# ON THE LOCAL STRUCTURE OF MAHLER SYSTEMS

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Abstract. This paper is a first step in the direction of a better understanding of the structure of the so-called Mahler systems : we classify these systems over the field $\mathscr{H}$ of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$. As an application of (a variant of) our main result, we give an alternative proof of the following fact : if, for almost all primes $p$, the reduction modulo $p$ of a given Mahler equation with coefficients in $\mathbb{Q}(z)$ has a full set of algebraic solutions over $\mathbb{F}_p(z)$, then the given equation has a full set of solutions in $\overline{\mathbb{Q}}(z)$ (this is analogous to Grothendieck's conjecture for differential equations).

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## 1. Introduction and main results

There is a fast growing literature on the theory of Mahler systems, *i.e.*, on the functional systems of the form

$$Y(z^{\ell}) = A(z)Y(z) \tag{1}$$

with $\ell \in \mathbb{Z}_{\geq 2}$ and $A(z) \in \mathrm{GL}_n(\overline{\mathbb{Q}}(z))$. This theory started in the late 1920s with celebrated papers by Mahler [Mah29, Mah30a, Mah30b] about the arithmetic properties of the values taken by solutions of Mahler systems

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(usually called Mahler functions) at algebraic numbers. For instance, Mahler considered the function

$$\mathfrak{f}(z)=\sum_{k\geq 0} z^{2^k},$$

which is easily seen to satisfy

$$\mathfrak{F}(z^2)=\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}\mathfrak{F}(z) \text{ with } \mathfrak{F}(z)=\begin{pmatrix} 1 \\ \mathfrak{f}(z) \end{pmatrix},$$

and proved that $\mathfrak{f}(\alpha)$ is transcendental for any nonzero algebraic number $\alpha$ such that $|\alpha|<1$. Since these pioneering works of Mahler, Mahler systems and functions have attracted the attention of many authors, including Kubota [Kub77], Loxton and van der Poorten [LvdP78], Masser [Mas82], Randé [Ran92], Dumas [Dum93], Becker [Bec94], Nishioka [Nis96], Dumas and Flajolet [DF96], Zannier [Zan98], Corvaja and Zannier [CZ02], Allouche and Shallit [AS03], Pellarin [Pel09], Nguyen [Ngu11, Ngu12], Philipon [Phi15], Shaëfke and Singer [SS16], Brent, Coons and Zudilin [BCZ16], Adamczewski and Bell [AB17], Dreyfus, Hardouin and Roques [DHR18], Chyzak, Dreyfus, Dumas and Mezzarobba [CDDM18], Bell, Chyzak, Coons and Dumas [BCCD18], Adamczewski and Faverjon [AF17, AF18], Fernandes [Fer18], to name just a few. Note that the abundance of recent papers devoted to Mahler systems is partly due to their connections with automata theory : if $f(z)=\sum_{k\geq 0} f_k z^k$ is the generating series of an $\ell$-automatic sequence $(f_k)_{k\geq 0}\in\overline{\mathbb{Q}}^{\mathbb{Z}_{\geq 0}}$, then the column vector

$$\begin{pmatrix} f(z) \\ f(z^{\ell}) \\ \vdots \\ f(z^{\ell^{n-1}}) \end{pmatrix}$$

satisfies a Mahler system of the form (1) for suitable $n\geq 1$ and $A(z)\in \mathrm{GL}_n(\overline{\mathbb{Q}}(z))$.

Despite this important activity around Mahler systems in the last decades, very little is known about their structure. The present paper is a first step in the direction of a better understanding of the local structure of these systems at 0 : we give the complete classification of the Mahler systems over the field $\mathscr{H}$ of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$ (for the definition of this field, see Section 2). Before stating our main result, we recall the notion of $\mathscr{H}$-equivalence for Mahler systems.

**Definition 1.** *Two Mahler systems $Y(z^{\ell})=A(z)Y(z)$ and $Y(z^{\ell})=B(z)Y(z)$ with $A(z),B(z)\in\mathrm{GL}_n(\mathscr{H})$ are $\mathscr{H}$-equivalent if there exists $F(z)\in\mathrm{GL}_n(\mathscr{H})$ such that*

$$A(z)F(z)=F(z^{\ell})B(z).$$

*In this context, such an $F(z)$ is called a gauge transformation.*

The raison d'être of this notion is the following : if $A(z)$, $B(z)$ and $F(z)$ are as in the previous definition, then $U(z)$ is a column vector solution of $Y(z^{\ell})=B(z)Y(z)$ if and only if $F(z)U(z)$ is a column vector solution of $Y(z^{\ell})=A(z)Y(z)$. It is easily seen that "being $\mathscr{H}$-equivalent" is an equivalence relation. The classification of the Mahler systems over $\mathscr{H}$ aims at describing the equivalence classes of Mahler systems for this equivalence relation. This is achieved with our main result :

**Theorem 2.** *Any Mahler system $Y(z^\ell) = A(z)Y(z)$ with $A(z) \in \mathrm{GL}_n(\mathscr{H})$ is $\mathscr{H}$-equivalent to a Mahler system with constant coefficients, i.e., of the form $Y(z^\ell) = A_0 Y(z)$ for some $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$. The matrix $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ is unique up to conjugation by an element of $\mathrm{GL}_n(\overline{\mathbb{Q}})$.*

**Remark 3.** *Using the change of variable $z \mapsto z^{-1}$, we can deduce from Theorem 2 the classification of Mahler systems over the field of Hahn series at $\infty$.*

**Remark 4.** *Besides $0$ and $\infty$, it is also natural to look for the local structure of the Mahler systems at $1$ (because $1$ is a fixed point of the endomorphism $z \mapsto z^\ell$ of $\mathbb{P}^1(\overline{\mathbb{Q}})$, the other two fixed being $0$ and $\infty$). Using the change of variable $z = e^u$, we see that this is equivalent to the study of the local structure of $q$-difference equations (with $q = \ell$) at $u = 0$, which is well understood; see the works of van der Put, Ramis, Reversat, Sauloy, Zhang* [RSZ13, Sau00, Sau04, vdPR07].

