic existence and uniqueness theorem for linear systems of ordinary differential equations, we can find a full set of solutions in M_ζ. □

According to the above lemma, we say that a ∂-module W has apparent singularities if all the singularities of an associated differential system are apparent.

Corollary 29. *A ∂-module W over M has apparent singularities if and only if it is trivial.*

Proof. Since all the singularities are apparent, one can continue solutions meromorphically along paths indefinitely. Since $\mathbb{C}$ is simply connected, this yields a single-valued solution for any initial conditions at 0. □

6.2. **Periodicity and monodromy.** Suppose now that $B \in \mathfrak{gl}_n(K_\Lambda)$ and all the singularities of (6.1) are apparent. Let U be a fundamental matrix for (6.1) in M. If $\omega \in \Lambda$ then $U(z+\omega)$ also satisfies (6.1) so there exists a matrix $M(\omega) \in \mathrm{GL}_n(\mathbb{C})$ such that

$$U(z+\omega) = U(z)M(\omega). \tag{6.2}$$

The map $\omega \mapsto M(\omega)$ is a homomorphism $\Lambda \to \mathrm{GL}_n(\mathbb{C})$, the *monodromy representation*. Since Λ is abelian, its image, the *monodromy group*, is an abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$.

If we replace $U(z)$ by another fundamental matrix $U(z)T$, with $T \in \mathrm{GL}_n(\mathbb{C})$, then the monodromy representation undergoes conjugation: $\omega \mapsto T^{-1}M(\omega)T$. Thus intrinsically, the monodromy representation is well-defined only up to conjugation.

Lemma 30. *Let $Z \in \mathrm{GL}_n(M)$ be another matrix of meromorphic functions satisfying* (6.2). *Then $U(z) = Q(z)Z(z)$ for a matrix $Q(z) \in \mathrm{GL}_n(K_\Lambda)$. If S is a K_Λ-subalgebra of M containing the entries of Z then the entries of U are in S as well.*

Proof. $Q(z) = U(z)Z(z)^{-1}$ is invariant under translation by Λ. It follows that its entries are Λ-elliptic. □

6.3. **Consequences of ϕ-isomonodromy.** Assume that W is a (ϕ, ∂)-module over K, so that W is a ϕ-isomonodromic ∂-module. Fix a basis of W over K and identify it with K^n, where Φ and ∇ are given by matrices A and B as above.

Lemma 31. *All the singularities of* (W, ∇) *are apparent, hence* W *becomes* ∂*-trivial over* M*.*

Proof. W is defined over some K_Λ. Since $W^{(\phi)}$ and W are isomorphic, if ζ is a regular point of W (*i.e.* an apparent singularity of the system (6.1)), so is $q\zeta$. For some $\varepsilon > 0$, every ζ in the punctured disk $0 < |\zeta| < \varepsilon$ is regular, so we deduce that every $\zeta \neq 0$ is regular. But then $\zeta = 0$ is regular too, because any $0 \neq \omega \in \Lambda$ is regular and B is Λ-periodic. The Lemma follows from Corollary 29. □

By the discussion in the previous subsection we may associate with W a monodromy representation $\omega \mapsto M(\omega)$ of Λ, well-defined up to conjugation by $\mathrm{GL}_n(\mathbb{C})$.

Proposition 32. *The monodromy representation* $\omega \mapsto M(\omega)$ *is potentially unipotent: there exists a sublattice* $\Lambda' \subset \Lambda$ *such that* $M(\omega)$ *is unipotent for every* $\omega \in \Lambda'$*.*

Proof. Let $U(z) \in \mathrm{GL}_n(M)$ be a fundamental matrix for (W_M, ∇) and $U^{(\phi)}(z) = U(qz)$ the corresponding fundamental matrix for $(W_M^{(\phi)}, \nabla^{(\phi)})$. Let $\iota : W^{(\phi)} \simeq W$ be an isomorphism of ∂-modules, explicitly

$$\iota(y) = A^{-1} y$$

where $A \in \mathrm{GL}_n(K)$ satisfies the consistency condition $q\phi(B) = ABA^{-1} + \partial(A)A^{-1}$. Since $\iota \circ \nabla^{(\phi)} = \nabla \circ \iota$, the homomorphism ι maps $U^{(\phi)}$ to a fundamental matrix for W, so for some $C \in \mathrm{GL}_n(\mathbb{C})$ we have

$$A(z)^{-1} U(qz) = U(z)C.$$

Let Λ be a lattice of periodicity for A and B. Substituting $z + \omega$ ($\omega \in \Lambda$) for z we get

$$A(z)^{-1} U(qz) M(\omega)^q = U(z) M(\omega) C = A(z)^{-1} U(qz) C^{-1} M(\omega) C.$$

It follows that

$$M(\omega) = C M(\omega)^q C^{-1}$$

and that the set of eigenvalues of $M(\omega)$ is invariant under raising to power q. Since there are only finitely many eigenvalues, these eigenvalues must be roots of unity, so $M(\omega)$ is potentially unipotent. Since the monodromy representation is determined by the (commuting) matrices $M(\omega_1)$ and $M(\omega_2)$, the Proposition follows. □

7. ∂-integrability and solvability

The last Proposition has a consequence for the solvability of the *difference Galois group* of a ∂-integrable system

$$\phi(y) = Ay. \tag{7.1}$$

Definition 33. A ϕ-module (W, Φ) over K is *solvable* if there exists a filtration

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = W$$

by ϕ-submodules such that $\dim_K W_i = i$.

If (W, Φ) is associated to the system (3.1) then it is solvable if and only if A is gauge equivalent to an upper-triangular matrix.

Theorem 34. *If* (W, Φ) *is* ∂*-integrable, it is solvable. In fact, if* ∇ *is a compatible* ∂*-connection on* W*, a filtration as above exists where each* W_i *is a* (ϕ, ∂)*-submodule.*

Proof. It is enough to find W_1, because then we can apply induction on the dimension to W/W_1 and lift the filtration found there to complete the filtration of W.

Let (W, Φ, ∇) be a (ϕ, ∂)-module structure on W as in Lemma 12. Fix matrices A and B as above, with respect to some basis of W over K. Let Λ be a lattice so that all the entries of A and B lie in K_Λ, *i.e.*, are Λ-elliptic. Regarding (W, ∇) as a ϕ-isomonodromic ∂-module, and replacing Λ by a smaller lattice if necessary, we conclude, by the previous section, that the monodromy representation

$$\Lambda \to \mathrm{GL}_n(\mathbb{C}), \quad \omega \mapsto M(\omega),$$

attached to (W, ∇) is unipotent, and that (W, ∇) has a fundamental matrix $U \in \mathrm{GL}_n(M)$. Let $T \in \mathrm{GL}_n(\mathbb{C})$ be such that $TM(\omega)T^{-1}$ are all upper-triangular with 1's along the diagonal. Replacing the fundamental matrix $U \in \mathrm{GL}_n(M)$ by UT^{-1}, $M(\omega)$ is replaced by $TM(\omega)T^{-1}$. We may therefore assume, without loss of generality, that $M(\omega)$ are already upper-triangular unipotent. In particular, the first column of U is a column vector

$$u \in K_\Lambda^n \subset M^n,$$

because it satisfies $u(z+\omega) = u(z)$ for all $\omega \in \Lambda$. But the column vectors of U form a basis of W_M^∇, and the first column is, as we have just seen, in W. It follows that W^∇ is non-zero.

The relation $q\Phi \circ \nabla = \nabla \circ \Phi$ implies that the $\mathbb{C}$-space W^∇ is Φ-invariant, hence there exists an eigenvector $e_1 \in W^\nabla$ for Φ. The 1-dimensional subspace $W_1 = Ke_1$ is the desired (Φ, ∇)-submodule. $\square$

Corollary 35. *Let G be the difference Galois group of the system*

$$\phi(y) = Ay$$

over K. Assume that (W, Φ) is ∂-integrable. Then G is solvable.

Proof. This is a direct consequence of the Tannakian correspondence, Proposition 21, see also [S-vdP97], Theorem I.1.21. Since W is a solvable ϕ-module, the standard representation of G on the solution space $\mathcal{U}$ is solvable too, and G is contained in the Borel subgroup of upper triangular matrices. $\square$

8. ∂-Triviality over the ring S_0

Let $A \in \mathrm{GL}_n(K_\Lambda)$, $B \in \mathfrak{gl}_n(K_\Lambda)$ be a consistent pair of matrices, and W the corresponding (ϕ, ∂)-module. As we have seen, the system

$$\partial(y) = By$$

has only apparent singularities and, hence, it has a complete set of solutions over M. Let $U \in \mathrm{GL}_n(M)$ be a fundamental matrix of solutions, and $\omega \mapsto M(\omega)$ its monodromy representation. As we have also seen, replacing Λ by a sublattice, we may assume that $M(\omega)$ are unipotent for all ω. Let ω_1, ω_2 be an oriented basis of Λ and $M_i = M(\omega_i)$.

Theorem 36. *The entries of $U(z)$ are in the ring $S_0 = K[z, \zeta(z, \Lambda)]$.*

Proof. According to Lemma 30 we only have to exhibit a matrix $Z(z) \in \mathrm{GL}_n(S_0)$ with the same monodromy as $U(z)$, *i.e.*, satisfying

$$Z(z + \omega_i) = Z(z)M_i. \tag{8.1}$$

Without loss of generality we may assume that $\omega_2 = 1$ and $\omega_1 = \tau \in \mathfrak{H}$ (the upper half plane). Since M_2 is unipotent,

$$N_2 = \log(M_2) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(M_2 - 1)^k$$

is a nilpotent matrix satisfying $\exp(N_2) = M_2$. Note that N_2 commutes with M_1, because M_1 and M_2 commute. The matrix $Z(z)$ satisfies (8.1) if and only if $\widetilde{Z}(z) = Z(z)\exp(-zN_2)$ satisfies

$$\widetilde{Z}(z+1) = \widetilde{Z}(z), \;\; \widetilde{Z}(z+\tau) = \widetilde{Z}(z)V \tag{8.2}$$

with $V = M_1 \exp(-\tau N_2)$. Since the entries of $\exp(-zN_2)$ are polynomials, hence lie in S_0, it is enough to find $\widetilde{Z}(z) \in \mathrm{GL}_n(S_0)$ satisfying (8.2). Note that V is also unipotent, because M_1 and N_2 commute, so $\log(V)$ is nilpotent.

