Introduction to scheme theory

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The parts of the text written in tiny are complementary material, not discussed during the lectures.

1 Introduction: the language of algebraic geometry

1.1 Systems of polynomial equations

Let k be a field, for instance $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{F}_p$, $\mathbb{Q}(T)$, etc. The purpose of algebraic geometry is to study the sets of solutions of systems of the form

$$\begin{cases} P_1(X_1, \dots, X_n) = 0 \\ \vdots \\ P_r(X_1, \dots, X_n) = 0 \end{cases} \tag{1}$$

where $P_i \in k[X_1, \dots, X_n]$. The case $n = 1$ suggests considering first the case where k is algebraically closed. So we assume from now on in this section that k is algebraically closed. Actually it will be more convenient to encode the system (1) by an ideal of $k[X_1, \dots, X_n]$. Let I be the ideal of $k[X_1, \dots, X_n]$ generated by the polynomials $P_1, \dots, P_r$. That is

$$I = (P_1, \dots, P_r) \coloneqq \{Q_1P_1 + \dots + Q_rP_r \mid Q_1, \dots, Q_r \in k[X_1, \dots, X_n]\}.$$

Let

$$V(I) \coloneqq \{(x_1, \dots, x_n) \in k^n \mid \forall P \in I,\ P(x_1, \dots, x_n) = 0\}.$$

Then $V(I)$ is also the set of solutions of the system (1). The following definitions are then certainly useful for our problem.

Definition 1.1. *An* algebraic subset *of k^n is a subset of the form $V(I)$ where I is an ideal of $k[X_1, \dots, X_n]$. An algebraic subset of k^n is also called an* affine algebraic variety*.*

Conversely, if $W \subset k^n$ is a subset, we define

$$I(W) \coloneqq \{P \in k[X_1, \dots, X_n] \mid \forall x \in W,\ P(x) = 0\}.$$

We can easily check that $I(W)$ is an ideal of $k[X_1, \dots, X_n]$ and, if $W \subset k^n$ is an algebraic subset, we have

$$W = V(I(W)).$$

Exercice 1.1. Check that, if $W \subset k^n$ is an algebraic subset, then $W = V(I(W))$.

In the other direction, if I is an ideal of $k[X_1, \dots, X_n]$, we have $I \subset I(V(I))$.

Exercice 1.2. Check that if I is an ideal of $k[X_1, \dots, X_n]$ then $I \subset I(V(I))$. Give an example where $I \subsetneq I(V(I))$.

The case of equality is governed by Hilbert's Nullstellensatz.

Theorem 1.2. *Let I be an ideal of $k[X_1, \dots, X_n]$. Then we have*

$$I(V(I)) = \sqrt{I} = \{P \in k[X_1, \dots, X_n] \mid \exists m \geqslant 1,\ P^m \in I\}.$$

Corollary 1.3. *The map which sends an algebraic subset $V \subset k^n$ to the ideal $I(V)$ induces a bijection from the set of algebraic subsets of k^n onto the set of radical ideals of $k[X_1, \dots, X_n]$. The inverse map is given by $I \mapsto V(I)$.*

If V is an algebraic subset of k^n, we say that a map $f : V \to k$ is *polynomial* if there exists $P \in k[X_1, \dots, X_n]$ such that

$$\forall x \in V, \quad f(x) = P(x).$$

Let $k[V]$ be the k-algebra of all polynomial maps from V to k. By definition the map $k[X_1, \dots, X_n] \to k[V]$ sending P to $P_{|V}$ is surjective. Its kernel is $I(V)$ so that we have an isomorphism of k-algebras

$$k[X_1, \dots, X_n]/I(V) \xrightarrow{\sim} k[V].$$

1.2 Morphisms between affine algebraic varieties

Let $V \subset k^n$ and $W \subset k^m$ be two affine algebraic varieties. We define a *morphism* $f : V \to W$ as a collection of polynomial functions $f = (f_1, \dots, f_m)$ such that $f(V) \subset W$. By definition each f_i is the restriction to V of a polynomial $P_i \in k[X_1, \dots, X_n]$. If $h \in k[W]$ is a polynomial function on W, then $h \circ f$ is a polynomial function on V. Namely if $h = F_{|W}$ for $F \in k[X_1, \dots, X_m]$, then $h \circ f$ is given by the polynomial

$$F(P_1(X), \dots, P_m(X)) \in k[X_1, \dots, X_n].$$

We denote by $f^* : k[W] \to k[V]$ the map sending h to $h \circ f$. It is easy to check that f^* is a morphism of k-algebras.

Exercice 1.3. Let $W \subset k^m$ be an algebraic subset and let $f : V \to k^m$ be a polynomial map given by $f = (f_1, \dots, f_m)$ with $f_i = P_{i|V}$. Let $\varphi : k[X_1, \dots, X_m] \to k[X_1, \dots, X_n]$ be the morphism of k-algebras sending F to $F(P_1, \dots, P_m)$. Check that $f(V) \subset W$ if and only if $\varphi(I(W)) \subset I(V)$, so that the map φ factors through a map $k[X_1, \dots, X_m]/I(W) \to k[X_1, \dots, X_n]/I(V)$. Check that it coincides with f^*.

We have the following result which we give without proof as it will be generalized later.

Proposition 1.4. *The construction $f \mapsto f^*$ induces a bijection from the set* $\mathrm{Hom}(V, W)$ *of morphisms from V to W to the set* $\mathrm{Hom}_{k\text{-alg}}(k[W], k[V])$ *of morphisms of k-algebras from $k[W]$ to $k[V]$.*

Exercice 1.4. Prove Proposition 1.4.

Let $\mathcal{A}ff$ be the category whose objects are affine algebraic varieties and whose morphisms are polynomial maps between them. Let $\mathcal{A}lg$ be the category of reduced k-algebras of finite type, that is the k-algebras which are isomorphic to a quotient $k[X_1, \dots, X_n]/I$ for some radical ideal I. Proposition 1.4 can be reformulated as follows.