As an application of (a variant of) Theorem 2, we give, in the last section of the present paper, a new proof of an analogue of the so-called Grothendieck's conjecture for Mahler systems, which was first proved in [Roq17]. Let us recall the statement of this result. Consider a Mahler equation of the form

$$a_n(z)y(z^{\ell^n}) + a_{n-1}(z)y(z^{\ell^{n-1}}) + \cdots + a_0(z)y(z) = 0 \tag{2}$$

with coefficients $a_0(z), \ldots, a_n(z) \in \mathbb{Q}(z)$ such that $a_0(z)a_n(z) \neq 0$. For almost all (*i.e.*, for all but finitely many) primes $p$, we can reduce the coefficients of equation (2) modulo $p$, and we obtain the equation

$$a_{n,p}(z)y(z^{\ell^n}) + a_{n-1,p}(z)y(z^{\ell^{n-1}}) + \cdots + a_{0,p}(z)y(z) = 0 \tag{3}$$

with coefficients $a_{0,p}(z), \ldots, a_{n,p}(z) \in \mathbb{F}_p(z)$, where $\mathbb{F}_p$ is the field with $p$ elements. The analogue of Grothendieck's conjecture proven in [Roq17, Theorem 1] is :

**Theorem 5.** *Assume that, for almost all primes $p$, the equation (3) has $n$ $\mathbb{F}_p$-linearly independent solutions in $\mathbb{F}_p((z))$ algebraic over $\mathbb{F}_p(z)$. Then, the equation (2) has $n$ $\overline{\mathbb{Q}}$-linearly independent solutions in $\overline{\mathbb{Q}}(z)$.*

**Remark 6.** *The conclusion of Grothendieck's original conjecture for linear differential equations involes algebraic solutions, not rational solutions. In the case of Mahler equations, there is no distinction between algebraic and rational solutions : a solution $f(z) \in \overline{\mathbb{Q}}((z))$ of* (2) *is algebraic if and only if it is rational, see* [Nis96, Theorem 5.1.7]*. Similarly, according to* [Roq17, Theorem 2]*, any solution in $\mathbb{F}_p((z))$ algebraic over $\mathbb{F}_p(z)$ of* (3) *actually belongs to $\mathbb{F}_p(z)$. The hypothesis of Theorem 5 is thus equivalent to the fact that* (3) *has $n$ $\mathbb{F}_p$-linearly independent solutions in $\mathbb{F}_p(z)$. This fact will be used in the proof of Theorem 5.*

The proof of Theorem 5 given in Section 6 heavily relies on Theorem 2. The first step of the proof consists in applying (a variant of) Theorem 2 to the

Mahler system $Y(z^\ell) = A(z)Y(z)$ associated to the Mahler equation (2): we obtain in this way $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ and $F(z) \in \mathrm{GL}_n(\mathscr{H})$ such that $A(z)F(z) = F(z^\ell)A_0$. We then show that the hypothesis of Theorem 5 implies that $A_0 = I_n$ and that the first line of $F(z)$ are $n$ $\overline{\mathbb{Q}}$-linearly independent rational solutions of (2). Note that, in the course of the proof, we have to reduce the coefficients of $F(z)$ modulo certain prime ideals; this is made possible by Theorem 20, which is a variant of Theorem 2. We refer the reader to Section 6 for details.

This paper is organized as follows. In Section 2, we recall the definition of the field $\mathscr{H}$ of Hahn series. In Section 3, we recall the notion of Mahler modules and its relationship with the notion of Mahler systems. In Section 4, we prove that any Mahler system is $\mathscr{H}$-equivalent to an upper triangular Mahler system with constant diagonal coefficients; this is a first step toward the proof of Theorem 2. The end of the proof of Theorem 2 is given in Section 5. In Section 6, we state a variant of Theorem 2 and outline a proof of Theorem 5.

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## 2. The field $\mathscr{H}$ of Hahn series

We denote by

$$\mathscr{H} = \overline{\mathbb{Q}}((z^{\mathbb{Q}}))$$

the field of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$. An element of $\mathscr{H}$ is a sequence $(f_\gamma)_{\gamma\in\mathbb{Q}} \in \overline{\mathbb{Q}}^{\mathbb{Q}}$ whose support

$$\mathrm{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) = \{\gamma \in \mathbb{Q} \mid f_\gamma \neq 0\}$$

is well-ordered (*i.e.*, any nonempty subset of $\mathrm{supp}(f)$ has a least element) with respect to the restriction to $\mathrm{supp}((f_\gamma)_{\gamma\in\mathbb{Q}})$ of the usual order on $\mathbb{Q}$. An element $(f_\gamma)_{\gamma\in\mathbb{Q}}$ of $\mathscr{H}$ is usually (and will be) denoted by

$$f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma.$$

The sum and product of two elements $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ and $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma$ of $\mathscr{H}$ are given by

$$f + g = \sum_{\gamma\in\mathbb{Q}} (f_\gamma + g_\gamma) z^\gamma$$

and

$$fg = \sum_{\gamma\in\mathbb{Q}} \left( \sum_{\gamma'+\gamma''=\gamma} f_{\gamma'} g_{\gamma''} \right) z^\gamma.$$

(Note that there are only finitely many $(\gamma', \gamma'') \in \mathbb{Q}\times\mathbb{Q}$ such that $\gamma'+\gamma'' = \gamma$ and $f_{\gamma'} g_{\gamma''} \neq 0$, so the sums $\sum_{\gamma'+\gamma''=\gamma} f_{\gamma'} g_{\gamma''}$ are meaningful.)

## 3. Mahler systems and Mahler modules

It is sometimes useful to work with Mahler modules instead of Mahler systems. For the convenience of the reader, we shall now recall what Mahler modules are and explain their links with Mahler systems. It is also the occasion to introduce some notation for later use.

We denote by $\phi$ the field automorphism of $\mathscr{H}$ defined by

$$\phi(f) = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^{\ell\gamma}$$

for any $f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$.

We denote by

$$\mathcal{D}_{\mathscr{H}} = \mathscr{H}\langle \phi, \phi^{-1} \rangle$$

the Öre algebra of noncommutative Laurent polynomials with coefficients in $\mathscr{H}$ such that

$$\phi f = \phi(f)\phi$$

for all $f \in \mathscr{H}$. A left $\mathcal{D}_{\mathscr{H}}$-module of finite length will be called a Mahler module (over $\mathscr{H}$, but we will omit this precision in this paper because we will only consider Mahler modules over $\mathscr{H}$). Note that a left $\mathcal{D}_{\mathscr{H}}$-module has finite length if and only if the $\mathscr{H}$-vector space obtained by restriction of scalars has finite dimension; by definition, the rank of a Mahler module is its dimension as an $\mathscr{H}$-vector space.