Let $\ell \in S_0$ satisfy $\ell(z+1) = \ell(z)$ and $\ell(z+\tau) = \ell(z)+1$. By Lemma 3, such an ℓ can be taken to be a $\mathbb{C}$-linear combination of z and $\zeta(z,\Lambda)$. We now set

$$\widetilde{Z}(z) = \exp(\ell(z)\log(V)).$$

We have $\widetilde{Z} \in \mathrm{GL}_n(S_0)$, because its entries are polynomials in $\ell(z)$. In addition, (8.2) is satisfied. This concludes the proof. □

Corollary 37. *Suppose $f \in F$ satisfies a linear homogeneous diferential equation with coefficients from K, as well as a linear homogeneous ϕ-difference equation over K. Then $f \in S_0$.*

Proof. Let $W \subset F$ be the (ϕ, ∂)-module spanned by f, as in Example 15, and let $e_1, \ldots, e_n$ be a basis of W over K, with $e_1 = f$. Let $A^* = (a_{ij}) \in \mathrm{GL}_n(K)$ and $B^* = (b_{ij}) \in \mathfrak{gl}_n(K)$ be defined by

$$\phi e_i = \sum_j a_{ij} e_j, \;\; \partial e_i = \sum_j b_{ij} e_j.$$

These are *not* the matrices A and B that were associated to W before, but rather ${}^tA^{-1}$ and $-{}^tB$, and they define the *dual* (ϕ, ∂)-module W^*. It is easily checked that A^* and B^* are consistent. This follows formally from the consistency of (A, B), or by duality if we think of the pair (A^*, B^*) as the pair associated with W^*. The last Theorem, applied to (A^*, B^*), shows that the equation $\partial y = B^* y$ has a full set of solutions in S_0, *i.e.*, there exists a matrix $U \in \mathrm{GL}_n(S_0)$ satisfying $\partial U = B^* U$. In *any ∂-field extension* of S_0*, whose constants are still* $\mathbb{C}$, we find the same solution set for the last system, namely the $\mathbb{C}$-linear combinations of the columns of U. Since $u = {}^t(e_1, \ldots, e_n) \in F^n$ is such a solution, the e_i, and in particular $f = e_1$, lie in S_0. □

9. Galois theory and algebraic independence, rank one

9.1. **The desired theorems.** Our ultimate goal is to prove the following stength-ening of Corollary 37. Recall that $f \in F$ is called *∂-algebraic* over K if it satisfies an equation

$$P(f, \partial(f), \partial^2(f), ..., \partial^r(f)) = 0$$

for some non-zero $P \in K[X_0, X_1, ..., X_r]$. If f is not ∂-algebraic, it is called ∂-transcendental, or *hypertranscendental*.

Theorem 38 (Main Theorem)**.** *Assume that $f \in F$ satisfies a linear homogeneous ϕ-difference equation over K. If f is ∂-algebraic over K, then $f \in S$.*

In the rank one case, we can be more precise.

Theorem 39. *Assume that $f \in F$ satisfies*

$$\phi(f) = af$$

with $a \in K$. If f is ∂-algebraic over K, then $z^{-r} f \in K$ for some $r \in \mathbb{Z}$.

We shall deduce the rank one case from the following Proposition, whose proof is postponed to Section 9.3, and which holds for an arbitrary (ϕ, δ)-field $\mathcal{F}$ with algebraically closed field of ϕ-constants, in lieu of F.

Proposition 40. *Let $\mathcal{F}$ be a (ϕ, δ)-field containing $K' = K(z)$ with $\phi \circ \delta = \delta \circ \phi$ and $\mathcal{F}^{\phi} = \mathbb{C}$. Let $f \in \mathcal{F}^{\times}$ satisfy $\phi(f) = af$ with $a \in K^{\times}$. Then f is δ-algebraic over K' if and only if*

$$a = c\frac{\phi(b)}{b}$$

for some $c \in \mathbb{C}^{\times}$ and $b \in K^{\times}$.

Proof. (That Proposition 40 implies Theorem 39). First, note that

- f is ∂-algebraic over K if and only if
- tr.deg.$K(f, \partial f, \partial^2 f, ...)/K < \infty$, if and only if
- tr.deg.$K(z, f, \delta f, \delta^2 f, ...)/K' < \infty$, if and only if
- f is δ-algebraic over K'.

Assume that we have proved the Proposition and f is as in Theorem 39. Assume that f is ∂-algebraic over K, hence δ-algebraic over K'. By the Proposition $a = c\phi(b)/b$ for some $c \in \mathbb{C}^{\times}$ and $b \in K^{\times}$. It follows that $g = f/b \in F$ satisfies

$$\phi(g) = cg.$$

Since g is a power series in z with constant coefficients, this forces $g = Cz^r$ for $C \in \mathbb{C}$ and $r \in \mathbb{Z}$ (and $c = q^r$). Thus $z^{-r} f = Cb \in K$ as desired. Note that the fact that $F = \mathbb{C}((z))$ enters the proof only at the last step. □

9.2. **Some properties of divisors.** For an abelian group R let

$$\mathcal{D}_{\Lambda}(R) = \mathrm{Div}(\mathbb{C}/\Lambda; R)$$

be the group of Λ-periodic divisors with values in R. We write $D \in \mathcal{D}_{\Lambda}(R)$ either as a finite linear combination

$$D = \sum_{\xi \in \mathbb{C}/\Lambda} r_{\xi}[\xi]$$

or as a Λ-periodic, discretely supported function $D : \mathbb{C} \to R$, $D(\xi) = r_{\xi}$. The support $\mathrm{supp}(D)$ is the set of $\xi \in \mathbb{C}$ where $D(\xi) \neq 0$.

We let $\mathcal{D}^0_{\Lambda}(R)$ be the subgroup of divisors of degree 0, where

$$\deg_{\Lambda} : \mathcal{D}_{\Lambda}(R) \to R, \ \ \deg_{\Lambda}(D) = \sum_{\xi \in \mathbb{C}/\Lambda} r_{\xi}.$$

We let $\mathcal{P}_\Lambda \subset \mathcal{D}^0_\Lambda(\mathbb{Z})$ be the subgroup of principal divisors, *i.e.*, of divisors of the shape $\mathrm{div}(f)$ for $f \in K_\Lambda^\times$. By the Abel-Jacobi theorem a $\mathbb{Z}$-valued divisor $D = \sum_{\xi\in\mathbb{C}/\Lambda} r_\xi[\xi]$ is principal if and only if $\deg_\Lambda(D) = 0$ and

$$s_\Lambda(D) = \sum_{\xi\in\mathbb{C}/\Lambda} r_\xi \xi = 0 \in \mathbb{C}/\Lambda.$$

We let $\phi \in End(\mathcal{D}_\Lambda)$ be defined as $\phi(D)(\xi) = D(q\xi)$; alternatively, if $D = \sum_{\xi\in\mathbb{C}/\Lambda} r_\xi[\xi]$, then

$$\phi(D) = \sum_{\xi\in\mathbb{C}/\Lambda} r_{q\xi}[\xi].$$

Note:

- $\mathrm{div}(\phi(f)) = \phi(\mathrm{div}(f))$ for $f \in K_\Lambda^\times$.
- $\deg_\Lambda(\phi(D)) = q^2 \deg_\Lambda(D)$.
- $q s_\Lambda(\phi(D)) = q^2 s_\Lambda(D)$.

To prove the last point, let $D = \sum_{\xi\in\mathbb{C}/\Lambda} r_\xi[\xi]$. Then

$$q s_\Lambda(\phi(D)) = q \sum_{\xi\in\mathbb{C}/\Lambda} r_{q\xi}\xi = \sum_{\xi\in\mathbb{C}/\Lambda} r_{q\xi} q\xi = q^2 \sum_{\eta\in\mathbb{C}/\Lambda} r_\eta \eta = q^2 s_\Lambda(D)$$

because for every η there are q^2 values of ξ with $q\xi = \eta$.

Lemma 41. *(i) For any abelian group R, $\phi - 1 \in End(\mathcal{D}_\Lambda(R))$ is injective.*
(ii) If $D \in \mathcal{D}_\Lambda(\mathbb{R})$ and $(\phi-1)(D) \in \mathcal{D}_\Lambda(\mathbb{Z})$, then $D \in \mathcal{D}_\Lambda(\mathbb{Z})$.
(iii) If $D \in \mathcal{D}_\Lambda(\mathbb{R})$ and $(\phi-1)(D) \in \mathcal{P}_\Lambda$, then $D \in \mathcal{P}_{\Lambda'}$ for $\Lambda' = q(q-1)\Lambda$.

Proof. (i) If D is Λ-periodic and non-zero, choose $0 \neq \xi \in \mathrm{supp}(D)$. If $D = \phi(D) = \phi^2(D) = \cdots$ then $\xi, \xi/q, \xi/q^2, ...$ are all in the support of D, contradicting the fact that the support is discrete.

(ii) If the claim is false, the image of D in $\mathcal{D}_\Lambda(\mathbb{R}/\mathbb{Z})$ is annihilated by $\phi - 1$ and non-zero, contradicting (i).

