ENS Lyon and Université Claude Bernard Lyon 1, M2A
December 19th, 2025

Calculus of Variations and Elliptic PDEs

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Final Exam

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3h duration. All kind of documents (notes, books...) are authorized, but not communication devices. The total number of points is much larger than 20, so attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (7 points)**.** Given a function $f \in L^1([0,1])$, consider the optimization problem

$$\min \left\{ \int_0^1 e^{-t} \left(\frac{|u'(t)|^2}{2} + f(t)u(t) \right) dt \,:\, u \in C^1([0,1]) \right\}.$$

Find a necessary and sufficient condition on f so that the problem above admits a solution. Find this solution in the case $f(t) = t - a$ for the only value of a which allows to satisfy such a condition.

Solution: The condition is $\int_0^1 e^{-t} f(t)dt = 0$. Indeed, if this condition is met, the functional becomes invariant by adding a constant, so that we can assume that a minimizing sequence is made of fucntions with zero mean, and we can apply the Poncaré-Wirtinger inequality to obtain that it is bounded in H^1. It is also necessary because otherwise adding a constant to u does not change the first term but allows to let the second tend to $-\infty$, and the inf would not be finite.

To be sure that when $\int_0^1 e^{-t} f(t)dt = 0$ the problem admits a C^1 minimizer we can define the function u via

$$u'(t) = e^t \int_0^t e^{-s} f(s)ds,$$

and we observe that it satisfies the Euler-Lgrange equation $(e^{-t}u'(t))' = e^{-t}f(t)$ together with the boundary conditions $u'(0) = u'(1) = 0$. The proble being convex, this is enough for being a minimizer. Morover, u' is continuous since it is defined by taking the primitive of an L^1 function.

In the case $f(t) = t-a$ the condition on a is given by $0 = \int_0^1 e^{-t}(t-a)dt = (1-2e^{-1}) - a(1-e^{-1})$, i.e. $a = \frac{e-2}{e-1}$. In this case the solution is given by solving the Euler-Lgrange equation, which can be re-written as

$$u''(t) - u'(t) = t - a$$

with Neumann boundary conditions. The general solution of the homogeneous equation is of the form $A + Be^t$ and a particular solution is given by $-\frac{1}{2}t^2 + (a-1)t$. The solution is then given by $A + Be^t - \frac{1}{2}t^2 + (a-1)t$ and we need to impose $u'(0) = u'(1) = 0$. The value of A is arbitrary and the value of B is given by $B = \frac{1}{e-1}$.

Exercice 2 (8 points)**.** Given a bounded and smooth domain $\Omega \subset \mathbb{R}^d$ and a constant $m \in \mathbb{R}$ consider the problem

$$\min \left\{ \int_\Omega \left(|\nabla v|^2 + v|\nabla u|^2 + vu^2 \right) dx \,:\, v, u \in H^1(\Omega), v-1, u-1 \in H^1_0(\Omega), v \geq m \right\}.$$

1. Prove that for the problem admits a solution if $m > 0$ and does not admit solutions if $m < 0$. What is the difficulty for the case $m = 0$?

2. When a solution exists, prove that it can be taken such that we have $0 \leq u \leq 1$ and $m \leq v \leq 1$.

3. Prove that the minimizers (u, v) satisfy distributionally inside Ω

$$\begin{cases} \nabla \cdot (v \nabla u) = vu, \\ -\Delta v + \frac{1}{2}(|\nabla u|^2 + u^2) \geq 0, \end{cases}$$

but also satisfy the equality

$$\int_\Omega (v - m)(|\nabla u|^2 + u^2)\eta dx + 2 \int_\Omega \nabla v \cdot \nabla[(v - m)\eta] dx = 0$$

for every smooth function $\eta \geq 0$ such that the support of η is compactly contained in the open set Ω.

4. If $m > 0$ prove that for any optimal (u, v) the function u is locally Hölder continuous inside Ω.

Solution:

1. If $m > 0$ then we take a minimizing sequence (v_n, u_n) and, since all terms are positive, we obtain a bound on $||v_n||_{H^1}$ and $||u_n||_{H^1}$ (for this second bound, we use $v_n \geq m$). We can then extract weakly converging subsequences (and the condition $u - 1, v - 1 \in H_0^1$ passes to the limit), and all the terms are convex in the gradient variables and continuous in the others, which means that the functional is l.s.c. If, instead, $m < 0$, the inf is $-\infty$. Indeed, we can choose an admissible v which is equal to $-m$ on a ball $B \subset \Omega$ far from the boundary, and an arbitrary function $\phi \in C_c^1(B)$ and take $u = 1 + \phi$. The functional is then equal to

$$\int_\Omega |\nabla v|^2 + \int_{\Omega \setminus B} v - m \int_B (|\nabla \phi|^2 + (1 + \phi)^2),$$

which can be made as negative as we want by multplying ϕ times a constant.

