Minimal mass subsolutions to the Lane-Emden equation

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Motivation : the parabolic-elliptic Keller-Segel system

The Keller–Segel equation.
An aggregation-diffusion equation arising from mathematical biology **[KS]**, with many interesting mathematica features in connection with functional inequalities and optimal transport:

$$\begin{cases} \partial_t \rho = \Delta \rho^m - \operatorname{div}(\rho \nabla u) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ -\Delta u = \rho & \text{in } (0,\infty)\times\mathbb{R}^d. \end{cases}$$

We take $u = \Gamma * \rho$ where Γ is the fundamental solution, $-\Delta\Gamma = \delta_0$.
The critical exponent (object of this poster) is $m = 2 - \frac{2}{d}$ for $d \geq 2$.

The mass $M = \int_{\mathbb{R}^d} \rho$ is preserved in time and for M small enough there is global existence, while for large M there could be blow-up in finite time. For $d = 2$ the critical mass is $M_c = 8\pi$ and *all* solutions with $M > 8\pi$ and $\int_{\mathbb{R}^d} |x|^2 d\rho_0(x) < +\infty$ explode in finite time.

The density ρ undergoes diffusion but at the same time moves towards points where the chemoattractant u, produced by ρ itself, is maximal

Gradient flow interpretation. The system is the gradient flow in $W_2(\mathbb{R}^d)$ of the *free energy*

$$F(\rho) = \int_{\mathbb{R}^d} f(\rho(x))dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx,$$

where $f(\rho) = \rho\log\rho$ (if $d = 2$) or $f(\rho) = \frac{\rho^m}{m-1}$ (if $d > 2$). The critical mass M_c is the maximal mass such that F is lower bounded on bounded sets of $W_2(\mathbb{R}^d)$. For $d > 2$ this bound comes from (see **[BCL]**):

$$\int_{\mathbb{R}^d} |\nabla u|^2 = \int_{\mathbb{R}^d} \rho u \leq C_* \|\rho\|_{L^m}^m \|\rho\|_{L^1}^{2-m}.$$

The extremals in this inequality are radial functions $\bar\rho$ such that $\Delta \frac{m}{m-1}\bar\rho^{m-1} + \bar\rho = 0$ in $B(0,R)$, with $\bar\rho = 0$ outside $B(0,R)$.
This condition is invariant by mass-preserving scaling, when we replace $\bar\rho$ with $\bar\rho_\lambda(x) := \lambda^d \bar\rho(\lambda x)$.

Aronson–Bénilan estimates

The equation can be written as $\partial_t \rho = \nabla\cdot(\rho\nabla v)$ for $v = p - u$, where $p = f'(\rho)$ is the pressure

$$p = \begin{cases} \log\rho & \text{if } m = 1 \quad (d = 2), \\ \frac{m}{m-1}\rho^{m-1} & \text{if } m > 1 \quad (d > 2). \end{cases}$$

We want to consider Δv:

$$\Delta v = \tfrac{m}{m-1}\Delta\rho^{m-1} + \rho$$

and prove bounds of the form

$$\Delta v(t,\cdot) \geq -O\left(\frac{1}{t}\right)$$

analogously to **[AB]** estimates for **porous-medium, fast diffusion and heat equations** $\partial_t \rho = \Delta\rho^m \Rightarrow \Delta p \geq -\frac{1}{(m-1+\frac{2}{d})t}$.

$\delta(t) := \inf \Delta v(t,\cdot)$

Computations show that we have

$$\delta'(t) \geq \delta^2(t) - \text{remainders}.$$

In order to estimate the remainder, prove that it is smaller than δ^2, and deduce estimates on ρ from estimates on δ, we want to prove

$$\|\rho\|_{L^\infty} \leq C|\delta|.$$

We expect this for $M < M_c$.
Variant for $M = M_c$: for every compact set $K \subset \mathcal{P}(\mathbb{R}^d)$ not including Dirac masses

$$\exists C \,:\, \rho \in K \Rightarrow \|\rho\|_{L^\infty} \leq C(1 + |\delta|).$$