(iii) Assume that $(\phi-1)(D) \in \mathcal{P}_\Lambda$. By (ii), $D \in \mathcal{D}_\Lambda(\mathbb{Z})$. Since

$$0 = \deg_\Lambda((\phi-1)(D)) = (q^2-1)\deg_\Lambda(D)$$

we must have $D \in \mathcal{D}^0_\Lambda(\mathbb{Z})$. Now

$$q(q-1)s_\Lambda(D) = q s_\Lambda((\phi-1)(D)) = 0.$$

Let Π be a fundamental parallelogram for Λ, and $m = q(q-1)$. If $D = \sum_{\xi\in\Pi} n_\xi[\xi_\Lambda]$ where $\xi_\Lambda = \xi \mod \Lambda$, then

$$s_{\Lambda'}(D) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{\xi\in\Pi} n_\xi(\xi + i\omega_1 + j\omega_2) \mod \Lambda' = m^2 \sum_{\xi\in\Pi} n_\xi \xi \mod \Lambda',$$

where ω_1 and ω_2 span Π (recall $\sum n_\xi = 0$). But $m\sum_{\xi\in\Pi} n_\xi\xi \in \Lambda$, so

$$m^2 \sum_{\xi\in\Pi} n_\xi \xi \in m\Lambda = \Lambda'.$$

□

9.3. **Proof of Proposition 40.**

9.3.1. *First steps.* Assume that $f \in \mathcal{F}$ satisfies

$$\phi(f) = af.$$

If $a = c\phi(b)/b$ for $c \in \mathbb{C}^\times$ and $b \in K^\times$, then replacing f by f/b we may assume that $a \in \mathbb{C}^\times$. Then

$$\phi(\frac{\delta f}{f}) = \frac{\delta(\phi(f))}{\phi(f)} = \frac{\delta f}{f},$$

so $\delta f/f \in \mathcal{F}^\phi = \mathbb{C}$ and f is δ-algebraic over K'.

Conversely, assume that f is δ-algebraic over K'. Let $u = \frac{\delta f}{f} \in \mathcal{F}$, so that

$$\phi(u) = u + \frac{\delta a}{a}.$$

Since u, like f, is also δ-algebraic over K', Theorem C.8 of [D-H21][5] shows that there exists a monic operator $\mathscr{L} \in \mathbb{C}[\delta]$ and a $v \in K'$ such that

$$\mathscr{L}(\frac{\delta a}{a}) = \phi(v) - v.$$

Embed $K(z)$ in $K((z))$ via the completion at 0, where we regard K as the field of constants. Extend ϕ to $K((z))$ so that $\phi(z) = qz$. Then the embedding is ϕ-compatible. (Warning: $K((z))$ can not be regarded as a subfield of $F = \mathbb{C}((z))$, despite the fact that $K \subset F$.) Write

$$v = \sum_{i \geq s} v_i z^i \in K((z))$$

with $v_i \in K$. Since $a \in K$, $\mathscr{L}(\frac{\delta a}{a})$ in fact lies in $K[z] \subset K(z)$ and as a polynomial in z with coefficients from K it looks like

$$\mathscr{L}(\frac{\delta a}{a}) = \partial^\ell(\frac{\partial a}{a})z^{\ell+1} + \text{lower terms}.$$

Here $\ell = \deg(\mathscr{L})$. Comparing the expansions we see that

$$\partial^\ell(\frac{\partial a}{a}) = q^{\ell+1}\phi(v_{\ell+1}) - v_{\ell+1}. \tag{9.1}$$

9.3.2. *Completion of the proof.* We shall deduce Proposition 40 from equation (9.1). Quite generally, if $h \in K_\Lambda$ and $z_0 \in \mathbb{C}$, and if

$$h(z) = \sum_{n >> -\infty} a_n(h, z_0)(z - z_0)^n$$

is the Laurent expansion of h at z_0, we let for $\ell \geq 1$,

$$a_{-\ell}(h) = \sum_{z_0 \in \mathbb{C}/\Lambda} a_{-\ell}(h, z_0)[z_0] \in \mathcal{D}_\Lambda(\mathbb{C}).$$

We call it the degree $-\ell$ polar divisor of h. Note that $a_n(h, z_0)$ are Λ-periodic in z_0, so this divisor is well-defined. Since for z near z_0 we have

$$\phi h(z) = h(qz) = \sum_{n >> -\infty} a_n(h, qz_0)(qz - qz_0)^n = \sum_{n >> -\infty} q^n a_n(h, qz_0)(z - z_0)^n,$$

we get

$$a_n(\phi h, z_0) = q^n a_n(h, qz_0),$$

[5]See also the proof of Lemma 48 below, where the same argument appears again.

hence

$$a_{-\ell}(\phi h) = q^{-\ell} \sum_{z_0 \in \mathbb{C}/\Lambda} a_{-\ell}(h, qz_0)[z_0] = q^{-\ell}\phi(a_{-\ell}(h)).$$

Now, (9.1) yields

$$a_{-\ell-1}(\partial^\ell(\frac{\partial a}{a})) = q^{\ell+1} a_{-\ell-1}(\phi(v_{\ell+1})) - a_{-\ell-1}(v_{\ell+1}) = (\phi - 1)(a_{-\ell-1}(v_{\ell+1})).$$

On the other hand

$$a_{-\ell-1}(\partial^\ell(\frac{\partial a}{a})) = (-1)^\ell \ell! a_{-1}(\frac{\partial a}{a}) = (-1)^\ell \ell! \mathrm{div}(a).$$

It follows that

$$\mathrm{div}(a) = (\phi - 1)(D)$$

where $D = (-1)^\ell a_{-\ell-1}(v_{\ell+1})/\ell!$. By part (iii) of Lemma 41 we deduce that there exists $b \in K^\times_{\Lambda'}$, $\Lambda' = q(q-1)\Lambda$, such that $D = \mathrm{div}(b)$. In particular

$$\mathrm{div}(a) = \mathrm{div}\frac{\phi(b)}{b},$$

so

$$a = c\frac{\phi(b)}{b}$$

for some $c \in \mathbb{C}^\times$, as desired.

10. Simple Galois group

10.1. **Comparison of the Galois groups over K and $K(z)$.** Consider the system (3.1), with $A \in \mathrm{GL}_n(K)$, and its Galois group G. We want to compare G with the Galois group H of the same system over K'.

In order to compare H and G, we adapt the proof of [DHR, Proposition 1.6] which was written for Mahler difference systems. Let R' be a Picard-Vessiot ring for $\phi(y) = Ay$ over K' and let L' be the associated PV-extension. For $U \in \mathrm{GL}_n(R')$ a fundamental solution matrix, it is easily seen that $R = K[U, \det(U)^{-1}]$ is a PV-ring for $\phi(y) = Ay$ over K and we denote by L the associated PV-extension. The inclusion $R \subset R'$ yields a closed immersion $\iota : H \hookrightarrow G$. Identifying H with its image in G, we can consider $L^H = {L'}^H \cap L = K' \cap L$. Since L^H/K is a sub-ϕ-extension of K'/K, it is easily seen that there exists an integer N such that $L^H = K(z^N)$. Now the Galois correspondence yields that H is normal in G and that G/H is isomorphic to $\mathrm{Aut}_\phi(L^H/K)$ which is trivial if N equals zero and $\mathbb{G}_m$ otherwise.

Lemma 42. *Assume (by replacing ϕ by some ϕ^r) that L is a field. Then:*

i) G is connected.
ii) If G is simple[6], then $H = G$.
iii) G is solvable if and only if H is solvable.

Proof. i) If G^0 is the connected component of G, then it is normal, and the finite group G/G^0 is, by the Galois correspondence, the Galois group of a finite extension of K contained in L to which ϕ extends as an automorphism. But such an extension must be K itself by Lemma 2, so $G = G^0$.

ii) If the normal subgroup H is trivial, G would be $\mathbb{G}_m$, so by the simplicity of G we must have $H = G$.

[6] By "G is simple", we mean that it is noncommutative and has no proper nontrivial normal closed subgroup.

iii) This follows from the fact that G/H is either trivial or $\mathbb{G}_m$.

□

Note that the group H need not be connected. For example, let $(M, N) = 1$, $y = x^M$, $z = x^N$ and $q = q_1^N$. If $A = (q_1^M)$ and $A' = (q_1^M) \oplus (q_1^N)$ then $L = K(y)$ with $\phi(y) = q_1^M y$, $K' = K(z)$ with $\phi(z) = q_1^N x$ and $L' = K(x)$ with $\phi(x) = q_1 x$. We would then have $H \simeq \mu_N \subset G = \mathbb{G}_m$.

10.2. **The G simple case.** Consider now the δ-parametrized PV extension

$$\mathcal{L}' = K' \langle U \rangle_\delta$$

and the δ-parametrized Galois group scheme $\mathcal{H}$ of $\mathcal{L}'$ over K'.

As explained before, $\mathcal{H}$ is a Zariski-dense differential subgroup scheme of $H = \mathrm{Aut}_\phi(L'/K')$, in the sense that its points $\mathcal{H}(\widetilde{C})$ in a δ-closure $\widetilde{C}$ of $\mathbb{C}$ form a Zariski dense subset of $H(\widetilde{C})$. We continue to assume that L, L' and $\mathcal{L}'$ are fields.

Proposition 43. *Under the assumption that* $G = \mathrm{Aut}_\phi(L/K)$ *is simple, we have*

$$\mathcal{H} = [\delta]_* H.$$

Consequently, $\mathrm{tr.deg.}(L/K) = \mathrm{tr.deg.}(L'/K') = \delta\mathrm{tr.deg.}(\mathcal{L}'/K')$.

This Proposition seems identical to Lemma 5.1 of [Arr-S] and similar to Proposition 4.11 of [A-D-H-W]. However, because of the need to descend, first from a differentially closed field of ϕ-constants to $\mathbb{C}$, and then from K' to K, we are forced to go through some of the arguments with care.