Corollary 1.5. *The functor $V \mapsto k[V]$ induces a contravariant equivalence of categories between $\mathcal{A}ff$ and $\mathcal{A}lg$.*

Exercice 1.5. Prove Corollary 1.5. Use Proposition 1.4 to prove that the functor is fully faithful and prove directly that it is essentially surjective.

Let $x = (x_1, \ldots, x_n) \in V$. Then the set $\{x\}$ is algebraic as $\{x\} = V(\mathfrak{m}_x)$ where $\mathfrak{m}_x$ is the maximal ideal $(X_1 - x_1, \ldots, X_n - x_n)$ in $k[X_1, \ldots, X_n]$. Hilbert's Nullstellensatz says that all maximal ideals of $k[X_1, \ldots, X_n]$ are of this form for some $(x_1, \ldots, x_n) \in k^n$. In particular there is a bijection between k^n and the set of maximal ideals of $k[X_1, \ldots, X_n]$. Note that $x \in V$ if and only if $I(V) \subset \mathfrak{m}_x$. Therefore this bijection restricts to a bijection between V and the set of maximal ideals of $k[X_1, \ldots, X_n]$ containing $I(V)$, or equivalently a bijection between V and the set of maximal ideals in $k[V]$.

If $f : V \to W$ is a polynomial map, is there a way to translate the value $f(x)$ in terms of f^*? The question is equivalent to finding the elements of $k[X_1, \ldots, X_m]$ vanishing on $f(x)$. For $P \in k[X_1, \ldots, X_m]$, we have

$$P(f(x)) = 0 \Leftrightarrow f^*(P)(x) = 0 \Leftrightarrow f^*(P) \in I(\{x\}) = \mathfrak{m}_x.$$

Therefore we have

$$\mathfrak{m}_{f(x)} = I(\{f(x)\}) = (f^*)^{-1}(\mathfrak{m}_x).$$

This implies two things:

- we can characterize the effect of f purely in terms of the morphism of k-algebras f^*;
- if $\varphi : A \to B$ is a morphism of reduced k-algebras of finite type, then for every maximal ideal $\mathfrak{m} \subset B$, $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal of A.

Therefore we have completely reduced the study of affine algebraic varieties to the study of reduced k-algebras of finite type, the points corresponding to maximal ideals.

1.3 Algebraic varieties

In order to motivate the following definitions, let us recall that a differentiable manifold is a topological space with a collection of "charts" on which we define the differentiable maps, giving rise to the notion of diffeomorphism, so that the transition maps between the charts are diffeomorphisms. The charts are usually modelled on open subsets of $\mathbb{R}^n$.

In algebraic geometry the charts are modelled on algebraic subsets, which are also called *affine varieties*. In order to continue the analogy with differential geometry we need to introduce some topology. As the algebraic subsets are defined by "closed conditions" (being zeros of polynomials), it is expected that they should be closed subsets. Actually the following result shows that they are exactly the closed subsets of a topology. The result will be generalized later so we state it without proof.

Proposition 1.6. *On k^n the set of all subsets of the form $k^n \setminus V(I)$, for I an ideal of $k[X_1, \ldots, X_n]$, is the set of open subsets for a topology on k^n.*

The topology on k^n whose open subsets are the complements of algebraic subsets is called the *Zariski topology*.

Example 1.7. If $f \in k[X_1, \ldots, X_n]$, the subset

$$D(f) := \{x \in k^n \mid f(x) \neq 0\}$$

is open for the Zariski topology: this is the complement of $V(f)$.

Not all open subsets of k^n are of the form $D(f)$, but these open subsets form a base of the topology: an open subset of k^n is a union of open subsets of the form $D(f)$.

Exercice 1.6. Let $U \subset k^n$ be an open subset for the Zariski topology. Let $x \in U$. Prove that there exists some $f \in k[X_1, \ldots, X_n]$ such that $x \in D(f) \subset U$.

Now we can define the analogue of differentiable maps in our context.

Definition 1.8. *Let $f \in k[X_1, \ldots, X_n]$. A map $h : D(f) \to k$ is called* regular *if there exists $P \in k[X_1, \ldots, X_n]$ and an integer $m \geqslant 0$ such that*

$$\forall x \in D(f), \quad h(x) = \frac{P(x)}{f(x)^m}.$$

Let $U \subset k^n$ be an open subset. A map $h : U \to k$ is called regular *if for all $x \in U$ there exists $g \in k[X_1, \ldots, X_n]$ with $x \in D(g) \subset U$ such that the restriction of h to $D(g)$ is regular.*

Finally if $V \subset k^n$ is an affine variety and $U \subset V$ is an open subset of V, a map $h : U \to k$ is called regular *if it is, locally on U, the restriction of a regular map on an open subset of k^n.*

Remark 1.9. It could appear that our two definitions of regularity on an open subset of the form $D(f)$ are not equivalent. Actually they are: a map on $D(f)$ which is locally regular is regular. This is a nice exercise, but it will be developed later in a more general context.

Now we can define an isomorphism between two open subsets of affine varieties $U \subset V \subset k^n$ and $U' \subset V' \subset k^m$. This is a bijective map $F : U \to U'$ of the form $F = (F_1, \ldots, F_m)$ such that the F_i are regular and such that F^{-1} has the same properties. With this notion of isomorphism, it is possible to define an algebraic variety as a topological space X with a covering by open subsets $(U_i)_{i \in I}$ and homeomorphisms $h_i : U_i \xrightarrow{\sim} W_i$ onto open subsets $W_i \subset V_i \subset k^{n_i}$ of affine varieties such that for all i, j, the map $h_i(U_i \cap U_j) \xrightarrow{\sim} h_j(U_i \cap U_j)$ given by $h_j \circ h_i^{-1}$ is an isomorphism.

This is roughly the starting point of algebraic geometry.

1.4 Why schemes?

We could continue in this way but the purpose of this course is the study of scheme theory, which is more general. Why do we need this extra generality? Here are some reasons.