There is a correspondence between Mahler systems and Mahler modules that we shall now recall.

One can associate to any Mahler system

$$\phi Y = AY \text{ with } A \in \mathrm{GL}_n(\mathscr{H}) \tag{4}$$

a Mahler module $M_A$ as follows. We consider the map $\Phi_A : \mathscr{H}^n \to \mathscr{H}^n$ defined by

$$\Phi_A(m) = A\phi(m)$$

(here $\phi$ acts component-wise on the elements of $\mathscr{H}^n$ seen as column vectors). The Mahler module $M_A$ is then defined as follows : the underlying abelian group is $\mathscr{H}^n$ (its elements being seen as column vectors) and the action of $L = \sum a_i \phi^i \in \mathcal{D}_{\mathscr{H}}$ on $m \in M_A$ is given by

$$Lm = (\sum a_i \phi^i) m = \sum a_i \Phi_A^i(m).$$

Conversely, we can attach to any Mahler module $M$, a Mahler system *via* the choice of a $\mathscr{H}$-basis $\mathcal{B} = (e_1, \ldots, e_n)$ of $M$ : the Mahler system associated to $M$ with respect to $\mathcal{B}$ is $\phi Y = AY$ where $A \in \mathrm{GL}_n(\mathscr{H})$ represents the action of $\phi$ on $\mathcal{B}$ (*i.e.*, the $j$th column of $A$ represents $\phi(e_j)$ in the basis $\mathcal{B}$). We have $M \cong M_A$.

It is easily seen that two Mahler systems $\phi Y = AY$ and $\phi Y = BY$ with $A, B \in \mathrm{GL}_n(\mathscr{H})$ are $\mathscr{H}$-equivalent if and only if the corresponding Mahler modules $M_A$ and $M_B$ are isomorphic.

Last, we will freely use the following classical result, known as the cyclic vector lemma, ensuring that any Mahler module "comes form" an equation.

**Proposition 7.** *For any Mahler module $M$, there exists $L \in \mathcal{D}_{\mathscr{H}}$ such that $M \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}} L$.*

For a proof, see for instance [HS99, Theorem B.2].

## 4. Triangularization of Mahler systems

The aim of this section is to prove the following result :

**Theorem 8.** *Consider a Mahler system $\phi Y = AY$ with $A \in \mathrm{GL}_n(\mathscr{H})$.*

*(i) The system $\phi Y = AY$ is $\mathscr{H}$-equivalent to $\phi Y = BY$ for some upper triangular matrix*

$$B = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \in \mathrm{GL}_n(\mathscr{H})$$

*with constant diagonal coefficients (i.e., $c_1, \ldots, c_n \in \overline{\mathbb{Q}}^{\times}$).*

*(ii) The list of diagonal coefficients of $B$ does not depend, up to permutation, on the chosen matrix $B$.*

Strictly speaking, we will not prove this result directly, but we will prove a reformulation of Theorem 8 in terms of Mahler modules.

### 4.1. Reformulation of Theorem 8 in terms of Mahler modules.

Theorem 8 can be reformulated in terms of Mahler modules as follows :

**Theorem 9.** *Let $M$ be a Mahler module of rank $n \geq 1$.*

*(i) There exists a filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

*by Mahler sub-modules of $M$ such, for all $i \in \{0, \ldots, n-1\}$,*

$$M_{i+1}/M_i \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi - c_i)$$

*for some $c_i \in \overline{\mathbb{Q}}^{\times}$ .*

*(ii) The list $c_1, \ldots, c_n$ does not depend, up to permutation, on the chosen filtration.*

Let us explain why this result is equivalent to Theorem 8.

Let $M$ be a Mahler module of rank $n \geq 1$. As recalled in (and with the notation of) Section 3, there exists $A \in \mathrm{GL}_n(\mathscr{H})$ such that $M \cong M_A$. If Theorem 8 is true, then $M \cong M_B$ for some upper triangular $B \in \mathrm{GL}_n(\mathscr{H})$ with constant diagonal coefficients $c_1, \ldots, c_n \in \overline{\mathbb{Q}}^{\times}$. Of course, the existence of a filtration of $M$ as in Theorem 9 is equivalent to the existence of a similar filtration for $M_B$. It is clear that $M_B$ has such a filtration : if $(e_1, \ldots, e_n)$ is the canonical basis of $\mathscr{H}^n$ then

$$\{0\} = N_0 \subset N_1 = \mathscr{H} e_1 \subset N_2 = \mathscr{H} e_1 + \mathscr{H} e_2 \subset \\ \cdots \subset N_n = \mathscr{H} e_1 + \cdots + \mathscr{H} e_n = M_B$$

is a filtration by submodules of $M_B$ such that

$$N_{i+1}/N_i \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi - c_i)$$

for all $i \in \{0, \ldots, n-1\}$.

Conversely, let $A \in \mathrm{GL}_n(\mathscr{H})$ and consider the Mahler module $M_A$. If Theorem 9 is true then there exists a filtration

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_n = M_A$$

by submodules of $M_A$ such that, for all $i \in \{0, \dots, n-1\}$,

$$N_{i+1}/N_i \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi - c_i).$$

Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis of $M$ such that, for all $i \in \{1, \dots, n\}$, $(e_1, \dots, e_i)$ is a basis of $N_i$ satisfying

$$\phi(e_i) \in c_i e_i + N_{i-1}.$$

Then, the Mahler system $\phi Y = BY$ associated to $M$ with respect to the basis $\mathcal{B}$ (see Section 3) is upper triangular with diagonal coefficients $c_1, \dots, c_n \in \mathbb{C}^\times$. Since the Mahler systems $\phi Y = AY$ and $\phi Y = BY$ are $\mathscr{H}$-equivalent, this yields the desired result.

This proves the equivalence between the existence statements (i) in Theorem 8 and 9. The equivalence between the uniqueness properties (ii) in these theorems can be seen similarly; the details are left to the reader.

The proof of Theorem 9, given in Section 4.4, will follow, *via* the cyclic vector Lemma, from a factorization property of Mahler operators that we shall now state and prove.

4.2. **Factorization of Mahler operators.** In this section, we consider

$$L = \sum_{i=0}^{n} a_i \phi^i \in \mathcal{D}_{\mathscr{H}}$$

where $n \geq 1$, $a_0, \dots, a_n \in \mathscr{H}$ and $a_0 a_n \neq 0$.