Proof. As we have seen in Lemma 42, when G is simple, so is H, and $H \simeq G$.

Let $\widetilde{K'}$ be a (ϕ, δ)-extension of K' whose field of ϕ-constants, $\widetilde{C} = (\widetilde{K'})^\phi$ is a differential closure of $\mathbb{C}$. It is known that $\widetilde{C}^\delta$ is just $\mathbb{C}$ ([M-MTDF], Lemma 2.11). Let $\widetilde{L}'$ be the PV extension over $\widetilde{K'}$, and $\widetilde{\mathcal{L}}'$ the δ-parametrized PV extension. See § 4.2.3 for the construction of $\widetilde{K'}, \widetilde{L}'$ and $\widetilde{\mathcal{L}}'$, starting from a differential closure $\widetilde{C}$ of $\mathbb{C}$, and their relation to K', L' and $\mathcal{L}'$.

Let $\widetilde{H}$ and $\widetilde{\mathcal{H}}$ be the difference Galois group and the δ-parametrized difference Galois group over $\widetilde{K}'$. We regard them as subgroups of $\mathrm{GL}_n(\widetilde{C})$, having fixed a fundamental matrix U. Since $\widetilde{H} = H(\widetilde{C})$ and $\widetilde{\mathcal{H}} = \mathcal{H}(\widetilde{C})$ (see § 4.2.3) we only need to show that $\widetilde{\mathcal{H}} = [\delta]_* \widetilde{H}$.

Since the field of ϕ-constants $\widetilde{C}$ is differentially closed and $\widetilde{H}$ is simple, Theorem 22 implies that if $\widetilde{\mathcal{H}} \subsetneq [\delta]_* \widetilde{H}$, there exists a matrix $E \in \mathrm{GL}_n(\widetilde{C})$ such that

$$\widetilde{\mathcal{H}} \subset E \cdot \mathrm{GL}_n(\mathbb{C}) \cdot E^{-1}.$$

Letting $D = \delta(E)E^{-1} \in \mathfrak{gl}_n(\widetilde{C})$ this implies that for any $\sigma \in \widetilde{\mathcal{H}}$,

$$\delta(V_\sigma) = DV_\sigma - V_\sigma D.$$

As in Proposition 27 this implies that the system $\phi(y) = Ay$ is δ-integrable over $\widetilde{K}'$. In fact, the matrix

$$B = \delta(U)U^{-1} - UDU^{-1}$$

which gives the δ-integrability relation

$$\delta(A) = \phi(B)A - AB \tag{10.1}$$

lies in $\mathfrak{gl}_n(\widetilde{C}\otimes_{\mathbb{C}}\mathcal{R}')$ by construction, as U, U^{-1} and $\delta(U)$ have entries in $\mathcal{R}'\subset\mathcal{L}'$ and D has entries in $\widetilde{C}$. It is shown, in Proposition 27, to be invariant under every

$$\sigma\in\widetilde{\mathcal{H}}=\mathcal{H}(\widetilde{C})=\mathrm{Aut}_{\phi,\delta}(\widetilde{C}\otimes_{\mathbb{C}}\mathcal{R}'/\widetilde{C}\otimes_{\mathbb{C}}K').$$

By a refinement of the parametrized Galois correspondence, the matrix B therefore lies in $\mathfrak{gl}_n(\widetilde{C}\otimes_{\mathbb{C}}K')$ and not just in $\mathfrak{gl}_n(\widetilde{K}')$ (see Proposition 53 in Appendix 14). Now let $\{\omega_\alpha\}$ be a basis of $\widetilde{C}$ as a vector space over $\mathbb{C}$ and write $B=\sum_\alpha\omega_\alpha\otimes B_\alpha$ with $B_\alpha\in\mathfrak{gl}_n(K')$. We may assume that $\omega_{\alpha_0}=1$. Since $\phi(\omega_\alpha)=\omega_\alpha$, and since A has entries in K', decomposing the δ-integrability relation to its α-components we get

$$\delta(A)=\phi(B_{\alpha_0})A-AB_{\alpha_0}.$$

Thus without loss of generality we may assume that $B\in\mathfrak{gl}_n(K')$. In other words, we have descended the integrability of the system over $\widetilde{K}'$ to the integrability of the same system over K'.

Equivalently, (10.1) may be written

$$\partial A=q\phi(z^{-1}B)A-A(z^{-1}B). \tag{10.2}$$

Consider the field $K((z))$, to which ϕ is extended as an automorphism so that $\phi(z)=qz$. Embedding $K(z)$ in $K((z))$ via the completion at 0, we expand

$$z^{-1}B=\sum_{-\infty<<\ell}B_\ell z^\ell,\ \ \phi(z^{-1}B)=\sum_{-\infty<<\ell}q^\ell\phi(B_\ell)z^\ell,$$

where $B_\ell\in\mathfrak{gl}_n(K)$. Substituting into (10.2) and comparing the $\ell=0$ terms we get (as $A\in\mathrm{GL}_n(K)$)

$$\partial A=q\phi(B_0)A-AB_0.$$

It follows that the system (3.1), proven to be δ-integrable over K', is also ∂-integrable over K.

We may now apply Corollary 35 to conclude that G is solvable. This contradicts the assumption that G is simple.

The assertion on the transcendence degrees follows from

$$\mathrm{tr.deg.}(L/K)=\dim G,\ \ \mathrm{tr.deg.}(L'/K')=\dim H,\ \ \delta\mathrm{tr.deg.}(\mathcal{L}'/K')=\delta\dim\mathcal{H},$$

(the first two equalities are well-known consequences of the *torsor theorem*; for the last one, a consequence of the same theorem in the δ-parametrized setup, see [H-S08], Proposition 6.26), and $\delta\dim[\delta]_*H=\dim H$ by [Kol, Prop. 10 p.200]. □

11. The Main Theorem in the irreducible case

11.1. **Reduction steps.** To prove Theorem 38 it is enough to prove the following theorem.

Theorem 44. *Let $A\in\mathrm{GL}_n(K)$ and consider the system (3.1). Let $u={}^t(u_1,...,u_n)\in F^n$ be a solution of*

$$\phi(u)=Au,$$

such that the u_i are ∂-algebraic over K. Then every $u_i\in S=K[z,z^{-1},\zeta(z,\Lambda)]$.

Indeed, suppose this is proven, and $f\in F$ is a solution of the linear homogenous ϕ-difference equation

$$\phi^n(y)+a_1\phi^{n-1}(y)+\cdots+a_{n-1}\phi(y)+a_ny=0$$

$(a_i \in K)$. We may assume that $a_n \neq 0$, otherwise f satisfies a similar equation of order $n-1$. The *companion matrix*

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix},$$

therefore lies in $\mathrm{GL}_n(K)$, and

$$u = {}^t(f, \phi(f), ..., \phi^{n-1}(f))$$

is a solution of $\phi(u) = Au$. If f is ∂-algebraic over K, then so is every $\phi^i(f)$, and we may conclude that the entries of u, and in particular f, lie in S.

We recall that the u_i are ∂-algebraic over K if and only if they are δ-algebraic over $K' = K(z)$.

Lemma 45 (Reduction Lemma)**.** *Without loss of generality, we may assume in Theorem 44:*

(1) The PV extension L (respectively L') of (3.1) over K (sesp. K'), and the δ-parametrized PV extension $\mathcal{L}'$ are fields and we have $L \subset L' \subset \mathcal{L}'$.
(2) The field $K' \langle u \rangle_\delta = K(z, u_i, \delta(u_i), \delta^2(u_i), ...) \subset F$ is embedded as a subfield of $\mathcal{L}'$.
(3) The difference Galois group $G = \mathrm{Aut}_\phi(L/K)$ is connected.

Proof. A-priori, a PPV extension $\mathcal{L}'$ has the form

$$\mathcal{L}' = \mathcal{L}'_1 \times \cdots \times \mathcal{L}'_r$$

where the $\mathcal{L}'_i$ are fields, $\phi(\mathcal{L}'_i) = \mathcal{L}'_{i+1 \mod r}$ and $\delta(\mathcal{L}') \subset \mathcal{L}'_i$. Replacing ϕ by ϕ^r and A by $A^{(r)} = \phi^{r-1}(A) \ldots \phi(A)A$, keeping u unchanged, $\mathcal{L}'$ gets replaced by $\mathcal{L}'_1$(see § 4.2.2). We may therefore assume that $\mathcal{L}'$ is a field, and as $L' \subset \mathcal{L}'$, the ordinary PV extension L' is a field too. Moreover, for $U \in \mathrm{GL}_n(L')$ a fundamental solution matrix, the subfield $L = K(U)$ of L' satisfies $L^\phi = L'^\phi = \mathbb{C}$ and is thereby a PV extension for (3.1) over K. This settles (1). Point (2), the embedding of $K'(u)$ in L', and of $K' \langle u \rangle_\delta$ in $\mathcal{L}'$, is proven in [A-D-H], Proposition 4.10. See also Remark 17. The vector u becomes then a linear combination, with constant coefficients, of the columns of the fundamental matrix U. Point (3) has already been observed before, as a consequence of the fact that K does not admit non-trivial finite ϕ-field extensions. □

11.2. **The Main Theorem in the irreducible case.**

Proposition 46. *Assume that the ϕ-module W associated with (3.1) over K is irreducible. If $\mathcal{U}_a \neq 0$, i.e., if there exists a non-zero solution u all of whose coordinates are ∂-algebraic over K, then the rank n must be equal to 1.*

Furthermore, if the given ∂-algebraic solution u lies in F^n, then Theorem 44 holds true.

Proof. Assume that W is irreducible. By Lemma 9, $W_{K'}$ is an irreducible (K', ϕ)-module. By the Tannakian correspondence, Proposition 21, the solution space

$$\mathcal{U} = U\mathbb{C}^n = W^\Phi_{L'} = W'^\Phi_{\mathcal{L}} \subset W_{\mathcal{L}'} = \mathcal{L}'^n$$

is an irreducible representation of $H = \mathrm{Aut}_\phi(L'/K')$.