$\|\rho\|_{L^\infty} \leq C(M)|\delta|$

We fix the mass M and argue by **contradiction**.
☞ **Sequence**. $\frac{m}{m-1}\Delta\rho_n^{m-1} + \rho_n \geq \delta_n$ and $\|\rho_n\|_{L^\infty} \geq n|\delta_n|$.
☞ **Scaling**. Consider $\eta_n(x) = \lambda_n^d \rho_n(\lambda_n x)$ so that $\int_{\mathbb{R}^d} \eta_n = M$ and choose λ_n in order to have $\|\eta_n\|_{L^\infty} = 1$ (hence $|\delta_n|\lambda_n^d \leq \frac{1}{n}$). We then have

$$\frac{m}{m-1}\Delta\eta_n^{m-1} + \eta_n = \lambda_n^d\left(\frac{m}{m-1}\Delta\rho_n(\lambda_n\cdot)^{m-1} + \rho_n(\lambda_n\cdot)\right) \geq \delta_n\lambda_n^d \to 0.$$

☞ **Compactness**. Up to translations, the sequence η_n converges to $\eta \not\equiv 0$ such that

$$\frac{m}{m-1}\Delta(\eta^{m-1}) + \eta \geq 0 \quad \text{where} \quad \int \eta \leq M.$$

We then obtain a **subsolution of a Lane–Emden** equation with mass $\leq M$. If we can exclude the existence of subsolutions with this mass we have the L^∞ bound (note that global solutions to this equation with this choice of exponents do not exist **[GS]**, but $\bar\rho$ is a subsolution).

Subsolutions of the Lane–Emden equation

- **Problem.** Find the minimal mass

$$M_c^* = \inf\left\{\int_{\mathbb{R}^d}\rho \,:\, \frac{m}{m-1}\Delta(\rho^{m-1}) + \rho \geq 0, \rho \geq 0, \rho \not\equiv 0\right\}$$

- Change of variables $h := c\rho^{m-1}$, $q = \frac{d}{d-2}$:
- What is the **minimal L^q norm** of subsolutions of this Lane–Emden equation?

$$\inf\left\{\int_{\mathbb{R}^d} h^q : h \in \mathcal{S}\setminus\{0\}\right\},\ \mathcal{S} = \{h \in L^\infty, h \geq 0 : \Delta h + h^q \geq 0\}$$

- Can we prove $M_c^* = M_c$? this means that the optimal h would be the radial function $\bar h$ solving $\Delta\bar h + \bar h^q = 0$ on its support.

Existence and positivity of the mass

- The existence of an optimal h (or ρ) can be obtained via variational arguments, after scaling a minimizing sequence h_n so that $h_n(0) = \max h_n = 1$. The condition $\Delta h_n \geq -1$ provides both the desired compactness and the existence of $\max h_n$.
- Lower bounds on the mass: testing the equation with h gives $\|\nabla h\|_{L^2}^2 \leq \|h\|_{L^{q+1}}^{q+1}$. Then use a **Gagliardo-Nirenberg inequality**:

$$\|h\|_{L^{q+1}} \leq C_{GN}\|\nabla h\|_{L^2}^{\frac{d-2}{d-1}}\|h\|_{L^q}^{\frac{1}{d-1}} \leq C_{GN}\|h\|_{L^{q+1}}\|h\|_{L^q}^{\frac{1}{d-1}}.$$

Inspiration from the case $d = 2$

In dimension $d = 2$ the question is the minimal possible mass when $\Delta\log\rho + \rho \geq 0$. Set $\Omega_t = \{\rho > t\}$ and $g(t) = \int_{\Omega_t}\rho$. We use $\frac{1}{t}\int_{\partial\Omega_t}|\nabla\rho| = \int_{\Omega_t} -\Delta\log\rho \leq g(t)$, then $-g'(t) = t\int_{\partial\Omega_t}\frac{1}{|\nabla\rho|}$ and the **isoperimetric inequality**, so that we obtain:

$$4\pi|\Omega_t| \leq \operatorname{Per}(\Omega_t)^2 \leq \int_{\partial\Omega_t}\frac{1}{|\nabla\rho|}\int_{\partial\Omega_t}|\nabla\rho| \leq -g'(t)g(t).$$