We shall now introduce some notation and terminologies. Let $r \neq 0, a$ be elements of some difference field extension of $\mathscr{H}$ such that $\phi(r) = ar$. We will denote by $L^{[r]}$ the operator defined by

$$L^{[r]} := r^{-1} L r = \sum_{i=0}^{n} a\phi(a) \cdots \phi^{i-1}(a) a_i \phi^i,$$

so that $L^{[r]}(f) = 0$ if and only if $L(rf) = 0$. In particular :

— for any $\mu \in \mathbb{Q}$, we consider $\theta_\mu$ such that $\phi(\theta_\mu) = z^\mu \theta_\mu$ so that

$$L^{[\theta_\mu]} = \sum_{i=0}^{n} z^{(1+\ell+\cdots+\ell^{i-1})\mu} a_i \phi^i;$$

we can and will take

$$\theta_\mu = z^{\frac{\mu}{\ell-1}} \in \mathscr{H}^\times;$$

— for any $c \in \overline{\mathbb{Q}}^\times$, we consider $e_c$ such that $\phi(e_c) = c e_c$ so that

$$L^{[e_c]} = \sum_{i=0}^{n} c^i a_i \phi^i.$$

**Definition 10.** *We define the Newton polygon $\mathcal{N}(L)$ of $L$ as the convex hull in $\mathbb{R}^2$ of*

$$\{(i,j) \in \mathbb{Z} \times \mathbb{R} \mid j \geq v_z(a_{n-i})\}$$

*where $v_z : \mathscr{H} \to \mathbb{Q} \cup \{+\infty\}$ denotes the $z$-adic valuation. This polygon is delimited by two vertical half lines and by $k$ vectors $(r_1, d_1), \ldots, (r_k, d_k) \in \mathbb{N}^* \times \mathbb{Q}$ having pairwise distinct slopes, called the Newton-slopes of $L$. For any $i \in \{1, \ldots, k\}$, $r_i$ is called the multiplicity of the Newton-slope $\frac{d_i}{r_i}$.*

**Lemma 11.** *There exists a unique $\mu_1 \in \mathbb{Q}$ such that the greatest Newton-slope of $L^{[\theta_{\mu_1}]}$ is $0$.*

*Proof.* The fact that the greatest Newton-slope of $L^{[\theta_{\mu_1}]}$ is $0$ means that, for all $i \in \{1, \ldots, n\}$,

$$v_z(a_i) + (1 + \ell + \cdots + \ell^{i-1})\mu_1 \geq v_z(a_0)$$

and that this inequality is an equality for some $i \in \{1, \ldots, n\}$. Obviously, there exists a unique $\mu_1 \in \mathbb{Q}$ with these properties. □

**Definition 12.** *The rational number $\mu_1$ given by Lemma 11 will be called the first theta-slope of $L$. Setting $L^{[\theta_{\mu_1}]} = \sum_{i=0}^{n} b_i \phi^i$, we define the characteristic polynomial associated to the first theta-slope $\mu_1$ of $L$ as $\sum_{i=0}^{n} \left(b_i z^{-v_z(b_0)}\right)_{|z=0} X^i \in \overline{\mathbb{Q}}[X]$; this is a polynomial of degree $\geq 1$ with nonzero constant coefficient.*

In what follows, we will denote by $\mathscr{H}^{\geq 0}$ the valuation ring of $\mathscr{H}$ with respect to the $z$-adic valuation $v_z$, *i.e.*,

$$\mathscr{H}^{\geq 0} = \{f \in \mathscr{H} \mid v_z(f) \geq 0\}.$$

It is a local domain with maximal ideal

$$\mathscr{H}^{>0} = \{f \in \mathscr{H} \mid v_z(f) > 0\}.$$

**Lemma 13.** *Let $\mu_1$ be the first theta-slope of $L$ and let $c_1$ be a root of the corresponding characteristic polynomial. Then, there exists $f_1 \in 1 + \mathscr{H}^{>0}$ such that $L(\theta_{\mu_1} e_{c_1} f_1) = 0$.*

*Proof.* We set $\mu = \mu_1$, $c = c_1$ and

$$L^{[\theta_\mu]} = \sum_{i=0}^{n} b_i \phi^i$$

with

$$b_i = z^{(1+\ell+\cdots+\ell^{i-1})\mu} a_i = \sum_{j \in \mathbb{Q}} b_{i,j} z^j \in \mathscr{H}.$$

Using the fact that the greatest Newton-slope of $L^{[\theta_\mu]}$ is $0$, we see that, up to left multiplication by some monomial in $z$, we can assume that $b_0, \ldots, b_n \in \mathscr{H}^{\geq 0}$ and $b_{0,0} \neq 0$. The characteristic polynomial attached to the first theta-slope $\mu$ of $L$ is given, up to multiplication by some constant in $\overline{\mathbb{Q}}^{\times}$, by $\sum_{i=0}^{n} b_{i,0} X^i$. For $f = \sum_{\gamma \in \mathbb{Q}_{\geq 0}} f_\gamma z^\gamma \in 1 + \mathscr{H}^{>0}$, we have

$$L(\theta_\mu e_c f) = \theta_\mu e_c \sum_{i \in \{0,\ldots,n\}, j, \gamma \in \mathbb{Q}_{\geq 0}} b_{i,j} c^i f_\gamma z^{j+\gamma \ell^i} = 0$$

if and only if, for all $m \in \mathbb{Q}_{\geq 0}$,

$$\sum_{\substack{i\in\{0,\ldots,n\},j,\gamma\in\mathbb{Q}_{\geq 0}\\ j+\gamma\ell^i=m}} b_{i,j}c^i f_\gamma = 0. \tag{5}$$

This equation is automatically satisfied for $m=0$ because

$$\sum_{\substack{i\in\{0,\ldots,n\},j,\gamma\in\mathbb{Q}_{\geq 0}\\ j+\gamma\ell^i=0}} b_{i,j}c^i f_\gamma = \left(\sum_i b_{i,0}c^i\right) f_0$$

and $\sum_i b_{i,0}c^i = 0$ because $c$ is a root of the characteristic polynomial. For $m \in \mathbb{Q}_{>0}$, the equation (5) can be rewritten as follows

$$\sum_{\substack{i\in\{0,\ldots,n\},j\in\cup_{i\in\{0,\ldots,n\}}\operatorname{supp}(b_i),\gamma\in\mathbb{Q}_{\geq 0}\\ \gamma<m,j+\gamma\ell^i=m}} b_{i,j}c^i f_\gamma = -b_{0,0}f_m. \tag{6}$$