The difficulty in proving existence for $m = 0$ comes from the fact that we cannot bound ∇u_n in L^2. Any form of weak convergence of ∇u_n would be enough for semicontinuity, but we would also need to guarantee that the limit is in the space of admissible competitors.

2. When a solution exists, we can truncate it, i.e. we replace (u, v) with $(\max\{0, \min\{1, u\}\}, \max\{m, \mathrm{m}$ and the value of the functional does not increase. Actually, the truncation in v strictly decreases this vlaue (so any solution must satisfy $m \leq v \leq 1$) but wa can't say the same for u in case $m = 0$ (for which we do not exclude the existence of a minimizer) on the set $\{v = 0\}$.

3. We take a minimizer (u, v), freeze v and look at this problem as a variational problem on u. The Euler-Lagrange equation on u provides the first condition. For v it is more delicate, since we have a constraint $v \geq m$. thus, we can only test the optimality of v against perturbations of the form $v + \varepsilon\phi$ with $\phi \geq 0$. The first-order expansion provides

$$\int 2\nabla v \cdot \nabla \phi + \phi(|\nabla u|^2 + u^2) \geq 0,$$

which corresponds to the claimed inequality. Yet, it is also possible to take $\phi < 0$ on the set where $v > m$, which should provide the equality on the set $\{v > m\}$. To make this rigorous, we consider a smooth function $\eta \geq 0$ as in the statement and $\varepsilon \geq 0$ small enough so that $\varepsilon||\eta||_\infty \leq 1$. We then compare v to $v_\varepsilon := (1-\varepsilon\eta)v + \varepsilon\eta m$, which amounts at obtaining the same first-order expansion as above for $\phi = \eta(m-v)$. Since this function is nonpositive, the above inequality holds for $-\phi$, which finally provides the desired equality.

4. If $m > 0$ the equation satisfied by u can be treated using DeGiorgi-Nash-Moser's theory for equations with bounded coeeficients, anf the right-hand side uv is L^∞ (hence, it belogs to L^p and as such is a divergence of a $W^{1,p}$ function, so in particular of a function in L^p, and this holds for every p).

Exercice 3 (10 points)**.** On the torus $\mathbb{T}^d$ (in dimension $d \geq 2$), given a Lipschitz continuous function $f : \mathbb{T}^d \to \mathbb{R}$ and an exponent $p > 1$, consider the two nonlinear elliptic PDEs

$$\text{(Q)} \quad \Delta_p u = u + f, \qquad\qquad \text{(W)} \quad \Delta_p u = -u + u^3 + f.$$

1. Prove by variational methods that both of them admit a weak solution in $W^{1,p}$.
2. Prove that the solution of (Q) is unique, while (W) could have several solutions.
3. Prove that any $W^{1,p}$ solution u of (Q) satisfies $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ as soon as $p \geq 2$.
4. Prove that any $W^{1,p}$ solution u of (W) satisfies $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ as soon as
$$p \geq 4\frac{d}{d+2}.$$
5. Prove that any $W^{1,p}$ solution of a PDE of the form $\Delta_p u = g(u) + f$ for g nondecreasing satisfies $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ as soon as $f \in W^{1,p'}$ and deduce that we can obtain the same answer to the two previous questions even if we remove the assumption $p \geq 2$ in the case of (Q) and we only assume $p \geq 2$ in the case of (W).

Solution:

1. The equation (Q) and (W) are the Euler-Lagrange equations of the minimization of
$$u \mapsto \int \frac{1}{p}|\nabla u|^p + \frac{u^2}{2} + fu \quad \text{and} \quad u \mapsto \int \frac{1}{p}|\nabla u|^p + \frac{u^4}{4} - \frac{u^2}{2} + fu,$$
respectively. For the first probem, any minimzing sequence (u_n) is bounded in L^2 and its gradient in L^p, and this ie enough to extract a converging subsequence. In the second, the L^4 part dominates the negative L^2 part, and the same occurs. The functional is not convex in the variable u (only in ∇u) but u_n will converge strongly in L^p, hence a.e., and it is enough to use Fatous's lemma on the sequence $\frac{u_n^4}{4} - \frac{u_n^2}{2}$, which is bounded from below by $-1/4$.
2. The solution of (Q) is unique because the corresponding functional is trictly convex, which is not the case for (W). In particular, for $f = 0$ we have at least three solutions, the constants $0, 1, -1$.
3. For solutions of $\Delta_p u = F$ the condition $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ is satisfied as soon as $F \in W^{1,p'}$. Here $F = u + f$, f is Lipschitz, and $u \in W^{1,p} \subset W^{1,p'}$ because $p \geq 2 \geq p'$..
4. Since $d \geq 2$ the exponent $4\frac{d}{d+2}$ is at least 2, so we have $-u + f \in W^{1,p'}$ as before. But we need to guarantee as well $u^3 \in W^{1,p'}$. We treat separaely the case $p = 2, d = 2$. In all the other cases we have $p > 2$ and we write
$$\int |\nabla u^3|^{p'} = C \int u^{2p'}|\nabla u|^{p'} \leq C \left(\int u^{2p'q'}\right)^{1/q'} \left(\int \nabla u|^{p'q}\right)^{1/q},$$