- We would like to consider fields which are not algebraically closed. This can be achieved by forgetting points altogether and replacing them by maximal ideals. Namely the polynomial $X^2 + Y^2 + 1$ has no zeros in $\mathbb{R}^2$, but the $\mathbb{R}$-algebra $\mathbb{R}[X,Y]/(X^2+Y^2+1)$ has maximal ideals.

- We would also like to do some intersection theory. Let $P_1, P_2 \in k[X,Y]$ be two irreducible polynomials which are linearly independent. They define algebraic curves $V(P_1)$ and $V(P_2)$ in k^2. They intersect properly and a famous theorem of Bézout predicts that, in the projective plane and *counted with multiplicities*, they intersect in $(\deg P_1)(\deg P_2)$ points. What are these multiplicities? Let us take an example with $P_1 = Y - X^2$ and $P_2 = Y$. The intersection of $V(P_1)$ and $V(P_2)$ is the vanishing locus of the ideal $I = (P_1) + (P_2) = (Y - X^2, Y) = (Y, X^2)$. Then the corresponding algebra is $k[X,Y]/I \simeq k[X]/(X^2)$. It is not reduced, which means that the ideal I is not radical, and it has dimension 2 over k. This reflects the fact that $V(P_1)$ and $V(P_2)$ intersect with multiplicity 2 at the point $(0,0)$. This suggests that in order to account for intersection multiplicities, it is reasonable to work not only with radical ideals but also with non radical ideals.

- We would also like to consider equations with coefficients in rings, not only in fields. This can be very useful in arithmetic. Let $f : \mathbb{Z} \to \mathbb{Q}$ be the inclusion. The inverse image of the maximal ideal (0) of $\mathbb{Q}$ is the ideal (0) of $\mathbb{Z}$, which is prime but not maximal. Therefore the pullback of a maximal ideal by a morphism of rings is not always a maximal ideal. This is linked to the fact that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-algebra, but it means that in order to work with general rings, working with maximal ideals won't be enough.

All these considerations lead us to the more general notion of *scheme*.

2 Affine schemes

2.1 The spectrum of a commutative ring

All rings are assumed to be commutative with unit.

Definition 2.1. *Let A be a ring. The* spectrum *of A, denoted* $\mathrm{Spec}(A)$*, is the set of prime ideals of A.*

Let $f : A \to B$ be a morphism of commutative rings. If $\mathfrak{p} \in \mathrm{Spec}\, B$, then $f^{-1}(\mathfrak{p})$ is an ideal of A and f induces a map $A/f^{-1}(\mathfrak{p}) \to B/\mathfrak{p}$ which is injective by construction. As $B/\mathfrak{p}$ is a domain, the quotient $A/f^{-1}(\mathfrak{p})$ is one too and thus $f^{-1}(\mathfrak{p})$ is a prime ideal of A. We have therefore defined a map ${}^a f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$.

Example 2.2. Let A be a commutative ring and let $I \subset A$ be an ideal. Let $\pi_I : A \to A/I$ be the quotient map. Then ${}^a\pi_I : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is injective and its image is the set of $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $I \subset \mathfrak{p}$.

Example 2.3. Let k be a field. Then k has a unique prime ideal, namely (0). Therefore $\mathrm{Spec}(k)$ is a point.

Example 2.4. Let k be a field and let $A = k[X]$. All ideals of A are principal. The prime ideals of A are either (0) or (P) where P is an irreducible polynomial. The maximal ideals are the ones of the form (P) for P irreducible.

Example 2.5. The prime ideals of $\mathbb{Z}$ are (0) and (p) for p a prime number. The maximal ideals are exactly the prime ideals of the form (p) with p prime.

Example 2.6. Let k be a field and let $A = k[X]/(X^2)$. The prime ideals of A are in bijection with the prime ideals of $k[X]$ containing (X^2), that is with the prime ideals containing (X); thus A has only one prime ideal, the image of (X). Therefore $\mathrm{Spec}(A)$ is a point, even if A is not a field. Note that here A is not reduced.

If $\mathfrak{p} \in \mathrm{Spec}(A)$, the ring $A/\mathfrak{p}$ is a domain. Let $k(\mathfrak{p})$ be the fraction field of $A/\mathfrak{p}$. This is a field called the *residue field of A at $\mathfrak{p}$*. The natural map $A \to k(\mathfrak{p})$ induces a map $\mathrm{Spec}(k(\mathfrak{p})) \to \mathrm{Spec}(A)$ whose image is exactly $\{\mathfrak{p}\}$. If $f \in A$, then f is to be considered as a function on $\mathrm{Spec}(A)$. The image of f in $k(\mathfrak{p})$ is denoted $f(\mathfrak{p})$ and is called the *value of f at $\mathfrak{p}$*. Note that $f(\mathfrak{p}) = 0$ if and only if $f \in \mathfrak{p}$.

Example 2.7. Let $A = k[X]$ for k a field and let $\mathfrak{p}$ be the maximal ideal $(X - a)$ for some $a \in k$. Then $k(\mathfrak{p}) \simeq A/\mathfrak{p} \simeq k$. If $P \in A$, the image of P in $k(\mathfrak{p})$ is exactly $P(a)$ so that we have $P(\mathfrak{p}) = P(a)$ and the notation is consistent.

2.2 Zariski topology

If S is a subset of A we denote

$$V(S) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid S \subset \mathfrak{p}\}.$$

The subset $V(S)$ is the subset of $\mathrm{Spec}(A)$ where the elements of S vanish. Usually we replace S by the ideal I generated by S, we then have $V(S) = V(I)$.

Proposition 2.8. *The set of all subsets of* $\mathrm{Spec}(A)$ *of the form* $V(I)$*, for* $I \subset A$ *an ideal, is the set of closed subsets of a topology on* $\mathrm{Spec}(A)$*.*

Proof. We have to check the following facts.

