ENS Lyon and Université Claude Bernard Lyon 1, M2A
December 19th, 2025

Calculus of Variations and Elliptic PDEs

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Final Exam

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3h duration. All kind of documents (notes, books...) are authorized, but not communication devices. The total number of points is much larger than 20, so attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (7 points)**.** Given a function $f \in L^1([0,1])$, consider the optimization problem

$$\min\left\{\int_0^1 e^{-t}\left(\frac{|u'(t)|^2}{2}+f(t)u(t)\right)dt\ :\ u\in C^1([0,1])\right\}.$$

Find a necessary and sufficient condition on f so that the problem above admits a solution. Find this solution in the case $f(t)=t-a$ for the only value of a which allows to satisfy such a condition.

Exercice 2 (8 points)**.** Given a bounded and smooth domain $\Omega\subset\mathbb{R}^d$ and a constant $m\in\mathbb{R}$ consider the problem

$$\min\left\{\int_\Omega\left(|\nabla v|^2+v|\nabla u|^2+vu^2\right)dx\ :\ v,u\in H^1(\Omega), v-1,u-1\in H^1_0(\Omega), v\geq m\right\}.$$

1. Prove that for the problem admits a solution if $m>0$ and does not admit solutions if $m<0$. What is the difficulty for the case $m=0$?
2. When a solution exists, prove that it can be taken such that we have $0\leq u\leq 1$ and $m\leq v\leq 1$.
3. Prove that the minimizers (u,v) satisfy distributionally inside Ω

$$\begin{cases}\nabla\cdot(v\nabla u)=vu,\\ -\Delta v+\frac{1}{2}(|\nabla u|^2+u^2)\geq 0,\end{cases}$$

   but also satisfy the equality

$$\int_\Omega (v-m)(|\nabla u|^2+u^2)\eta dx+2\int_\Omega\nabla v\cdot\nabla[(v-m)\eta]dx=0$$

   for every smooth function $\eta\geq 0$ such that the support of η is compactly contained in the open set Ω.
4. If $m>0$ prove that for any optimal (u,v) the function u is locally Hölder continuous inside Ω.

Look at the back for next two exercises

Exercice 3 (10 points)**.** On the torus $\mathbb{T}^d$ (in dimension $d \geq 2$), given a Lipschitz continuous function $f : \mathbb{T}^d \to \mathbb{R}$ and an exponent $p > 1$, consider the two nonlinear elliptic PDEs

$$(\mathrm{Q}) \quad \Delta_p u = u + f, \qquad\qquad (\mathrm{W}) \quad \Delta_p u = -u + u^3 + f.$$

1. Prove by variational methods that both of them admit a weak solution in $W^{1,p}$.
2. Prove that the solution of (Q) is unique, while (W) could have several solutions.
3. Prove that any $W^{1,p}$ solution u of (Q) satisfies $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ as soon as $p \geq 2$.
4. Prove that any $W^{1,p}$ solution u of (W) satisfies $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ as soon as
$$p \geq 4\frac{d}{d+2}.$$
5. Prove that any $W^{1,p}$ solution of a PDE of the form $\Delta_p u = g(u) + f$ for g nondecreasing satisfies $|\nabla u|^{\frac{p}{2}-1}\nabla u \in H^1$ as soon as $f \in W^{1,p'}$ and deduce that we can obtain the same answer to the two previous questions even if we remove the assumption $p \geq 2$ in the case of (Q) and we only assume $p \geq 2$ in the case of (W).

Exercice 4 (6 points)**.** Consider a sequence of functionals $F_n : X \to \mathbb{R}$, defined on a metric space (X, d), which is Γ-converging to a functional F. Suppose that on a subset $D \subset X$ the functionals F_n uniformly converge to the restriction to D of a functional $G : X \to \mathbb{R}$. For each of the following proposition say whether they are true or false, and explain why.

1. If $D = X$ then $F = G$.
2. If G is continuous and D is dense, then $F \leq G$.
3. If G is continuous and D is dense, then $F \geq G$.
4. If X is compact, D is dense, and F_n, G are continuous, then $\min_X F = \min_X G$.
5. If D is open, then $F = G$ on D.
6. If G is lower-semicontinuous, then $F = G$ on the interior of D.