ENS Lyon and Université Claude Bernard Lyon 1, M2A
February 5th, 2026

Calculus of Variations and Elliptic PDEs

–

Resit examination

–

3h duration. All kind of documents (notes, books…) are authorized, but not communication devices. The total number of points is much larger than 20, so attacking only some exercises could be a reasonable option. The exercises are not necessarily ordered by difficulty.

Exercice 1 (7 points)**.** Given an integer n and two numbers $c_1, c_2 \in \mathbb{R}$ find the solution to the optimization problem

$$\min\left\{\int_1^2 \left(t^{n+1}\frac{|u'(t)|^2}{2} + t^{n-1}\frac{|u(t)|^2}{2}\right) dt \;:\; u \in C^1([0,1]),\ u(1) = c_1,\ u(2) = c_2\right\}.$$

Exercice 2 (8 points)**.** Let $h : \mathbb{R} \to \mathbb{R}$ be the function given by $h(s) = |s| + |s|^2 + \frac{1}{3}|s|^3$ and $H : \mathbb{R}^d \to \mathbb{R}$ be given by $H(v) = h(|v|)$. Compute h^* and H^* and find the dual of the optimization problem

$$\min\left\{\int_{\mathbb{T}^d} H(\mathbf{v}(x))dx \;:\; \nabla \cdot \mathbf{v} = f\right\}$$

where $\mathbb{T}^d$ is the d-dimensional torus, and $f \in W^{-1,3}(\mathbb{T}^d)$ is a scalar distribution with zero average. Prove that for every f an optimal $\mathbf{v}$ exists, is unique, and belongs to $H^1(\mathbb{T}^d)$ if $f \in W^{1,3}(\mathbb{T}^d)$.

Exercice 3 (5 points)**.** Given two functions $f, g \in L^1_{loc}(\Omega)$, let $u \in H^1_{loc}$ be a weak solution of the following PDE

$$\nabla \cdot ([2 + \arctan(f + u)]\nabla u) = \arctan(g + u).$$

Prove that u is locally $C^{0,\alpha}$ for some exponent $\alpha > 0$.
If moreover $f, g \in C^{k,\beta}$ prove that u is locally $C^{k+2,\beta}$.

Exercice 4 (8 points)**.** Let $L_N, L : \mathbb{R}^d \to \mathbb{R}$ be the functions defined via $L_N(z) := \sum_{k=0}^N \frac{|z|^{2k}}{k!}$ and $L(z) = e^{|z|^2}$. Let P be a polinomial in the variable s with x-dependent bounded coefficients, i.e. $P(x,s) := \sum_{k=0}^M a_k(x)s^k$ for some natural number M and some bounded functions $a_k : \Omega \to \mathbb{R}$, where Ω is a given bounded and smooth open subset of $\mathbb{R}^d$. Consider the functionals

$$F_N(u) := \int_\Omega (L_N(\nabla u(x)) - P(x, u(x)))\, dx, \quad F(u) := \int_\Omega (L(\nabla u(x)) - P(x, u(x)))\, dx,$$

defined for $u \in X := H^1_0(\Omega) \cap L^M(\Omega)$ (these functionals can take the value $+\infty$ according to the integrability of ∇u).

1. We endow X with the strong L^1 convergence. Prove that F_N Γ-converges to F when $N \to \infty$ according to this convergence.

2. Prove that each functional F_N admits a minimizer on X, at least for large N.

3. Prove that, up to a subsequence, any sequence u_N of minimizers of F_N converges in L^1 to a minimizer of F over X.

4. Prove that at least a solution u of $\min\{F(u) : u \in X\}$ satisfies

$$\int_\Omega |\nabla u(x)|^2 L(\nabla u(x))dx < +\infty$$