- For $I = (0)$, we have $V(I) = \mathrm{Spec}(A)$ and for $I = A$, we have $V(I) = \emptyset$.
- Let (I_i) be a family of ideals of A. Then we have $V(\sum_i I_i) = \bigcap_i V(I_i)$.
- Let I_1 and I_2 be two ideals in A. As $I_1 I_2 \subset I_1 \cap I_2 \subset I_1$, we have
$$V(I_1) \subset V(I_1 \cap I_2) \subset V(I_1 I_2)$$
and similarly
$$V(I_2) \subset V(I_1 \cap I_2) \subset V(I_1 I_2)$$
so that
$$V(I_1) \cup V(I_2) \subset V(I_1 \cap I_2) \subset V(I_1 I_2).$$
Conversely assume that $\mathfrak{p} \notin V(I_1) \cup V(I_2)$. Then there exist $f_1 \in I_1$ and $f_2 \in I_2$ such that $f_1 \notin \mathfrak{p}$ and $f_2 \notin \mathfrak{p}$. As $\mathfrak{p}$ is a prime ideal in A, we have $f_1 f_2 \notin \mathfrak{p}$ so that $\mathfrak{p} \notin V(I_1 I_2)$. Therefore we have
$$V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2).$$ □

Remark 2.9. In the proof of Proposition 2.8 we have proved the following useful facts:

- for a family of ideals (I_i), $\bigcap_i V(I_i) = V\left(\sum_i I_i\right)$;
- for finitely many ideals $I_1, \ldots, I_r$, we have
$$V(I_1) \cup \cdots \cup V(I_r) = V(I_1 \cdots I_r).$$

Definition 2.10. *The topology on* $\mathrm{Spec}(A)$ *whose closed subsets are the* $V(I)$ *with* I *an ideal of* A *is called the Zariski topology on* $\mathrm{Spec}(A)$*.*

Let $f : A \to B$ be a ring homomorphism. Then the map ${}^a f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is continuous for the Zariski topology. Namely we have, for I an ideal of A,

$$\begin{aligned}({}^a f)^{-1}(V(I)) &= \{\mathfrak{p} \in \mathrm{Spec}(B) \mid I \subset f^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(B) \mid f(I) \subset \mathfrak{p}\} \\ &= V(f(I)) = V(Bf(I)).\end{aligned}$$

Note that $f(I)$ is not an ideal of B in general but we have $V(f(I)) = V(Bf(I))$ where $Bf(I)$ is the ideal of B generated by $f(I)$.

Remark 2.11. If $f : A \to B$ and $g : B \to C$ are ring homomorphisms, we have ${}^a(g \circ f) = {}^a f \circ {}^a g$. In categorical terms, this means that $A \mapsto \mathrm{Spec}(A)$ is a contravariant functor from the category of rings to the category of topological spaces.

Proposition 2.12. *1. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then the topological closure of $\{\mathfrak{p}\}$ in $\mathrm{Spec}(A)$ is the closed subset $V(\mathfrak{p})$.*

2. The closed points of $\mathrm{Spec}(A)$ are exactly the maximal ideals of A.

Proof. Let us prove 1. Let $I \subset A$ be an ideal such that $V(I) = \overline{\{\mathfrak{p}\}}$. As $\mathfrak{p} \in V(I)$, we have $I \subset \mathfrak{p}$, hence $V(\mathfrak{p}) \subset V(I)$. On the other hand $V(\mathfrak{p})$ is a closed subset containing $\mathfrak{p}$, so that $V(I) = \overline{\{\mathfrak{p}\}} \subset V(\mathfrak{p})$. We conclude that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is the smallest closed subset containing $\mathfrak{p}$. This proves 1. Now $\{\mathfrak{p}\}$ is closed in $\mathrm{Spec}(A)$ if and only if $V(\mathfrak{p}) = \{\mathfrak{p}\}$, i.e. if and only if $\mathfrak{p}$ is maximal. This proves 2. $\square$

Example 2.13. Let k be a field. If k is algebraically closed, then

$$\mathrm{Spec}(k[X]) = \{(X-a) \mid a \in k\} \cup \{(0)\}.$$

The closed points of $\mathrm{Spec}(k[X])$ are therefore in bijection with the points of k. The point (0) is not closed and is dense in $\mathrm{Spec}(k[X])$, since $\overline{\{(0)\}} = V((0)) = \mathrm{Spec}(k[X])$. If k is not assumed to be algebraically closed, the points of $\mathrm{Spec}(k[X])$ are (0) and the (P), where P runs through the monic irreducible polynomials of $k[X]$. If $\overline{k}$ denotes an algebraic closure of k, the closed points of $\mathrm{Spec}(k[X])$ are then in bijection with the orbits of the group $\mathrm{Aut}(\overline{k}/k)$ acting on $\overline{k}$ (when k is perfect, this group is the Galois group $\mathrm{Gal}(\overline{k}/k)$). As every ideal of $k[X]$ is principal, the closed subsets of $\mathrm{Spec}(k[X])$ are therefore $\mathrm{Spec}(k[X])$ together with the subsets consisting of finitely many maximal ideals, and the open subsets are therefore the empty set together with the complements of finite sets of maximal ideals.

Example 2.14. We have

$$\mathrm{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}.$$

The situation is analogous to that of $\mathrm{Spec}(k[X])$. The closed points of $\mathrm{Spec}(\mathbb{Z})$ are in bijection with the prime numbers. The point (0) is not closed and is dense in $\mathrm{Spec}(\mathbb{Z})$. The open subsets are the empty set together with the complements of finite sets of prime numbers.

Example 2.15. Let k be an algebraically closed field. The prime ideals of $k[X,Y]$ are of three types. There are the maximal ideals, which are of the form $(X-a, Y-b)$ with $(a,b) \in k^2$, the prime ideals of the form $(f(X,Y))$ where $f(X,Y)$ is an irreducible polynomial, and (0). Only the point (0) is dense in $\operatorname{Spec}(k[X,Y])$. If $f(X,Y)$ is irreducible, the closure of the point $(f(X,Y))$ in $\operatorname{Spec}(k[X,Y])$ is then

$$\overline{\{(f(X,Y))\}} = \{(f(X,Y))\} \cup \{(X-a, Y-b) \mid (a,b) \in k^2 \text{ and } f(a,b) = 0\}.$$

Exercice 2.1. List the points of $\operatorname{Spec}(\mathbb{Z}[X])$ and determine their closures.