Since $M = \int\rho = g(0) = \int|\Omega_t|dt$, integrating gives $M \geq 8\pi$.
☞ We only used the positivity of $\int_{\Omega_t}\Delta\log\rho + \rho$ for every t, and inequalities which are sharp for radial functions.
☞ Also for $d > 2$ we could do the same and extend the problem to a new class $\mathcal{S}'$.

$$\mathcal{S}' = \left\{h \in L^\infty(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)\cap H^1_{loc}(\mathbb{R}^d),\ h \geq 0,\ \int_{\{h>t\}}\Delta h + h^q \geq 0 \text{ for every } t\right\}.$$

Moving to radial functions

- We want now to solve $\inf\{\int_{\mathbb{R}^d} h^q : h \in \mathcal{S}'\setminus\{0\}\}$.
- Belonging to the class $\mathcal{S}'$ can be characterized via $\int\nabla h\cdot\nabla\phi \leq \int h^q\phi$ for every $\phi \geq 0$ which is a non-decreasing function of h, and as such this condition is invariant by radial rearrangement (**Pólya-Szegő inequality**).
- After scaling so that $h(0) = \max h = 1$, the problem becomes

$$\min\left\{\int_0^{R_0} r^{d-1}f(r)^q dr \,:\, f(0) = 1,\ f \text{ is nonincreasing and } x \mapsto f(|x|) \text{ belongs to } \mathcal{S}'\right\}.$$

For radially decreasing functions f, the level sets are centered balls. Hence,

"$x \mapsto f(|x|)$ in $\mathcal{S}'$" means "$R^{d-1}f'(R) \geq -M(R)$ for $M(R) := \int_0^R r^{d-1}f(r)^q dr$".

An optimal control problem

Problem. We consider trajectories $(f, M) : [0, R_0] \to \mathbb{R}^2$ satisfying

$$\begin{cases} r^{d-1}f'(r) = -M(r) + \alpha(r), \\ M'(r) = r^{d-1}f^q(r), \\ f(0) = 1, \quad M(0) = 0, \end{cases}$$

where $\alpha : [0, R_0] \to \mathbb{R}_+$ is a measurable control, subject to the state-dependent constraint $\alpha(r) \in [0, M(r)]$.
➣ We want to minimize $M(R_0)$, where R_0 is the final horizon.
➣ The condition $\alpha \leq M$ imposes $f' \leq 0$ and provides suitable bounds to obtain existence of a minimizer.
➣ The control α does not represent of $\Delta h + h^q$, but only of its cumulated mass.
➣ We should a priori take $R_0 = +\infty$, but we will a posteriori prove that for R_0 large enough the optimal f will be compactly supported, thus making the choice of large R_0 irrelevant.

The Pontryagin's Principle

There exists a pair (p_1, p_2) of dual variables such that

$$\begin{cases} p_1'(r) = -qp_2(r)r^{d-1}f_+(r)^{q-1}, & p_1(R_0) = 0, \\ p_2'(r) = -(p_1(r))_- r^{1-d}, & p_2(R_0) = -1, \end{cases}$$

and the optimality implies

$$p_1 > 0 \Rightarrow \alpha = M; \quad p_1 < 0 \Rightarrow \alpha = 0.$$

Analyzing the above ODE system we obtain the existence of $R_1 \leq R_0$ and $\gamma \in [0, R_1)$ such that $p_1 > 0$ on $(0,\gamma)$, then $p_1 < 0$ on (γ, R_1), then $f = 0$ on $[R_1, R_0]$.

Hence, the optimal f is of the form f_γ, for $\gamma \geq 0$, given by $f_\gamma = 1$ on a *plateau* $[0,\gamma]$ and then

$$f_\gamma''(r) + \frac{d-1}{r}f_\gamma'(r) + f^q(r) = 0, \quad f_\gamma(\gamma) = 1,\ f_\gamma'(\gamma) = -\frac{\gamma}{d},$$

on an interval $(\gamma, R(\gamma))$ defined by $f_\gamma(R(\gamma)) = 0$.