choosing $q = p/p' = p - 1$. This gives $q' = (p-1)/(p-2)$ and $2p'q = 2p/(p-2)$. Since $\nabla u \in L^p$ we only need to guarantee $u \in L^{2p/(p-2)}$. If $p > d$ the $u \in L^\infty$ and if $p = d$ we have anyway $u \in L^r$ for every r. So the only case to be considered is $p < d$ and we need $2p/(p-2) \leq dp/(d-p)$, which is the exponent of the Sobolev injection. After simplification this gives exactly the desired condition on p.

We now come back to the case $p = 2, d = 2$ but in this case the equation becomes a linear Laplacian, and the right hand side belongs to all L^r spaces. We then deduce $u \in W^{2,r}$, which is stronger than what we have to prove.

5. The equation $\Delta_p u = g(u) + f$ is the Euler-Lagrange equation of the minimization of $\min \int \frac{1}{p}|\nabla u|^p + G(u) + fu$,, where G is the antiderivative of g, and is convex. We can consider the dual probelm which, by Fenchel-Rockafellar, is (written as a minimization) $\min \int \frac{1}{q}|v|^q + G^*(\nabla \cdot v - f)$. We then apply the usual regularity-via-duality argument to these problems, using $\frac{1}{q}|v|^q + \frac{1}{p}|w|^p \geq v \cdot w + c|v^{q/2} - w^{p/2}|^2$ but for G and G^* we only use their convexity. This provides the desired regularity result for solutions of $\Delta_p u = g(u) + F$ and in our cases we have to consider either $F = f$ (for the equation (Q), and f being Lipschitz satisfies the required assumptions) or $F = -u + f$ (for the equation (W), and we need $u \in W^{1,p'}$, which is true if $p \geq 2$).

Exercice 4 (6 points)**.** Consider a sequence of functionals $F_n : X \to \mathbb{R}$, defined on a metric space (X, d), which is Γ-converging to a functional F. Suppose that on a subset $D \subset X$ the functionals F_n uniformly converge to the restriction to D of a functional $G : X \to \mathbb{R}$. For each of the following proposition say whether they are true or false, and explain why.

1. If $D = X$ then $F = G$.
2. If G is continuous and D is dense, then $F \leq G$.
3. If G is continuous and D is dense, then $F \geq G$.
4. If X is compact, D is dense, and F_n, G are continuous, then $\min_X F = \min_X G$.
5. If D is open, then $F = G$ on D.
6. If G is lower-semicontinuous, then $F = G$ on the interior of D.

Solution:

1. False. We need G lsc for it to be true. Consider $F_n = G$ a non-lsc function. Then F is not G but its lower semicontinuous envelop.
2. True. This is a consequence of the facI that the $\Gamma - \limsup$ inequality can be verifies on a set which is dense in energy.
3. False. Take a set D which is dense, but $X \setminus D$ is also dense (for instance the set of rational points). Take $G = 1$ and $F_n = 1$ on D, $F_n = 0$ on $X \setminus D$. The $\Gamma-$limit of F_n is then the lower semicontinuous envelop $F = 0$.
4. True. If X is compact, then $\min_X F_n \to \min_X F$. But we also have $\min_X F_n = \inf_D F_n \to \inf_D G = \min_X G$ (we used the fact tha, on any set, the uniform convergence is enough to pass to the limit the value of the inf).
5. False. We need G lsc for it to be true. Again, take $F_n = G$ a non-lsc function, and $D = X$.
6. True. Take a point x in the interior of D and a sequence $x_n \to x$. Then we have $x_n \in D$ as well for $n \geq n_0$ and $F_n(x_n) \geq G(x_n) - ||F_n - G||_{\infty,D}$. Using the lower semicontinuity of G to obaint $\liminf F_n(x_n) \geq G(x)$. This proves that the $\Gamma - \liminf$ is larger than G. But the $\Gamma - \limsup$, using the constant sequence $x_n = x$, is smaller than G. Hence, $F = G$ on the interior of D.