Let $f \in A$. We write $V(f)$ for $V(\{f\}) \subset \operatorname{Spec}(A)$ and $D(f)$ for the complement of $V(f)$ in $\operatorname{Spec}(A)$. The subset $D(f)$ is an open subset of $\operatorname{Spec}(A)$. Let us recall that A_f is the localization of A with respect to the multiplicative subset $\{f^n \mid n \in \mathbb{N}\}$: it is the ring of elements of the form $\frac{a}{f^n}$ for $a \in A$ and $n \geqslant 0$, with the relation

$$\frac{a}{f^n} = \frac{b}{f^m} \Leftrightarrow \exists r \geqslant 0, \quad f^r(f^m a - f^n b) = 0.$$

We have a ring homomorphism $i_f : A \to A_f$. The map ${}^a i_f$ induces a bijection between the prime ideals of A_f and the prime ideals of A not containing f. In other words, ${}^a i_f$ induces a continuous bijection from $\operatorname{Spec}(A_f)$ onto $D(f)$. The open subsets of $\operatorname{Spec}(A)$ of the form $D(f)$, for $f \in A$, are called the *distinguished open subsets* of $\operatorname{Spec}(A)$.

Exercice 2.2. Show that ${}^a i_f$ induces a homeomorphism of $\operatorname{Spec}(A_f)$ onto $D(f)$.

Lemma 2.16. *Let A be a ring. The set of distinguished open subsets of* $\operatorname{Spec}(A)$ *forms a base of the topology of* $\operatorname{Spec}(A)$ *and is stable under finite intersection.*

Proof. Note that if $f_1, f_2 \in A$, we have $D(f_1 f_2) = D(f_1) \cap D(f_2)$, which proves that this set of open subsets is stable under intersection. Now let U be an open subset of $\operatorname{Spec}(A)$ and let $\mathfrak{p} \in U$. We have to show that there exists $f \in A$ such that $\mathfrak{p} \in D(f) \subset U$. Let I be an ideal of A such that $V(I) = \operatorname{Spec}(A) \smallsetminus U$. As $\mathfrak{p} \notin V(I)$, we have $I \not\subset \mathfrak{p}$, so we may choose $f \in I \smallsetminus \mathfrak{p}$. Then $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$ since every prime ideal containing I contains f; that is, $D(f) \subset U$. □

Lemma 2.17. *Let A be a ring.*

(i) Let $f \in A$. Then $D(f) = \emptyset$ if and only if f is nilpotent.

(ii) Let $f, g \in A$ be such that $D(f) \subset D(g)$. Then there exists some $n \geqslant 0$ such that $f^n \in (g)$.

Proof. Let us prove (i). Recall that ${}^a i_f$ induces a bijection from $\operatorname{Spec}(A_f)$ onto $D(f)$, so that $D(f) = \emptyset$ if and only if $\operatorname{Spec}(A_f) = \emptyset$. By Krull's theorem every nonzero ring has a maximal ideal, which is prime, so that the latter holds if and only if $A_f = 0$, that is if and only if $\frac{1}{1} = \frac{0}{1}$ in A_f. By definition of A_f this means that $f^r = 0$ for some $r \geqslant 0$, that is that f is nilpotent.

Now let us prove (ii). Let $\pi : A \to A/(g)$ be the quotient map. By Example 2.2 the map ${}^a\pi$ is a bijection from $\mathrm{Spec}(A/(g))$ onto $V(g)$. The inclusion $D(f) \subset D(g)$ implies $V(g) \subset V(f)$, so that every prime ideal of A containing g contains f. Therefore $\pi(f)$ belongs to every prime ideal of $A/(g)$, that is $D(\pi(f)) = \emptyset$ in $\mathrm{Spec}(A/(g))$. By (i) applied to the ring $A/(g)$, the element $\pi(f)$ is nilpotent, which is exactly the desired result. □

Corollary 2.18. *Let A be a ring and let I be an ideal of A. We have*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

Proof. Let $\pi : A \to A/I$ be the quotient map. By Example 2.2 the map ${}^a\pi$ is a bijection from $\mathrm{Spec}(A/I)$ onto $V(I)$. Let $f \in A$. Then f belongs to $\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$ if and only if $\pi(f)$ belongs to every prime ideal of A/I, that is if and only if $D(\pi(f)) = \emptyset$ in $\mathrm{Spec}(A/I)$. By Lemma 2.17 (i) this is equivalent to $\pi(f)$ being nilpotent, i.e. to $f \in \sqrt{I}$. □

Proposition 2.19. *Let A be a ring. The topological space* $\mathrm{Spec}(A)$ *is quasi-compact.*

Proof. Let $(U_i)_{i \in I}$ be an open covering of $\mathrm{Spec}(A)$. As the set of distinguished open subsets of $\mathrm{Spec}(A)$ forms a base of the topology, there is a finer covering of $\mathrm{Spec}(A)$ consisting of distinguished open subsets. We may therefore assume that $U_i = D(f_i)$ for all $i \in I$. We then have $\bigcap_{i \in I} V(f_i) = V(\{f_i \mid i \in I\}) = \emptyset$. Let J be the ideal of A generated by the f_i. We then have $V(J) = \emptyset$. Since every proper ideal of A is contained in a maximal, hence prime, ideal, this implies that $J = A$, that is $1 \in J$. There are therefore finitely many indices $i_1, \ldots, i_r \in I$ and elements $a_1, \ldots, a_r \in A$ such that $1 = a_1 f_{i_1} + \cdots + a_r f_{i_r}$. Hence we have

$$V(f_{i_1}) \cap \cdots \cap V(f_{i_r}) = V(f_{i_1}, \ldots, f_{i_r}) = \emptyset$$

that is $\mathrm{Spec}(A) = D(f_{i_1}) \cup \cdots \cup D(f_{i_r})$. We have therefore extracted a finite covering from our open covering. □

Exercice 2.3. Let A be a ring. Show that the following conditions are equivalent:

(i) the topological space $\mathrm{Spec}(A)$ is disconnected;

(ii) there exist nonzero idempotent elements $e_1, e_2 \in A$ (i.e. such that $e_1^2 = e_1$, $e_2^2 = e_2$) such that $e_1 e_2 = 0$ and $e_1 + e_2 = 1$;

(iii) the ring A is isomorphic to a product $A_1 \times A_2$ of two nonzero rings.