We want to prove that $\gamma = 0$ minimizes the function

$$\gamma \mapsto \mathrm{M}(\gamma) := \int_0^{R(\gamma)} r^{d-1}f_\gamma^q(r)dr.$$

Optimal length of the plateau

Consider $w = \frac{\partial f_\gamma}{\partial\gamma}$ on $(\gamma, R(\gamma))$, which solves

$$w''(r) + \frac{d-1}{r}w'(r) + qf^{q-1}(r)w(r) = 0 \quad w(\gamma) = \frac{\gamma}{d},\ w'(\gamma) = 0.$$

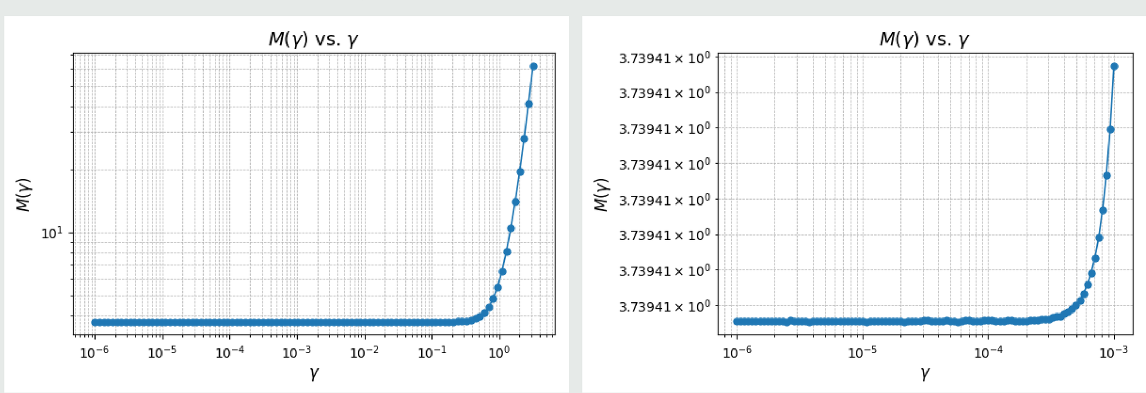
Moreover, we have

$$\mathrm{M}'(\gamma) = -R(\gamma)^{d-1}w'(R(\gamma)).$$

At the optimal γ we have Neumann b.c. on both sides and

$$\int r^{d-1}|w'(r)|^2 dr = q\int r^{d-1}f^{q-1}(r)|w(r)|^2 dr.$$

The density $r \mapsto r^{d-1}f^{q-1}(r)$ is log-concave and a **Brascamp-Lieb-type inequality** proves, when $d = q = 3$, that the only case of equality above is $w = 0$, hence $\gamma = 0$. For $d = 3$ we only have numerics on $\mathrm{M}(\gamma)$



Plot of $\mathrm{M}(\gamma)$ w.r.t. $\gamma \in [10^{-6}, 1]$ in $d = 4$ and zooming on $\gamma \in [10^{-6}, 10^{-3}]$.The graph seems increasing.

Conclusions and open problems

➣ For $d = 2$ the minimal mass of the subsolutions of the Liouville equation $\Delta h + e^h \geq 0$ is $\int\rho = \int e^h = 8\pi$.
➣ For $d = 3$ the minimal mass of the subsolutions of the Lane-Emden equation $\Delta h + h^3 \geq 0$ is realized by the radial function $\bar h$ which solves $\Delta\bar h + \bar h^3 = 0$ in the ball. Both for $d = 2$ and $d = 3$ we have $M_c^* = M_c$.
➣ For $d > 3$ the question is open, and the minimal mass in $\mathcal{S}'$ could a priori be attained by a function $f_\gamma, \gamma > 0$, which does not belong to $\mathcal{S}$.
➣ Numerics suggest on the other hand that the optimal γ is 0, i.e. $M_c^* = M_c$ for higher d as well.

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