Let us prove that this equation has a solution $f \in 1+\mathscr{H}^{>0}$. Let

$$j_0 = \min \cup_{i\in\{0,\ldots,n\}} \operatorname{supp}(b_i) \setminus \{0\} \in \mathbb{Q}_{>0}$$

(this minimum exists because the supports of the $f_i$ are well-ordered). Take $\epsilon > 0$ such that, for all $r \in \mathbb{Z}_{\geq 1}$,

$$m \in [0, j_0 + r\epsilon] \Rightarrow m - j_0, m/\ell \in ]-\infty, j_0 + (r-1)\epsilon];$$

it follows that, for any $m \in ]j_0+(r-1)\epsilon, j_0+r\epsilon] \cap \mathbb{Q}$, $i \in \{0,\ldots,n\}$, $j \in \cup_{i\in\{0,\ldots,n\}} \operatorname{supp}(b_i)$, $\gamma \in \mathbb{Q}_{\geq 0}$ , $\gamma < m$,

$$j + \gamma\ell^i = m \Rightarrow \gamma \in ]-\infty, j_0+(r-1)\epsilon] \cap \mathbb{Q}.$$

This choice of $\epsilon$ allows us to define a sequence $(f_\gamma)_{\gamma\in\mathbb{Q}}$ with support in $\mathbb{Q}_{\geq 0}$ as follows:

— $f_0 = 1$;
— for all $m \in ]0, j_0[\cap\mathbb{Q}$, $f_m = 0$;
— $f_{j_0} = \frac{-1}{b_{0,0}} \sum_{i\in\{0,\ldots,n\}} b_{i,j_0}c^i$;
— for all $r \geq 1$, for all $m \in ]j_0+(r-1)\epsilon, j_0+r\epsilon] \cap \mathbb{Q}$ ,

$$f_m = \frac{-1}{b_{0,0}} \sum_{\substack{i\in\{0,\ldots,n\},j\in\cup_{i\in\{0,\ldots,n\}}\operatorname{supp}(b_i),\gamma\in\mathbb{Q}_{\geq 0}\\ \gamma<m,j+\gamma\ell^i=m}} b_{i,j}c^i f_\gamma.$$

It is obvious that this sequence satisfies (6). We shall now prove that the support of $(f_\gamma)_{\gamma\in\mathbb{Q}}$ is well-ordered. First note that, for all $r \in \mathbb{Z}_{\geq 1}$, we have

$$\begin{aligned}&\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0+r\epsilon]\\ &\qquad\subset \cup_{i\in\{0,\ldots,n\}} \left(\operatorname{supp}(b_i) + \ell^i \cdot \operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0+(r-1)\epsilon]\right).\end{aligned}$$

Therefore, if $\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0+(r-1)\epsilon]$ is well-ordered then $\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0+r\epsilon]$ is well-ordered as well. But, for $r=0$,

$$\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0+r\epsilon] = \operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0] \subset \{0, j_0\}$$

is well-ordered. It follows by an obvious induction argument that, for all $r \in \mathbb{Z}_{\geq 1}$, $\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) \cap ]-\infty, j_0+r\epsilon]$ is well-ordered and, hence, that $\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}})$ is well-ordered. This concludes the proof of the Lemma. $\square$

**Lemma 14.** *Maintaining the notation of Lemma 13, we can factorize $L$ as follows*

$$L = L_2(\phi - c_1)(f_1\theta_{\mu_1})^{-1}$$

*for some $L_2 \in \mathcal{D}_{\mathscr{H}}$.*

*Proof.* By euclidean division (see [Ore33, Section 2]) of $L$ by the operator $(\phi - c_1)(f_1\theta_{\mu_1})^{-1}$, we obtain $L_2 \in \mathcal{D}_{\mathscr{H}}$ and $h \in \mathscr{H}$ such that $L = L_2(\phi - c_1)(f_1\theta_{\mu_1})^{-1} + h$. Since both $L$ and $L_2(\phi - c_1)(f_1\theta_{\mu_1})^{-1}$ annihilate $\theta_{\mu_1}e_{c_1}f_1$, we have $h = 0$. □

A repeated application of the previous lemma leads to the following result.

**Theorem 15.** *The operator $L$ admits a factorization of the form*

$$L = a_n\phi^n(f_1\theta_{\mu_1})\cdots\phi(f_n\theta_{\mu_n})(\phi - c_n)(f_n\theta_{\mu_n})^{-1}\cdots(\phi - c_1)(f_1\theta_{\mu_1})^{-1}$$

*where, for all $i \in \{1, \ldots, n\}$, $c_i \in \overline{\mathbb{Q}}^\times$, $\mu_i \in \mathbb{Q}$ and $f_i \in 1 + \mathscr{H}^{>0}$.*

4.3. **Mahler modules of rank 1.** We shall first study the Mahler modules of rank one. For any $\alpha \in \mathscr{H}^\times$, we denote by $I_\alpha$ the Mahler module of rank one defined by

$$I_\alpha = \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi - \alpha).$$

In what follows, we will denote by $\mathrm{cld}(\alpha)$ the coefficient of the term of lowest $z$-adic valuation of $\alpha \in \mathscr{H}^\times$. Note that $\mathrm{cld} : \mathscr{H}^\times \to \overline{\mathbb{Q}}^\times$ is a group morphism.