2.3 The structure sheaf of an affine scheme

Let A be a ring. We have defined a topological space $\mathrm{Spec}(A)$. Now we want to define what a function on $\mathrm{Spec}(A)$, or on an open subset of $\mathrm{Spec}(A)$, is. In geometry, the interesting classes of functions are local: being continuous, differentiable, holomorphic, etc. are local notions. In other words they are defined by a sheaf. Similarly we want to define a sheaf of functions $\mathcal{O}_{\mathrm{Spec}(A)}$ on $\mathrm{Spec}(A)$. It is natural to require that this sheaf be stable under elementary operations like addition and multiplication. Therefore we want to define a *sheaf of rings* $\mathcal{O}_{\mathrm{Spec}(A)}$. Another natural requirement is the characterization of invertible functions: we want a function $f \in \mathcal{O}_{\mathrm{Spec}(A)}(U)$ to be invertible (in the ring $\mathcal{O}_{\mathrm{Spec}(A)}(U)$) if and only if $f(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in U$. Finally, as already observed, we want the elements of A to give rise to functions defined globally on $\mathrm{Spec}(A)$. In view of all these requirements, the following construction is natural.

Let $U \subset \mathrm{Spec}(A)$ be an open subset and let $S_U \subset A$ be the subset of elements $s \in A$ such that $s(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in U$. As the relation $s(\mathfrak{p}) \neq 0$ takes place in the field $k(\mathfrak{p})$, it is clear that S_U is a multiplicative subset in A. Alternatively we can observe that

$$S_U = \bigcap_{\mathfrak{p} \in U} (A \smallsetminus \mathfrak{p}),$$

that $A \smallsetminus \mathfrak{p}$ is multiplicative and that an intersection of multiplicative subsets is multiplicative. Then we define

$$\mathcal{O}'_{\mathrm{Spec}(A)}(U) := S_U^{-1}A = \left\{ \frac{a}{b} \mid (a,b) \in A \times S_U \right\}$$

the localization of A with respect to S_U.

If $V \subset U$ is an inclusion of open subsets, we have $S_U \subset S_V$ and a natural ring homomorphism $\rho'_{U,V} : S_U^{-1}A \to S_V^{-1}A$. Moreover if $W \subset V \subset U$, we have clearly $\rho'_{V,W} \circ \rho'_{U,V} = \rho'_{U,W}$. Therefore we have defined a *presheaf* $U \mapsto \mathcal{O}'_{\mathrm{Spec}(A)}(U)$ on $\mathrm{Spec}(A)$.

In general, $\mathcal{O}'_{\mathrm{Spec}(A)}$ is not a sheaf and we denote by $\mathcal{O}_{\mathrm{Spec}(A)}$ the associated sheaf. The sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ is called the *structure sheaf* of $\mathrm{Spec}(A)$.

If $V \subset U$ are open subsets of $\mathrm{Spec}(A)$, we denote by $\rho_{U,V}$ the restriction map $\mathcal{O}_{\mathrm{Spec}(A)}(U) \to \mathcal{O}_{\mathrm{Spec}(A)}(V)$ and, if $s \in \mathcal{O}_{\mathrm{Spec}(A)}(U)$, we will simply denote $s_{|V}$ for $\rho_{U,V}(s)$ and call it the *restriction* of s to V.

Note also that all the $\mathcal{O}'_{\mathrm{Spec}(A)}(U)$ are rings and that the $\rho'_{U,V}$ are morphisms of rings. This implies that $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf of rings, that is that all the $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ are rings and the $\rho_{U,V}$ are ring homomorphisms.

Remark 2.20. Recall that if $f \in A$, then $D(f)$ is an open subset of $\mathrm{Spec}(A)$ which is homeomorphic to $\mathrm{Spec}(A_f)$. Under this homeomorphism, the restriction of the presheaf $\mathcal{O}'_{\mathrm{Spec}(A)}$ to $D(f)$ is isomorphic to the presheaf $\mathcal{O}'_{\mathrm{Spec}(A_f)}$. Namely if $U \subset D(f)$ is an open subset we have $f \in S_U$, so that the natural map $S_U^{-1}A \to S_U^{-1}A_f$ is an isomorphism. As the restriction of $\mathcal{O}_{\mathrm{Spec}(A)}$ to $D(f)$ is the sheaf associated to the restriction of $\mathcal{O}'_{\mathrm{Spec}(A)}$ to $D(f)$, we conclude that the restriction of $\mathcal{O}_{\mathrm{Spec}(A)}$ to $D(f)$ is canonically isomorphic to $\mathcal{O}_{\mathrm{Spec}(A_f)}$.

In order to understand this sheaf of rings better, let us compute its ring of global sections. For any open subset $U \subset \mathrm{Spec}(A)$, there is a natural ring homomorphism $A \to \mathcal{O}'_{\mathrm{Spec}(A)}(U)$ given by $a \mapsto \frac{a}{1}$. Composed with the natural map $\mathcal{O}'_{\mathrm{Spec}(A)}(U) \to \mathcal{O}_{\mathrm{Spec}(A)}(U)$, this gives us a ring homomorphism $\iota_U : A \to \mathcal{O}_{\mathrm{Spec}(A)}(U)$. Applied to $U = \mathrm{Spec}(A)$, we obtain a ring homomorphism

$$\iota_{\mathrm{Spec}(A)} : A \to \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)).$$

Lemma 2.21. *(i) The homomorphism $\iota_{\mathrm{Spec}(A)}$ is injective.*

(ii) For all $f \in A$, the homomorphism $A_f \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ extending $\iota_{D(f)}$ is injective.