By Corollary 26, the vector space $\mathcal{U}_a$ of differentially algebraic over K solutions is H-invariant. If $\mathcal{U}_a \neq 0$, we must have $\mathcal{U} = \mathcal{U}_a$, all the entries of U are δ-algebraic, and the field $\mathcal{L}'_a$ formed by the differentially algebraic over K elements coincides with $\mathcal{L}'$.

We claim that this forces H to be solvable, contradicting the irreducibility of $\mathcal{U}$, unless $n = 1$.

Suppose, therefore, that H is not solvable. Then G is not solvable either by Lemma 42 and as it is connected, it must have a simple quotient G_1 with $0 < \dim(G_1)$ (indeed, the quotient of G by its radical is non-trivial, connected and semi-simple, so admits a simple connected quotient). By the Galois correspondence theorem this quotient is the difference Galois group of a normal subextension $M_1 \subset L$, a PV extension of another system $\phi(y) = A_1 y$ over K. Since $M_1 \subset L \subset L' \subset \mathcal{L}'$, the δ-parametrized PV extension of $\phi(y) = A_1 y$ over K' is a subfield $\mathcal{M}'_1 \subset \mathcal{L}'$. It follows from the equality $\mathcal{L}'_a = \mathcal{L}'$ that

$$\delta\mathrm{tr.deg.}(\mathcal{M}'_1/K') = 0.$$

Proposition 43, applied to the system $\phi(y) = A_1 y$ and the simple Galois group G_1, yields the contradiction

$$0 < \dim(G_1) = \mathrm{tr.deg.}(M_1/K) = \delta\mathrm{tr.deg.}(\mathcal{M}'_1/K') = 0.$$

This shows that H must be solvable, and concludes the proof that $n = 1$.

Finally, if in addition $u \in F$, by Theorem 39 it belongs to S. □

12. Conclusion of the proof

12.1. **Reduction to the case of an inhomogeneous rank 1 equation.** Keeping the notation as above, we can now complete the proof of Theorem 44, and with it the proof of Theorem 38.

If W is irreducible, we have just proved the theorem. Assume that it is reducible, and let $W_1 \subset W$ be an *irreducible* ϕ-submodule of dimension $1 \leq n_1 < n$. In an appropriate basis, our system of equations and the solution u look like

$$\phi \begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} u' \\ u'' \end{pmatrix},$$

where $A_1 \in \mathrm{GL}_{n_1}(K)$, u' is a vector of length n_1 and u'' is a vector of length $n - n_1$. By induction on n, and since the coordinates of u'' are all ∂-algebraic over K and lie in F, we deduce that $u'' \in S^{n-n_1}$.

As in [A-D-H], §5.3 (see also [A-D-H-W], §4.1), we are going to show that $n_1 = 1$. Let

$$E = Frac(S) = K(z, \zeta(z, \Lambda)) \subset F$$

be the field of fractions of S. The equation (3.1) shows that the coordinates of u span a finite dimensional ϕ-invariant K-subspace of F. If they belong to E, then by Corollary 19, they must lie in S, and we are done.

Assume therefore that one of the coordinates of u' does not belong to E. Consider the system of equations (3.1) over E, and let $L_E = E(U) \subset \mathcal{L}_E$ be its PV and PPV extensions over E. As in Lemma 45, we can consider an iterate of the system $\phi(y) = Ay$ in order to assume that L_E is a field and $E(u')$ is a subfield of L_E. Since the ϕ-module W_1 is irreducible, its Galois group G_1 is irreducible. It is also connected as a quotient of the connected group $\mathrm{Aut}_\phi(L/K)$. Thus, for any positive integer r, the ϕ^r-module (W_1, Φ^r) is still irreducible. Indeed, its Galois group is G_1 by [S-vdP97, Cor. 1.17] and it acts irreducibly on the space of solution of $\phi^r(y) = A_1^{(r)} y$, which coincides with the space of solution of $\phi(y) = A_1 y$. The Tannakian equivalence yields the irreducibility of (W_1, Φ^r) as ϕ^r-module for any r. Finally, one can easily consider the PV extension L' over K' as a subfield of L_E and the PPV extension $\mathcal{L}'$ as a subfield of $\mathcal{L}_E$. Since $u' \notin E^{n_1}$, there exists a

$$\tau \in \mathrm{Aut}_\phi(L_E/E) \subset \mathrm{Aut}_\phi(L'/K') = H$$

with $v = \tau(u') - u' \neq 0$. Furthermore, by Corollary 26, and the assumption that the coordinates of u are δ-algebraic over K', the coordinates of v are δ-algebraic over K' as well. By Lemma 9, the ϕ-module $W_{1,K'}$ is irreducible over K'. As τ fixes $A_{12}u''$, our v satisfies

$$\phi(v) = A_1 v,$$

so the system corresponding to $W_{1,K'}$ admits a non-zero δ-algebraic solution. It follows now from Proposition 46 that $n_1 = 1$.

12.2. **Rank 1 inhomogeneous equations.**

12.2.1. *Reduction to the case $a \in \mathbb{C}^\times$.* We have arrived at the equation

$$\phi(w) = aw + b \tag{12.1}$$

where $a = a_{11} \in K$, $b = a_{12}u_2 + \cdots + a_{1n}u_n \in S$. We assume that $w = u_1 \in F$ satisfies it, and is ∂-algebraic over K. To conclude the proof of the Main Theorem we must show that $w \in S$. By Corollary 19, it is enough to show that $w \in E$.

We continue to work over E as a ground field. Let L_E and $\mathcal{L}_E$ be respectively the PV and the PPV extensions of (12.1) over E and let us consider the associated difference Galois group $G_E = \mathrm{Aut}_\phi(L_E/E)$ and the δ-parametrized Galois group scheme $\mathcal{G}_E$ of $\mathcal{L}_E$ over E. If $w \in E$, we are done. Thus we can assume that this is not the case. Our first goal is to show that we may assume $a \in \mathbb{C}^\times$. Since $w \notin E$, by the Galois correspondence theorem there exists a $\tau \in G_E$ such that $\tau(w) \neq w$. Letting $v = \tau(w) - w$ we arrive at

$$\phi(v) = av.$$

Since w is δ-algebraic over K', Corollary 26 shows that $\tau(w)$ is δ-algebraic as well, hence v is δ-algebraic over K'. Proposition 40 (with $\mathcal{F} = \mathcal{L}_E$) implies that

$$a = c\frac{\phi(\beta)}{\beta}$$

for some $c \in \mathbb{C}^\times$ and $\beta \in K^\times$. The original equation becomes equivalent to

$$\phi(\frac{w}{\beta}) = c\frac{w}{\beta} + \frac{b}{\phi(\beta)}$$

and $b/\phi(\beta)$ still lies in S. If we show $w/\beta \in S$, then $w \in S$ as desired. This allows us to assume, from the beginning, that $a \in \mathbb{C}^\times$.

12.2.2. *The proof when $a \in \mathbb{C}^\times$.* Recall that $v \in L_E$ is the fundamental matrix of the rank-1 equation $\phi(v) = av$. Contrary to

$$E_2 := E(w) \subset F,$$

and unless a is a power of q, the field $E_1 := E(v) \subset L_E$ can not be embedded ϕ-equivariantly in the Laurent power series field F. A-priori, we only know that $\delta(v) \in \mathcal{L}_E$.

Lemma 47. *We have $\delta(v) = cv$ for some $c \in \mathbb{C}$. Thus E_1 is a (ϕ, δ)-field.*

Proof. If $\delta(v) = 0$ this is clear. Otherwise, one has

$$\frac{\phi(\delta v)}{\delta v} = \frac{\delta(\phi v)}{\delta v} = \frac{\delta(av)}{\delta v} = a$$

since $a \in \mathbb{C}$. But $\frac{\phi(v)}{v} = a$ as well, so $\delta(v)/v$, being fixed by ϕ, must belong to $\mathcal{L}_E^\phi = \mathbb{C}$. ☐

Consider $U' = \begin{pmatrix} v & w \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(L_E)$ and $A' = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(E)$, so that

$$\phi(U') = A'U',$$

and $E(v, w) = E_1E_2 \subset L_E$ is the PV extension of the linear system $\phi(y) = A'y$ over E. Let $\mathcal{E} = E\langle v, w\rangle_\delta \subset \mathcal{L}_E$ be the δ-parametrized PV extension of this last system over E.

The proof of the following Key Lemma resembles the proof of [H-S08], Proposition 3.8. See also the proof, in the case of two difference operators, in [dS23], §5.4.

Lemma 48. *After possibly replacing w by $z^r w$, a by $q^r a$ and b by $q^r z^r b$ for some positive integer r, there exists a monic operator $\mathscr{L} \in \mathbb{C}[\delta]$, and a function $f \in S_0$, such that*

$$\mathscr{L}(b) = (\phi - a)(f). \tag{12.2}$$

Remark. We do not rule out $f = \mathscr{L}(b) = 0$. In fact, this may well be the solution if b is annihilated by some operator from $\mathbb{C}[\delta]$, a condition which is easily verified to hold if and only if b is a Laurent polynomial in z.