**Proposition 16.** *(i) For any $\alpha, \beta \in \mathscr{H}^\times$, the Mahler modules $I_\alpha$ and $I_\beta$ are isomorphic if and only if $\mathrm{cld}(\alpha) = \mathrm{cld}(\beta)$.*
*(ii) For any $\alpha \in \mathscr{H}^\times$, the Mahler modules $I_\alpha$ and $I_{\mathrm{cld}(\alpha)}$ are isomorphic.*
*(iii) For any Mahler module $M$ of rank 1, there exists a unique $c \in \overline{\mathbb{Q}}^\times$ such that $M$ is isomorphic to $I_c$.*

*Proof.* It is easily seen that the set of $\mathcal{D}_{\mathscr{H}}$-modules morphisms from $I_\alpha$ to $I_\beta$ is given by

$$\mathrm{Hom}(I_\alpha, I_\beta) = \{\varphi_u \mid u \in \mathscr{H}, \alpha u = \phi(u)\beta\}$$

where $\varphi_u : I_\alpha \to I_\beta$ is defined by $\varphi_u(\overline{P}) = \overline{Pu}$ and that $\varphi_u$ is an isomorphism if and only if $u \in \mathscr{H}^\times$. Therefore, $I_\alpha \cong I_\beta$ if and only if there exists $u \in \mathscr{H}^\times$ such that $\alpha u = \phi(u)\beta$. But

$$\{\phi(u)/u \mid u \in \mathscr{H}^\times\} = \ker(\mathrm{cld} : \mathscr{H}^\times \to \overline{\mathbb{Q}}^\times)$$

(indeed, the direct inclusion is obvious; for the converse inclusion, note that any $a \in \mathscr{H}^\times$ such that $\mathrm{cld}(a) = 1$ can be decomposed as $a = z^\mu f$ for some $\mu \in \mathbb{Q}$ and some $f \in 1 + \mathscr{H}^{>0}$, so $u = \theta_\mu \prod_{j\geq 0} \phi^j(f^{-1}) \in \mathscr{H}^\times$ satisfies $a = \phi(u)/u$, whence the desired result). So $I_\alpha \cong I_\beta$ if and only if $\mathrm{cld}(\alpha) = \mathrm{cld}(\beta)$. This proves (i). The remaining assertions follow easily. □

4.4. **Proof of Theorem 9.** According to the cyclic vector lemma (Proposition 7), there exists $L \in \mathcal{D}_{\mathscr{H}}$ such that $M \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}L$. Theorem 15 ensures that

$$L = c(\phi - c_n)g_n^{-1} \cdots (\phi - c_1)g_1^{-1}$$

for some $c \in \mathscr{H}^\times$, $c_i \in \overline{\mathbb{Q}}^\times$ and $g_i \in \mathscr{H}^\times$. We deduce from this factorization a filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

by Mahler sub-modules of $M$ such that, for all $i \in \{0, \ldots, n-1\}$, $M_{i+1}/M_i \cong \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi - c_i)g_i^{-1}$. Indeed, this follows immediately from the following classical result.

**Lemma 17.** *Let $P, Q, R \in \mathcal{D}_{\mathscr{H}}$ such that $P = QR$. Then, we have a natural exact sequence*

$$0 \to \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}Q \to \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}P \to \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}R \to 0.$$

*Proof.* We have $\mathcal{D}_{\mathscr{H}}P \subset \mathcal{D}_{\mathscr{H}}R$, whence a surjective morphism $\mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}P \to \mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}R$. Its kernel is $\mathcal{D}_{\mathscr{H}}R/\mathcal{D}_{\mathscr{H}}P$. The map $\mathcal{D}_{\mathscr{H}} \to \mathcal{D}_{\mathscr{H}}R$, $F \mapsto FR$ induces an isomorphism $\mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}Q \to \mathcal{D}_{\mathscr{H}}R/\mathcal{D}_{\mathscr{H}}P$. □

Now, it follows from Proposition 16 that $\mathcal{D}_{\mathscr{H}}/\mathcal{D}_{\mathscr{H}}(\phi - c_i)g_i^{-1} \cong I_{c_i}$ and this concludes the proof of the assertion (i) of Theorem 9.

It remains to prove the assertion (ii) of Theorem 9. By the Jordan-Hölder theorem, if

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_m = M$$

is another filtration of $M$ such that, for all $i \in \{0, \ldots, m-1\}$, $N_{i+1}/N_i \cong I_{d_i}$ for some $d_i \in \overline{\mathbb{Q}}^\times$, then $m = n$ and there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $M_{\sigma(i)+1}/M_{\sigma(i)} \cong N_{i+1}/N_i$. Proposition 16 ensures that $c_{\sigma(i)} = d_i$, whence (ii).

## 5. Proof of Theorem 2

### 5.1. Inhomogeneous equations of order 1 with constant coefficients.

**Lemma 18.** *For any subset $E$ of $\mathbb{Q}$, we set*

$$\mathrm{Sat}_\ell(E) = \{\ell^{-k}x \mid x \in E \cap \mathbb{Q}_{\leq 0}, k \geq 0\} \cup \{\ell^k x \mid x \in E \cap \mathbb{Q}_{\geq 0}, k \geq 0\}.$$

*If $E$ is a well-ordered subset of $\mathbb{Q}$, then $\mathrm{Sat}_\ell(E)$ is a well-ordered subset of $\mathbb{Q}$.*

*Proof.* Let $F$ be a subset of $\mathrm{Sat}_\ell(E)$.

Assume that $F \cap \mathbb{Q}_{<0} \neq \emptyset$ and consider $\gamma \in F \cap \mathbb{Q}_{<0}$. Since $E$ is bounded from below, there exists $M$ such that, for all $k \geq M$, for all $x \in E$, $\gamma < \ell^{-k}x$. Therefore, in order to prove that $F$ has a least element, it is sufficient to prove that

$$\{\ell^{-k}x \mid x \in E \cap \mathbb{Q}_{\leq 0}, k \in \{0, \ldots, M-1\}\} \cap F$$

has a least element. This follows from the facts that the latter set can be rewritten has the finite union $\cup_{k=0}^{M-1}(\ell^{-k}E \cap \mathbb{Q}_{\leq 0}) \cap F$ and that each $(\ell^{-k}E \cap \mathbb{Q}_{\leq 0}) \cap F$ has a least element (because $E$ and, hence, $\ell^{-k}E \cap \mathbb{Q}_{\leq 0}$ are well-ordered).

The case $F \cap \mathbb{Q}_{<0} = \emptyset$ is similar. □

**Proposition 19.** *For all $c,d \in \overline{\mathbb{Q}}^{\times}$, for all $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{H}$ with $g_0 = 0$, there exists $f \in \mathscr{H}$ such that $g = (c\phi - d)f$.*

*Moreover, if $c \neq d$, then, for all $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{H}$, there exists $f \in \mathscr{H}$ such that $g = (c\phi - d)f$.*

*Proof.* Dividing by $c$, it is clearly sufficient to consider the case $c = 1$.