Proof. Note that $A = \mathcal{O}'_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$, since $S_{\mathrm{Spec}(A)}$ is the group of units of A. So we need to prove that the ring homomorphism $\mathcal{O}'_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \to \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$ is injective. Let $a \in A$ and assume that $\iota_{\mathrm{Spec}(A)}(a) = 0$. By definition of the sheaf associated to $\mathcal{O}'_{\mathrm{Spec}(A)}$ this implies that there exists an open covering $(U_i)_{i \in I}$ of $\mathrm{Spec}(A)$ such that $\rho'_{\mathrm{Spec}(A),U_i}(\frac{a}{1}) = 0$ for all $i \in I$. Equivalently that $\frac{a}{1} = 0$ in $S_{U_i}^{-1}A$ for all $i \in I$. By definition of $S_{U_i}^{-1}A$, this means that, for all $i \in I$, there exists some $s_i \in S_{U_i}$ such that $s_i a = 0$ in A. As $s_i \in S_{U_i}$ we must have $U_i \subset D(s_i)$. As $\bigcup_{i \in I} U_i = \mathrm{Spec}(A)$, we have $\bigcup_{i \in I} D(s_i) = \mathrm{Spec}(A)$ and therefore

$$V\left(\sum_{i \in I}(s_i)\right) = \bigcap_{i \in I} V((s_i)) = \emptyset.$$

Therefore, as every proper ideal of A is contained in a maximal ideal, we must have $\sum_{i \in I}(s_i) = A$, so that we can write $1 = \sum_{i \in I} x_i s_i$ for a family $(x_i)_{i \in I}$ of elements of A with finite support. Then we have

$$a = \sum_{i \in I} x_i s_i a = 0.$$

This proves (i). The assertion (ii) is then a direct consequence of (i) applied to the ring A_f and of Remark 2.20. $\square$

Proposition 2.22. *The ring homomorphism $\iota_{\mathrm{Spec}(A)}$ is an isomorphism.*

Proof. By Lemma 2.21 (i) the homomorphism $\iota_{\mathrm{Spec}(A)}$ is injective, so it remains to prove that it is surjective. Let $s \in \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$. By definition of the sheaf associated to $\mathcal{O}'_{\mathrm{Spec}(A)}$, there exist an open covering $(U_i)_{i\in I}$ of $\mathrm{Spec}(A)$ and, for all $i \in I$, an element $s_i \in \mathcal{O}'_{\mathrm{Spec}(A)}(U_i)$ such that the image of s_i in $\mathcal{O}_{\mathrm{Spec}(A)}(U_i)$ is $s_{|U_i}$. Here we note that, as the distinguished open subsets form a basis of the topology (Lemma 2.16), we can refine the covering U_i by a covering of distinguished subsets. Moreover as $\mathrm{Spec}(A)$ is quasi-compact by Proposition 2.19, we can extract from $(U_i)_{i\in I}$ a finite cover of $\mathrm{Spec}(A)$. So from now we assume that I is finite and that each U_i is of the form $D(f_i)$ for some $f_i \in A$. If we can prove that there exists some $a \in A$ such that $s_i = \frac{a}{1}$ in $\mathcal{O}'_{\mathrm{Spec}(A)}(U_i)$ for all $i \in I$, then we are done. Namely if $\tilde{a}$ is the image of a in $\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$, then we will have $\tilde{a}_{|U_i} = s_{|U_i}$ for all $i \in I$ and then $\tilde{a} = s$ by the sheaf property.

By definition we have $\mathcal{O}'_{\mathrm{Spec}(A)}(U_i) = S_{U_i}^{-1}A$ so that we can write $s_i = \frac{a_i}{b_i}$ with $b_i \in S_{U_i}$. Therefore we have $U_i = D(f_i) \subset D(b_i)$. By Lemma 2.17 (ii), there exists an integer n_i and $c_i \in A$ such that $f_i^{n_i} = c_i b_i$ so that we can write $s_i = \frac{c_i a_i}{f_i^{n_i}}$. As moreover $D(f_i^{n_i}) = D(f_i)$, we can, after replacing f_i by $f_i^{n_i}$ and a_i by $c_i a_i$, assume that $s_i = \frac{a_i}{f_i}$ for some $a_i \in A$.

For $i, j \in I$, we have $D(f_i f_j) = D(f_i) \cap D(f_j)$. As $s_{|D(f_i)}$ is the image of s_i, it follows that $s_{|D(f_i f_j)}$ can be identified with the image of s_i via the map

$$A_{f_i} \to A_{f_i f_j} \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f_i f_j))$$

but also to the image of s_j via the map

$$A_{f_j} \to A_{f_i f_j} \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f_i f_j)).$$

Therefore we conclude from Lemma 2.21 (ii) that s_i and s_j have the same image in $A_{f_i f_j}$. Therefore there exists some integer $n_{i,j}$ such that

$$(f_i f_j)^{n_{i,j}} f_j a_i = (f_i f_j)^{n_{i,j}} f_i a_j.$$

As I is finite we can fix an integer $n \geqslant 0$ such that $n_{i,j} \leqslant n$ for all $i, j \in I$. So we can assume that, for all $i, j \in I$, we have

$$(f_i f_j)^n f_j a_i = (f_i f_j)^n f_i a_j.$$

As in the first part of the proof the equality $\mathrm{Spec}(A) = \bigcup_{i\in I} U_i$ implies that we can write $1 = \sum_{i\in I} x_i f_i^{n+1}$ for some $x_i \in A$ and define $a = \sum_i x_i f_i^n a_i$. Then we have, for all $j \in I$,

$$f_j^{n+1} a = \sum_i x_i f_j^{n+1} f_i^n a_i = \sum_i x_i f_i^{n+1} f_j^n a_j = f_j^n a_j.$$

This implies that the image of a in A_{f_j} coincides with $\frac{a_j}{f_j}$ so that the image of a in $\mathcal{O}_{\mathrm{Spec}(A)}(U_j)$ coincides with s_j. This is exactly the desired property. $\square$

Corollary 2.23. *Let A be a ring. Let $f \in A$. Then the natural homomorphism*

$$A_f \longrightarrow \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$$

is a ring isomorphism.