Proof. The δ-parametrized Galois group scheme $\mathcal{G}'$ of the system $\phi(y) = A'y$ over E is a linear differential subgroup scheme $\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 \right\}$, defined over $\mathbb{C}$. Its intersection with the unipotent radical $\left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 \right\}$, denoted $\mathcal{G}'_u$, is the δ-parametrized Galois group scheme of $\mathcal{L}_E$ over the (ϕ, δ)-field

$$E_1 = E(v).$$

This $\mathcal{G}'_u$ is a linear differential subgroup scheme of the additive group $[\delta]_*\mathbb{G}_a$ over $\mathbb{C}$. Since w is δ-algebraic, the torsor theorem for the parametrized Galois correspondence yields that $\delta\mathrm{tr.deg.}(\mathcal{E}/\mathrm{E}_1) = 0 = \delta \dim_{\mathbb{C}}(\mathcal{G}'_\mathrm{u})$, so, by [Cas72], Proposition 11, there must be a non-trivial linear operator $\mathscr{L}_1 \in \mathbb{C}[\delta]$ such that

$$\mathscr{L}_1(\beta_\tau) = 0$$

for every $\tau = \begin{pmatrix} 1 & \beta_\tau \\ 0 & 1 \end{pmatrix} \in \mathcal{G}'_u(C)$, for any δ-ring extension C of $\mathbb{C}$.

As

$$\phi(\frac{w}{v}) = \frac{w}{v} + \frac{b}{av}$$

and $b/av \in E_1$, $\mathcal{E}$ is a PPV extension for the equation

$$\phi(y) = y + b/av$$

over E_1. The action of $\tau \in \mathcal{G}'_u(C)$ is given by

$$\tau(\frac{w}{v}) = \frac{w}{v} + \beta_\tau.$$

It follows that for any $\tau \in \mathcal{G}'_u(C)$ we have, in the base-changed PPV ring

$$E_1[w, \delta(w), ..., w^{-1}] \otimes_{\mathbb{C}} C$$

the equation

$$\tau(\mathscr{L}_1(\frac{w}{v})) = \mathscr{L}_1(\frac{w}{v} + \beta_\tau) = \mathscr{L}_1(\frac{w}{v}).$$

By the Galois correspondence in the δ-parametrized framework, $\mathscr{L}_1(\frac{w}{v}) \in E_1$. Leibnitz' formula and the equation $\delta(v) = cv$ imply

$$v^{-1}\delta^k(w) = \sum_{j=0}^{k} \binom{k}{j} c^{k-j}\delta^j(\frac{w}{v}).$$

Since a linear combination, with constant coefficients, of the $\delta^j(\frac{w}{v})$, lies in E_1, so does a linear combination, with constant coefficients, of $v^{-1}\delta^k(w)$. As $v \in E_1$, we find that for some non-trivial $\mathscr{L} \in \mathbb{C}[\delta]$,

$$f = \mathscr{L}(w) \in E_1.$$

Since $\mathscr{L}$ commutes with ϕ, this f satisfies (12.2), but may not be in E.

By Lemma 4.7 of [A-D-H-W] (with E in the role of L *and* K, v in the role of x, and L_E in the role of L_A) we conclude that there exists an $f \in E$ (possibly different than $\mathscr{L}(w)$) satisfying (12.2). By Corollary 19, $f \in S$.

We have arrived at the two equations

$$\left\{ \begin{array}{c} (\phi - a)(w) = b \\ (\phi - a)(f) = \mathscr{L}(b). \end{array} \right. \tag{12.3}$$

Let Z be the operator of multiplication by z. We have the relations

$$Z \circ (\phi - a) = q^{-1}(\phi - qa) \circ Z, \;\; Z \circ \mathscr{L}(\delta) = \mathscr{L}(\delta - 1) \circ Z.$$

Multiplying the two equations by z^r, replacing a by $q^r a$, b by $q^r z^r b$, $\mathscr{L}(\delta)$ by $\mathscr{L}(\delta - r)$, w by $z^r w$ and f by $z^r f$, we get a similar pair of equations, but we may assume now that $f \in S_0$, not only in S. □

Corollary 8 implies now that for some $h \in S$ and $d \in \mathbb{C}$ we have

$$(\phi - a)(w - h) = dz^r,$$

with $d = 0$ unless $a = q^r$. As $w - h \in F$, it is easily verified that $w - h$ must be of the form ez^m (with $e \in \mathbb{C}$ and $e = 0$ unless $a = q^m$). It follows that $w \in S$, as desired.

13. Appendix A: Proof of Proposition 7

We recall the proposition whose proof we give here, with a slight change in notation.

Proposition. *Let $f, g \in S_\Lambda = K_\Lambda[z, \zeta(z,\Lambda)]$, $a, c \in \mathbb{C}$ and $p \in \mathbb{C}[z]$ be such that*

$$(\delta - c)(g) = (\phi - a)(f) + p. \tag{13.1}$$

Then $g = (\phi - a)(u) + \widetilde{p}$ for some $u \in S_\Lambda$ and $\widetilde{p} \in \mathbb{C}[z]$. Furthermore, if $a = q^r$ for some $r \geq 0$ we can take $\widetilde{p} = dz^r$, $d \in \mathbb{C}$, and otherwise we can take $\widetilde{p} = 0$.

We start with some lemmas. To ease notation, write from now on $\zeta := \zeta(z, \Lambda)$. Note that

$$\phi(\zeta) = q\zeta + f_\zeta$$

where $f_\zeta = \zeta(qz, \Lambda) - q\zeta(z,\Lambda) \in K_\Lambda$ is elliptic.

Recall that ζ is transcendental over K_Λ, consider the polynomial ring $K_\Lambda[\zeta]$, and write $K_\Lambda[\zeta]_{<d}$ for the polynomials of degree $< d$ in ζ.

Lemma 49. *Let $a \in \mathbb{C}$. If $a \neq q^r$, the operator $(\phi - a)$ is injective on $F = \mathbb{C}((z))$, while* $\ker(\phi - q^r) = \mathbb{C}z^r$. *A-fortiori, the same applies to S_Λ. If $a \neq 1$, the operator $(\phi - a)$ is injective on $K_\Lambda[\zeta]$, while* $\ker(\phi - 1) = \mathbb{C}$.

Proof. The first statement is clear, since $\phi(\sum a_n z^n) = \sum a_n q^n z^n$. The second follows from the fact that for $r \neq 0$, $z^r \notin K_\Lambda[\zeta]$. See Lemma 5. □

Lemma 50. *Let $f = \sum_{i=0}^d f_i \zeta^i \in K_\Lambda[\zeta]$ ($f_i \in K_\Lambda$, $f_d \neq 0$). Let $a \in \mathbb{C}^\times$. Then:*

i) If $a \neq q^d$, or $a = q^d$ and $f_d \notin \mathbb{C}$, then $(\phi - a)(f)$ has degree d as a polynomial in ζ and the coefficient of ζ^d is $q^d\phi(f_d) - af_d$.
ii) If $d = 0$, $a = 1$ and $f_0 \in \mathbb{C}$ then $(\phi - 1)(f) = 0$.
iii) If $d \geq 1$, $a = q^d$ and $f_d \in \mathbb{C}$ then $(\phi - q^d)(f)$ has degree $d - 1$ in ζ and the coefficient of ζ^{d-1} is

$$df_d q^{d-1} f_\zeta + q^{d-1}(\phi - q)(f_{d-1}).$$

Proof. We have $\phi(f) = \sum_{i=0}^d \phi(f_i)(q\zeta + f_\zeta)^i$, so the coefficient of ζ^d in $(\phi - a)(f)$ is $q^d\phi(f_d) - af_d$. If f_d is non-constant, it must have a pole at some $z_0 \neq 0$. If we take z_0 to be a non-zero pole with minimal absolute value, then $q^d\phi(f_d) - af_d$ has a pole at z_0/q, and in particular can not vanish. If f_d is constant, the coefficient of ζ^d vanishes only if $a = q^d$. This proves i), and ii) is obvious. In case iii) the coefficient of ζ^d vanishes and the next coefficient, of ζ^{d-1}, comes out as stated from the same computation. If it vanished, there would be an elliptic function h such that

$$(\phi - q)(\zeta) = f_\zeta = (\phi - q)(h).$$

This contradicts the injectivity of $(\phi - q)$ on $K_\Lambda[\zeta]$. □

The next Lemma says that a function of the form $u + r\zeta$ where u is elliptic, can not have a global meromorphic primitive, unless $r = 0$.

Lemma 51. *If $u \in K_\Lambda$, w is globally meromorphic, $r \in \mathbb{C}$ and $u + r\zeta = w'$ then* $r = 0$.

Proof. For $\omega \in \Lambda$ write $\chi(z,\omega) = w(z+\omega) - w(z)$. Differentiating with respect to z we get that $\chi'(z,\omega) = r\eta(\omega)$ where η is the Legendre η-function of the lattice Λ. Thus $\chi(z,\omega) = r\eta(\omega)z + \mu(\omega)$ for some $\mu(\omega) \in \mathbb{C}$. We get

$$\chi(z,\omega_1+\omega_2) = \chi(z+\omega_1,\omega_2)+\chi(z,\omega_1) = r\eta(\omega_2)(z+\omega_1)+r\eta(\omega_1)z+\mu(\omega_1)+\mu(\omega_2),$$

so

$$\mu(\omega_1+\omega_2) = r\eta(\omega_2)\omega_1 + \mu(\omega_1) + \mu(\omega_2).$$

If $r \neq 0$ this is absurd, since the left hand side is symmetric in ω_1 and ω_2, but for an oriented basis (ω_1,ω_2) of Λ the Legendre relation gives $\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2 = 2\pi i$, so the right hand side is not symmetric. □

Lemma 52. *Consider $g, f \in K_\Lambda[\zeta]$ and $a, \gamma \in \mathbb{C}$ such that*

$$g' = (q\phi - a)(f) + \gamma.$$

Then there exists a $u \in K_\Lambda[\zeta]$ and $\beta \in \mathbb{C}$ such that

$$g = (\phi - a)(u) + \beta.$$

Furthermore, if $a \neq 1$, we may take $\beta = 0$.

Proof. The last statement is clear, because if $a \neq 1$, the operator $(\phi - a)$ is surjective on the constants, so $\beta = (\phi - a)(u_0)$ for some $u_0 \in \mathbb{C}$, and replacing u by $u + u_0$ we may assume $\beta = 0$.