We first assume that $g_0 = 0$. We set $g^- = \sum_{\gamma\in\mathbb{Q}_{<0}} g_\gamma z^\gamma \in \mathscr{H}$ and $g^+ = \sum_{\gamma\in\mathbb{Q}_{>0}} g_\gamma z^\gamma \in \mathscr{H}$, so that $g = g^- + g^+$. We are going to prove that there exist $f^\pm \in \mathscr{H}$ such that $g^\pm = (\phi - d)f^\pm$. This will imply the desired result because $f = f^- + f^+ \in \mathscr{H}$ satisfies $g = (\phi - d)f$.

For all $\gamma \in \mathbb{Q}_{<0}$ such that $\ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g) \neq \emptyset$, we set $\gamma^- = \min \ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g)$ (it exists because $\mathrm{supp}(g)$ is well-ordered). We let $(f_\gamma^-)_{\gamma\in\mathbb{Q}_{<0}}$ be the unique element of $\overline{\mathbb{Q}}^{\mathbb{Q}_{<0}}$ such that, for all $\gamma \in \mathbb{Q}_{<0}$ such that $\ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g) \neq \emptyset$,

$$\begin{cases} f^-_{\gamma^-/\ell^{i+1}} = d f^-_{\gamma^-/\ell^{i}} + g_{\gamma^-/\ell^{i}} & \text{for } i \geq 0, \\ f^-_{\gamma^-/\ell^{i+1}} = 0 & \text{for } i \leq -1 \end{cases}$$

and, for all $\gamma \in \mathbb{Q}_{<0}$ such that $\ell^{\mathbb{Z}}\gamma \cap \mathrm{supp}(g) = \emptyset$,

$$f_\gamma^- = 0.$$

Then, $f^- = \sum_{\gamma\in\mathbb{Q}_{<0}} f_\gamma^- z^\gamma \in \mathscr{H}$ satisfies $(\phi - d)f^- = g^-$. The fact that $f^-$ belongs to $\mathscr{H}$ is a consequence of Lemma 18 because $\mathrm{supp}(f) \subset \mathrm{Sat}_\ell(\mathrm{supp}(g))$.

The construction of $f^+$ is similar.

We now assume that $c = 1 \neq d$. We set $g^- = \sum_{\gamma\in\mathbb{Q}_{<0}} g_\gamma z^\gamma \in \mathscr{H}$ and $g^+ = \sum_{\gamma\in\mathbb{Q}_{>0}} g_\gamma z^\gamma \in \mathscr{H}$, so that $g = g^- + g_0 + g^+$. We have already proved that there exist $f^\pm \in \mathscr{H}$ such that $g^\pm = (\phi - d)f^\pm$. Moreover, $f_0 = \frac{g_0}{1-d}$ satisfies $g_0 = (\phi - d)f_0$. So, $f = f^- + f_0 + f^+ \in \mathscr{H}$ satisfies $g = (\phi - d)f$. □

5.2. **Proof of Theorem 2.** Using Theorem 8, it is clear that the first (and main) statement of Theorem 2 will be proven if we manage to prove that, for any upper triangular

$$A(z) = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \in \mathrm{GL}_n(\mathscr{H})$$

with diagonal coefficients $c_1, \ldots, c_n \in \overline{\mathbb{Q}}^{\times}$, the system $\phi Y = AY$ is $\mathscr{H}$-equivalent to $\phi Y = A_0 Y$ for some

$$A_0 = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \in \mathrm{GL}_n(\overline{\mathbb{Q}}).$$

We shall now prove this property by induction on $n$.

The case $n = 1$ is true as a direct consequence of Theorem 8.

Assume that our claim is true for some $n \geq 1$ and consider

$$A(z) = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_{n+1} \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathscr{H})$$

with diagonal coefficients $c_1, \dots, c_{n+1} \in \overline{\mathbb{Q}}^\times$. We write

$$A(z) = \begin{pmatrix} B(z) & * \\ 0 & c_{n+1} \end{pmatrix}$$

with

$$B(z) = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \in \mathrm{GL}_n(\mathscr{H}).$$

The induction hypothesis ensures that

$$B(z)F(z) = F(z^\ell)B_0$$

for some $F(z) \in \mathrm{GL}_n(\mathscr{H})$ and some

$$B_0 = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \in \mathrm{GL}_n(\overline{\mathbb{Q}}).$$

We then have

$$A(z)G(z) = G(z^\ell)C_0$$

with

$$G(z) = \begin{pmatrix} F(z) & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathscr{H}) \text{ and } C_0 = \begin{pmatrix} B_0 & * \\ 0 & c_{n+1} \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathscr{H}).$$

So, the system $\phi Y = AY$ we started with is $\mathscr{H}$-equivalent to $\phi Y = C_0 Y$. We will now eliminate the (*a priori*) non-constant coefficients of $C_0$. In what follows, we denote by $E_{i,j}$ the matrix in $\mathrm{M}_{n+1}(\mathbb{Q})$ differing from the zero matrix by its $(i,j)$-coefficient which is equal to 1.

According to Proposition 19, there exists $f_1 \in \mathscr{H}$ and $\delta_1 \in \mathbb{C}$ such that $c_{n+1}\phi(f_1) - c_1 f_1 = -(C_0)_{1,n+1} + \delta_1$. Setting $F_1 = I_{n+1} + f_1 E_{1,n+1}$, we have

$$D_1 := F_1(z^\ell)C_0F_1(z)^{-1} = \begin{pmatrix} B_0 & V_1 \\ 0 & c_{n+1} \end{pmatrix}$$

with

$$V_1 = \begin{pmatrix} \delta_1 \\ * \\ \vdots \\ * \end{pmatrix} \in \mathscr{H}^n.$$

Similarly, according to Proposition 19, there exists $f_2 \in \mathscr{H}$ and $\delta_2 \in \mathbb{C}$ such that $c_{n+1}\phi(f_2) - c_2 f_2 = -(D_1)_{2,n+1} + \delta_2$. Then, setting $F_2 = I_{n+1} + f_2 E_{2,n+1}$, we have

$$D_2 := F_2(z^\ell)C_0F_2(z)^{-1} = \begin{pmatrix} B_0 & V_2 \\ 0 & c_{n+1} \end{pmatrix}$$

with

$$V_2 = \begin{pmatrix} \delta_1 \\ \delta_2 \\ * \\ \vdots \\ * \end{pmatrix} \in \mathscr{H}^n.$$

Iterating this process, we end up with a matrix $D_n \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ such that $\phi Y = D_n Y$ is $\mathscr{H}$-equivalent to $\phi Y = C_0 Y$. This concludes the proof of the first statement of Theorem 2 (with $A_0 = D_n$).