We shall prove the lemma by induction on ℓ, the degree of g as a polynomial in ζ. Write

$$g = \sum g_i\zeta^i, \quad f = \sum f_i\zeta^i \quad (g_i, f_i \in K_\Lambda).$$

Case $\ell = 0$. Assume that $g \in K_\Lambda$ and $g' = (q\phi - a)(f) + \gamma$ for $f \in K_\Lambda[\zeta]$,$\gamma \in \mathbb{C}$. By Lemma 50 we are in one of the following two sub-cases:

- $f = f_0 \in K_\Lambda$ and $g' = (q\phi - a)(f_0) + \gamma$,
- $a = q^2$, $f = f_0 + f_1\zeta$ with $0 \neq f_1 \in \mathbb{C}$ and $g' = q(\phi - q)(f_0) + qf_1f_\zeta + \gamma$.

Quite generally, if $\xi \in \mathbb{C}$ and $h(z)$ is meromorphic at ξ, let us say that ξ is a *residual point* of $h(z)$ if

$$Res_\xi h(z)dz \neq 0.$$

If ξ is a residual point of $h(z)$, then ξ/q is a residual point of $h(qz)$. A globally meromorphic function h admits a primitive, *i.e.*,. $h = w'$ for some globally meromorphic w, if and only if h has no residual points.

In the first sub-case, we claim that f_0 has no residual points. Otherwise, thanks to the periodicity of f_0, there would be a residual point $\xi \neq 0$, and we can take it to be of minimal absolute value. The function $(q\phi - a)(f_0) + \gamma$ then has ξ/q as a residual point, conradicting the fact that it has a meromorphic primitive g.

It follows that there exists a globally meromorphic primitive w with $w' = f_0$. Since f_0 is Λ-periodic,

$$\chi(\omega) = w(z+\omega) - w(z)$$

$(\omega \in \Lambda)$ does not depend on z, and is a homomorphism $\Lambda \to \mathbb{C}$. But every such homomorphism is supplied by a linear combination of z and ζ as well. It follows that for some $r, s \in \mathbb{C}$ and $u \in K_\Lambda$ we have

$$w = u + r\zeta + sz.$$

Integrating the given expression for g' we find that $g = (\phi - a)(w) + \gamma z + \beta$ for some $\beta \in \mathbb{C}$, or

$$g = (\phi - a)(u + r\zeta) + (sq - sa + \gamma)z + \beta.$$

As g is elliptic, the term with z must vanish (and in fact $r = 0$ unless $a = q$), proving the Lemma in this case.

We claim that the second sub-case *never occurs*. We can assume, dividing g by qf_1 to ease notation, that

$$g' = (\phi - q)(f_0 + \zeta) + \gamma.$$

Now, $f_0 + \zeta$ is not Λ-periodic, but its polar part is, so had there been a residual point ξ for it, so would be $\xi + \omega$ for any $\omega \in \Lambda$, and we could assume that $\xi \neq 0$. The same argument as before shows that $f_0 + \zeta$ has no residual points at all, hence

$$f_0 + \zeta = w'$$

for some meromorphic w. This contradicts Lemma 51.

Induction step. Let $\ell \geq 1$, and assume that the lemma had been proved up to degree $\ell - 1$.

Case $g_\ell \in \mathbb{C}$ **and** $a \neq q^\ell$. In this case

$$g = (\phi - a)(v) + \widetilde{g}$$

where $v = g_\ell (q^\ell - a)^{-1}\zeta^\ell \in K_\Lambda[\zeta]$ and $\widetilde{g} \in K_\Lambda[\zeta]_{<\ell}$ (has degree smaller than ℓ in ζ). Clearly $\widetilde{g}$ satisfies the assumption of the lemma on $\widetilde{g}'$, so by the induction hypothesis is of the form $\widetilde{g} = (\phi - a)(\widetilde{u}) + \gamma$, with $\widetilde{u} \in K_\Lambda[\zeta]$ and $\gamma \in \mathbb{C}$. It follows that so is g, with $u = \widetilde{u} + v$.

Case $g_\ell \in \mathbb{C}$ **and** $a = q^\ell$. We claim that this case *does not occur*. We have

$$g' = (g_\ell \ell \zeta' + g'_{\ell-1})\zeta^{\ell-1} \mod K_\Lambda[\zeta]_{<\ell-1}$$

and f must be, according to Lemma 50, of degree $\ell - 1$. Comparing coefficients of $\zeta^{\ell-1}$ we arrive at

$$g_\ell \ell \zeta' + g'_{\ell-1} = q^\ell(\phi - 1)(f_{\ell-1}) + \gamma_1$$

where $\gamma_1 = \gamma$ if $\ell = 1$ and $\gamma_1 = 0$ otherwise. By the same argument as above, on residual points, $f_{\ell-1} = w'$ for some meromorphic function w, and $w = u + r\zeta + sz$ for $u \in K_\Lambda,\ r, s \in \mathbb{C}$. Integrating we get that

$$g_\ell \ell \zeta + g_{\ell-1} = q^{\ell-1}(\phi - q)(w) + \gamma_1 z + \beta = q^{\ell-1}(\phi - q)(u) + rq^{\ell-1}f_\zeta + \gamma_1 z + \beta$$

($\beta \in \mathbb{C}$). This contradicts Lemma 5, as ζ does not show up on the right hand side.

Case $g_\ell \notin \mathbb{C}$ **and** $a \neq q^{\ell+1}, q^{\ell+2}$. In this case $g' \equiv g'_\ell \zeta^\ell \mod K_\Lambda[\zeta]_{<\ell}$. Since $a \neq q^{\ell+2}$, Lemma 50 shows that f must be of degree ℓ in ζ and, comparing coefficients of ζ^ℓ,

$$g'_\ell = (q^{\ell+1}\phi - a)(f_\ell).$$

As before, this implies that $f_\ell = w'$ for some meromorphic w, and that $w = v + r\zeta + sz$ for $v \in K_\Lambda$, $r, s \in \mathbb{C}$. Integrating this gives

$$g_\ell = (q^\ell \phi - a)(v + r\zeta + sz) + \alpha$$

($\alpha \in \mathbb{C}$). As $a \neq q^{\ell+1}$ we must have $r = s = 0$, since the left hand side is elliptic, $\phi(z) = qz$ and $\phi(\zeta) = q\zeta \mod K_\Lambda$. Note $(\phi - a)(v\zeta^\ell) \equiv (q^\ell\phi - a)(v)\zeta^\ell \mod K_\Lambda[\zeta]_{<\ell}$. Letting

$$\widetilde{g} = g - (\phi - a)(v\zeta^\ell) = \sum_{i=0}^{\ell} \widetilde{g}_i \zeta^i$$

we see that (a) $\widetilde{g}'$ satisfies the hypothesis of the Lemma, with $\widetilde{f} = f - (v\zeta^\ell)'$ (b) $\widetilde{g}_\ell = \alpha \in \mathbb{C}$, hence we are in a case studied before. Thus $\widetilde{g} = (\phi - a)(\widetilde{u}) + \beta$ (for some $\widetilde{u} \in K_\Lambda[\zeta]$, $\beta \in \mathbb{C}$), and $u = \widetilde{u} + v\zeta^\ell$ solves our problem.

Case $g_\ell \notin \mathbb{C}$ **and** $a = q^{\ell+1}$. Up to the point where

$$g_\ell = (q^\ell\phi - a)(v + r\zeta + sz) + \alpha$$

the proof is as in the previous case. Furthermore, we may assume that $s = 0$ since $(q^\ell\phi - a)(z) = 0$, and that $\alpha = 0$ because it is of the form $(q^\ell\phi - a)(\alpha_0)$ for $\alpha_0 \in \mathbb{C}$ so can be "swallowed" in v. Dividing by q^ℓ we may write

$$g_\ell = (\phi - q)(h + e\zeta)$$

with $h \in K_\Lambda$ and $e \in \mathbb{C}$. A direct computation shows that

$$g_\ell\zeta^\ell \equiv q^{-\ell}(\phi - q^{\ell+1})(h\zeta^\ell + \frac{e\zeta^{\ell+1}}{\ell+1}) \mod K_\Lambda[\zeta]_{<\ell}.$$

As before, letting $\widetilde{g} = g - q^{-\ell}(\phi - q^{\ell+1})(h\zeta^\ell + \frac{e\zeta^{\ell+1}}{\ell+1}) \in K_\Lambda[\zeta]_{<\ell}$, this $\widetilde{g}$ satisfies the hypothesis of the Lemma, so by the induction hypothesis is of the form $(\phi - q^{\ell+1})(\widetilde{u})$, with some $\widetilde{u} \in K_\Lambda[\zeta]$. It follows that $g = (\phi - q^{\ell+1})(u)$ for an appropriate $u \in K_\Lambda[\zeta]$.

Case $g_\ell \notin \mathbb{C}$ **and** $a = q^{\ell+2}$. By Lemma 50 f has degree ℓ or $\ell+1$ in ζ, and in the second case $f_{\ell+1} \in \mathbb{C}$. Assume first that we are in this second case, so $0 \neq f_{\ell+1} \in \mathbb{C}$. Comparing coefficients of ζ^ℓ in g' we get

$$g'_\ell = q^{\ell+1}(\phi - q)\{f_{\ell+1}(\ell + 1)\zeta + f_\ell\}.$$

As argued before, $h = f_{\ell+1}(\ell + 1)\zeta + f_\ell$, though not periodic, has a periodic polar part, so if ξ is a residual point for it, so is $\xi + \omega$ for $\omega \in \Lambda$. Had there been a residual point for h, we could therefore take such a point $0 \neq \xi$ of minimal absolute value, and then ξ/q would be a residual point for the right hand side. We conclude that h has no residual points at all, hence has a meromorphic primitive. This contradicts, however, Lemma 51. Thus we can not have $f_{\ell+1} \neq 0$, and f has degree ℓ.

It follows that

$$g'_\ell = q^{\ell+1}(\phi - q)(f_\ell).$$

As before, this implies that f_ℓ has no residual points, hence admits a primitive: $f_\ell = w'$, and $w = v + r\zeta + sz$ with $v \in K_\Lambda$, $r, s \in \mathbb{C}$. Thus

$$g_\ell = q^\ell(\phi - q^2)(v + r\zeta + sz) + \beta,$$

($\beta \in \mathbb{C}$), but we can "swallow" β in v, so we may assume $\beta = 0$. As the left hand side is elliptic, this forces $r = s = 0$, and $g_\ell = q^\ell(\phi - q^2)(v)$. This yields

$$g_\ell\zeta^\ell \equiv (\phi - q^{\ell+2})(v\zeta^\ell) \mod K_\Lambda[\zeta]_{<\ell}.$$

Replacing g by $\widetilde{g} = g - (\phi - q^{\ell+2})(v\zeta^\ell) \in K_\Lambda[\zeta]_{<\ell}$ we may apply induction to conclude that $\widetilde{g} = (\phi - q^{\ell+2})(\widetilde{u})$, for $\widetilde{u} \in K_\Lambda[\zeta]$, hence $g = (\phi - q^{\ell+2})(u)$ with $u = \widetilde{u} + v\zeta^\ell$. This concludes the proof of the Lemma. $\square$

We can now finish the proof of Proposition 7.

Proof. Set

$$g=\sum_{i=0}^{M} g_i z^i,\ \ f=\sum_{i=0}^{M} f_i z^i,\ \ p=\sum_{i=0}^{M} p_i z^i$$

where $g_i, f_i \in K_\Lambda[\zeta]$ and $p_i \in \mathbb{C}$. For $i \notin [0,M]$ let $g_i = f_i = p_i = 0$. Equating the coefficients of z^i in (13.1) we get

$$g'_{i-1} + (i-c)g_i = (q^i\phi - a)(f_i) + p_i. \tag{13.2}$$

For $i = M+1$ this gives $g_M \in \mathbb{C}$, so

$$g_M = (q^M\phi - a)(u_M) + \beta_M$$

with $u_M \in K_\Lambda[\zeta]$ and $\beta_M \in \mathbb{C}$ trivially (e.g. $u_M = 0$, $\beta_M = g_M$).

For $i = M$, substituting the latter expression for g_M in (13.2) we find that

$$g'_{M-1} = (q^M\phi - a)(\widetilde{f}_{M-1}) + \gamma_{M-1}$$

for some $\widetilde{f}_{M-1} \in K_\Lambda[\zeta]$ and $\gamma_{M-1} \in \mathbb{C}$. Lemma 52 ensures that

$$g_{M-1} = (q^{M-1}\phi - a)(u_{M-1}) + \beta_{M-1}$$

with $u_{M-1} \in K_\Lambda[\zeta]$ and $\beta_{M-1} \in \mathbb{C}$. Iterating, using the recursion formula (13.2) and Lemma 52, we solve succesively for

$$g_i = (q^i\phi - a)(u_i) + \beta_i,$$

with $u_i \in K_\Lambda[\zeta]$ and $\beta_i \in \mathbb{C}$, giving the desired equation

$$g = (\phi - a)\left(\sum_{i=0}^{M} u_i z^i\right) + \sum_{i=0}^{M} \beta_i z^i.$$

The final reduction of $\widetilde{p} = \sum_{i=0}^{M} \beta_i z^i$ to a monomial dz^r (if $a = q^r$) or 0 (otherwise), by a suitable modification of the u_i, is obvious, as $(\phi - a)(z^i) = (q^i - a)z^i$. □

14. Appendix B: Galois invariants

We want to prove a technical result, for which we could not find a reference.

Let K be a (ϕ,δ)-field with $K^\phi = C$ algebraically closed. Let $\widetilde{C}$ be a differential closure of C. It is unique up to isomorphism. Let

$$K^\dagger = \widetilde{C} \otimes_C K$$

(a domain, but in general not a field).

Let $\phi(y) = Ay$ be a difference system with $A \in \mathrm{GL}_n(K)$ and let $\mathcal{L} = K\langle U\rangle_\delta$ be a PPV extension for this system over K. We assume that $\mathcal{L}$ is a field. We consider

$$R = K[U, \det(U)^{-1}],\ \ \mathcal{R} = K\{U, \det(U)^{-1}\}_\delta$$

the PV ring and the δ-parametrized PV ring inside $\mathcal{L}$. Note that $R^\phi = \mathcal{R}^\phi = C$. Let

$$R^\dagger = \widetilde{C} \otimes_C R = K^\dagger[U, \det(U)^{-1}],\ \ \mathcal{R}^\dagger = \widetilde{C} \otimes_C \mathcal{R} = K^\dagger\{U, \det(U)^{-1}\}_\delta.$$

Let $\widetilde{\mathcal{G}} = \mathrm{Aut}_{\phi,\delta}(\mathcal{R}^\dagger/K^\dagger)$. This is the δ-parametrized Galois group over $\widetilde{C}$ and we view it as a subgroup of $\mathrm{GL}_n(\widetilde{C})$. If $\mathcal{G}$ is the δ-parametrized Galois group scheme attached to $\mathcal{R}$, then $\widetilde{\mathcal{G}} = \mathcal{G}(\widetilde{C})$.

Proposition 53. *If $a \in \mathcal{R}^\dagger - K^\dagger$, then there exists a $g \in \widetilde{\mathcal{G}}$ with $g(a) \neq a$.*

The Proposition is usually phrased when $K^\dagger$ is replaced by $Frac(K^\dagger)$ and $\mathcal{R}^\dagger$ by $Frac(\mathcal{R}^\dagger)$, the δ-parametrized PV extension. However, it is not a-priori clear that $\mathcal{R}^\dagger \cap Frac(K^\dagger) = K^\dagger$.

Proof. We have adapted slightly the arguments of [DVHW, Lemma 3.1] to our context. Consider

$$d = a \otimes 1 - 1 \otimes a \in \mathcal{R}^\dagger \otimes_{K^\dagger} \mathcal{R}^\dagger = \widetilde{C} \otimes_C (\mathcal{R} \otimes_K \mathcal{R}).$$

Since $a \notin K^\dagger$, $d \neq 0$. This is clear if $\mathcal{R}^\dagger$ and $K^\dagger$ are replaced by $\mathcal{R}$ and K, since K is a field. Expanding $a = \sum \omega_\alpha \otimes a_\alpha$ with $\{\omega_\alpha\}$ a basis of $\widetilde{C}$ over C, we see that $d = \sum \omega_\alpha \otimes (a_\alpha \otimes 1 - 1 \otimes a_\alpha)$. Since $a \notin K^\dagger$, there is some $a_{\alpha_0} \in \mathcal{R} - K$ so that $a_{\alpha_0} \otimes 1 - 1 \otimes a_{\alpha_0}$ is a nonzero element in $\mathcal{R} \otimes_K \mathcal{R}$.

The torsor theorem is equivalent to the fact that the inclusion $(\mathcal{R} \otimes_K \mathcal{R})^\phi \hookrightarrow \mathcal{R} \otimes_K \mathcal{R}$ extends to a $\mathcal{R}$-(ϕ, δ)-isomorphism

$$\Theta : (\mathcal{R} \otimes_K \mathcal{R})^\phi \otimes_C \mathcal{R} \to \mathcal{R} \otimes_K \mathcal{R}.$$

The C-δ-algebra $C\{\mathcal{G}\} = (\mathcal{R} \otimes_K \mathcal{R})^\phi$ represents the functor $\mathcal{G}$ (see Proposition 6.18 in [H-S08] for the case where C is differentially closed and the discussion after its proof to see how it can be adapted to our situation where $Frac(\mathcal{R})^\phi = C$.) An element $g \in \mathcal{G}(\widetilde{C})$ thereby corresponds to a C-δ-algebra morphism ψ from $C\{\mathcal{G}\}$ to $\widetilde{C}$ whose kernel is a prime δ-ideal $\mathfrak{m}_g$. Its action on an element $1 \otimes b \in \widetilde{C} \otimes_C \mathcal{R}$ is given by $(\psi \otimes id) \circ \Theta^{-1}(b \otimes 1)$ (see for instance the discussion in [DVHW, Lemma 3.1] in the analogous context where the roles of δ and ϕ are interchanged). A nonzero element $1 \otimes b$ of $\widetilde{C} \otimes_C \mathcal{R}$ is $\mathcal{G}(\widetilde{C})$-invariant if $\Theta^{-1}(b \otimes 1 - 1 \otimes b)$ belongs to $\cap_{g \in \mathcal{G}(\widetilde{C})} \mathfrak{m}_g \otimes_C \mathcal{R}$. Now, we claim that $\cap_{g \in \mathcal{G}(\widetilde{C})} \mathfrak{m}_g$ is the zero ideal in $C\{\mathcal{G}\}$. Indeed, since $C\{G\}$ is reduced, differentially finitely generated over C and $\widetilde{C}$ is differentially closed, this is a direct consequence of the differential Nullstellensatz (see [M-MTDF]). To conclude, we have proved that if a nonzero element $1 \otimes b \in \mathcal{R}^\dagger$ is invariant under $\mathcal{G}(\widetilde{C})$ then $b \otimes 1 - 1 \otimes b = 0$ in $\mathcal{R} \otimes_K \mathcal{R}$.

Therefore, there exists an element g in $\mathcal{G}(\widetilde{C})$ such that $g(\omega_{\alpha_0} \otimes a_{\alpha_0}) \neq \omega_{\alpha_0} \otimes a_{\alpha_0}$. This yields $g(a) \neq a$ and concludes the proof. ☐

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