In order to prove the second statement of Theorem 2, we have to prove that, if $\phi Y = AY$ and $\phi Y = BY$ with $A, B \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ are $\mathscr{H}$-equivalent, then $A$ and $B$ are conjugate by an element of $\mathrm{GL}_n(\overline{\mathbb{Q}})$. Consider two such matrices $A, B \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ and let $F = \sum_{\gamma \in \mathbb{Q}} F_\gamma z^\gamma \in \mathrm{GL}_n(\mathscr{H})$ be such that $AF(z) = F(z^\ell)B$. We have, for all $\gamma \in \mathbb{Q}$, $AF_\gamma = F_{\gamma/\ell} B$. So, for all $\gamma \in \mathbb{Q}$ and all $k \in \mathbb{Z}$, $A^k F_\gamma = F_{\gamma/\ell^k} B^k$. If $\gamma \in \mathbb{Q}_{>0}$, then $F_{\gamma/\ell^k} = 0$ for $k$ large enough (because the support of $F$ is well-ordered), so $F_\gamma = 0$. Similarly, $F_\gamma = 0$ for all $\gamma \in \mathbb{Q}_{<0}$. So, $F = F_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ and this yields the desired result.

## 6. A variant of Theorem 2 and proof of Theorem 5

6.1. **A variant of Theorem 2.** We let $\mathscr{H}_b$ be the subfield of $\mathscr{H}$ made of the $f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ whose coefficients $(f_\gamma)_{\gamma \in \mathbb{Q}}$ belong to some finitely generated $\mathbb{Z}$-subalgebra of $\overline{\mathbb{Q}}$.

One can easily check that, in all the previous results of the present paper, the field $\mathscr{H}$ can be replaced by $\mathscr{H}_b$. In particular, the following variant of our main result holds true.

**Theorem 20.** *Any Mahler system $Y(z^\ell) = A(z)Y(z)$ with $A(z) \in \mathrm{GL}_n(\mathscr{H}_b)$ is $\mathscr{H}_b$-equivalent to a Mahler system with constant coefficients, i.e., of the form $Y(z^\ell) = A_0 Y(z)$ for some $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$. The matrix $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ is unique up to conjugation by an element of $\mathrm{GL}_n(\overline{\mathbb{Q}})$.*

6.2. **An application : proof of Theorem 5.** We shall now indicate briefly how one can use Theorem 20 in order to give a variant of the proof of Theorem 5 below which was first proved in [Roq17].

We consider the difference system associated to the equation (2) :

$$\phi(Y) = AY, \text{ with } A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix}. \tag{7}$$

According to Section 6.1, there exist $F \in \mathrm{GL}_n(\mathscr{H}_b)$ and $A_0 \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ such that

$$AF = \phi(F)A_0. \tag{8}$$

Let $K$ be a number field containing the entries of $A_0$ and of the coefficients of $F$ and of $A$. We have, for almost all primes $\mathfrak{p}$ of $K$,

$$A_\mathfrak{p} F_\mathfrak{p} = \phi(F_\mathfrak{p}) A_{0,\mathfrak{p}},$$

where the subscript $\mathfrak{p}$ means that we have reduced the coefficients modulo $\mathfrak{p}$. Hence, the entries of $A_{0,\mathfrak{p}}$ belong to the residue field $\kappa_\mathfrak{p}$ of $K$ at $\mathfrak{p}$ and $A_\mathfrak{p}$ and $F_\mathfrak{p}$ are Hahn series with coefficients in $\mathrm{M}_n(\kappa_\mathfrak{p})$ and value group $\mathbb{Q}$.

On the other hand, according to [Roq17, Theorem 2], our hypotheses imply that, for almost all primes $p$, the equation (3) has $n$ $\mathbb{F}_p$-linearly independent solutions in $\mathbb{F}_p(z)$. So, for almost all primes $\mathfrak{p}$ of $K$, there exists $G_{\mathfrak{p}} \in \mathrm{GL}_n(\kappa_{\mathfrak{p}}(z))$ such that

$$A_{\mathfrak{p}} G_{\mathfrak{p}} = \phi(G_{\mathfrak{p}}).$$

Therefore, $H_{\mathfrak{p}} = G_{\mathfrak{p}}^{-1} F_{\mathfrak{p}}$ satisfies

$$H_{\mathfrak{p}} = \phi(H_{\mathfrak{p}}) A_{0,\mathfrak{p}}.$$

Setting $H_{\mathfrak{p}} = \sum_{\gamma \in \mathbb{Q}} H_{\mathfrak{p},\gamma} z^{\gamma}$ with $H_{\mathfrak{p},\gamma} \in \mathrm{M}_n(\kappa_{\mathfrak{p}})$, we get $H_{\mathfrak{p},\ell\gamma} = H_{\mathfrak{p},\gamma} A_{0,\mathfrak{p}}$, for all $\gamma \in \mathbb{Q}$. The fact that the support of $H_{\mathfrak{p}}$ is well-ordered implies that $H_{\mathfrak{p},\gamma} = 0$ for all $\gamma \in \mathbb{Q}^{\times}$ (provided that $A_{0,\mathfrak{p}}$ is invertible, which is true for almost all primes $\mathfrak{p}$ of $K$). So, $H_{\mathfrak{p}} = H_{\mathfrak{p},0}$ and $A_{0,\mathfrak{p}} = I_n$.

It follows that $A_0 = I_n$. It follows also that, for almost all primes $\mathfrak{p}$ of $K$, $F_{\mathfrak{p}} = G_{\mathfrak{p}} H_{\mathfrak{p}} = G_{\mathfrak{p}} H_{\mathfrak{p},0}$ has entries in $\kappa_{\mathfrak{p}}(z)$. But, the first line of $F$ is made of $n$ $\overline{\mathbb{Q}}$-linearly independent solutions $(f_1, \ldots, f_n)$ in $\mathscr{H}_b$ of the equation (2). These $f_i$ actually belong to $K((z))$ because, for almost all primes $\mathfrak{p}$ of $K$, the reductions modulo $\mathfrak{p}$ of the $f_i$ are elements of $\kappa_{\mathfrak{p}}(z) \subset \kappa_{\mathfrak{p}}((z))$. Then, [AB17, Lemma 5.3] ensures that the $f_i$ actually belong to $\overline{\mathbb{Q}}(z)$.

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