Proof. This is a direct consequence of Proposition 2.22 and Remark 2.20. $\square$

We can give some examples.

Example 2.24. Let k be a field, let $A = k[X]$ and let $U = \mathrm{Spec}(A) \smallsetminus V(P) = D(P)$ for some nonzero polynomial $P \in k[X]$. Then we have

$$\mathcal{O}_{\mathrm{Spec}(A)}(U) = k[X]_P = \left\{ \frac{Q(X)}{P(X)^n} \mid Q(X) \in k[X],\, n \geqslant 0 \right\}.$$

Example 2.25. Let $A = \mathbb{Z}$ and let $U = \mathrm{Spec}(\mathbb{Z}) \smallsetminus V(n) = D(n)$ for some nonzero $n \in \mathbb{Z}$. Then $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}(U)$ is the subring of $\mathbb{Q}$ whose elements are the fractions $\frac{a}{d}$ where the prime divisors of d divide n.

Exercice 2.4. Let k be a field and let $U = \mathrm{Spec}(k[X,Y]) \smallsetminus V(X,Y)$ be the open subset complementary to the closed point (X,Y). Show that

$$k[X,Y] \simeq \mathcal{O}_{\mathrm{Spec}(k[X,Y])}(U).$$

Let A be a ring and let $\mathfrak{p} \in \mathrm{Spec}(A)$. We recall that the *stalk* of $\mathcal{O}_{\mathrm{Spec}(A)}$ at $\mathfrak{p}$ can be defined as the ring of germs of functions at $\mathfrak{p}$ or, more algebraically,

$$\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} = \varinjlim_{U \ni \mathfrak{p}} \mathcal{O}_{\mathrm{Spec}(A)}(U).$$

As the family $(D(f))_{f \notin \mathfrak{p}}$ is a basis of open neighbourhoods of $\mathfrak{p}$ which is stable under finite intersection, we have

$$\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} = \varinjlim_{f \notin \mathfrak{p}} \mathcal{O}_{\mathrm{Spec}(A)}(D(f)).$$

By Corollary 2.23 we have therefore

$$\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} = \varinjlim_{f \notin \mathfrak{p}} A_f.$$

By definition, the images of the elements of $A \smallsetminus \mathfrak{p}$ in $\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$ are invertible so that the canonical ring homomorphism $A \to \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$ extends to a ring homomorphism $A_{\mathfrak{p}} \to \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$.

Proposition 2.26. *The ring homomorphism $A_{\mathfrak{p}} \to \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$ is an isomorphism.*

Proof. Exercise. □

Exercice 2.5. Prove Proposition 2.26.

Exercice 2.6. Describe the points and the structure sheaves of the following schemes: $\mathrm{Spec}(\mathbb{C}[X]/(X^2))$, $\mathrm{Spec}(\mathbb{C}[X]/(X^2-X))$, $\mathrm{Spec}(\mathbb{C}[X]/(X^3-X^2))$, $\mathrm{Spec}(\mathbb{R}[X]/(X^2+1))$, $\mathrm{Spec}(\mathbb{Z}/24\mathbb{Z})$.

2.4 Morphisms of affine schemes

We recall that a *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings over X. A *morphism* from a ringed space $(X, \mathcal{O}_X)$ to another one $(Y, \mathcal{O}_Y)$ is a pair (f, θ) where $f : X \to Y$ is a continuous map and $\theta : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings over Y. If $V \subset Y$ is an open subset, we denote by $\theta_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ the corresponding ring homomorphism. Recall that if $\mathcal{F}$ is a sheaf over X, then the presheaf $U \mapsto \mathcal{F}(f^{-1}(U))$ is a sheaf over Y called the *direct image* of $\mathcal{F}$ by f and denoted $f_*\mathcal{F}$.

Note that if (f, θ) is a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and (g, η) is a morphism of ringed spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, we can define a morphism of ringed spaces $(h, \mu) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ by $h = g \circ f$ and $\mu = g_*(\theta) \circ \eta$. Here we have used the equality $g_*(f_*\mathcal{O}_X) = (g \circ f)_*(\mathcal{O}_X)$.

Exercice 2.7. Check that the ringed spaces and the morphisms of ringed spaces form a category.

Let $(f, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $x \in X$. We want to attach to θ a ring homomorphism from the stalk of $\mathcal{O}_Y$ at $f(x)$ to the stalk of $\mathcal{O}_X$ at x. Let us first determine the stalk of $f_*\mathcal{O}_X$ at $f(x)$. This is

$$(f_*\mathcal{O}_X)_{f(x)} = \varinjlim_{V \ni f(x)} (f_*\mathcal{O}_X)(V) = \varinjlim_{V \ni f(x)} \mathcal{O}_X(f^{-1}(V)),$$

the inductive limit being taken over the set of all open subsets $V \subset Y$ containing $f(x)$. Note that, as f is continuous, all the $f^{-1}(V)$ are open subsets of X containing x. As a consequence there is a natural ring homomorphism

$$s_x : \varinjlim_{V \ni f(x)} \mathcal{O}_X(f^{-1}(V)) \longrightarrow \varinjlim_{U \ni x} \mathcal{O}_X(U) = \mathcal{O}_{X,x},$$

where U is varying among all the open subsets of X containing x. This provides a ring homomorphism $(f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$. As θ is a morphism of sheaves of rings, it induces a ring homomorphism

$$\mathcal{O}_{Y,f(x)} \longrightarrow (f_*\mathcal{O}_X)_{f(x)}$$

on the stalks at $f(x)$. Composing it with s_x we obtain a ring homomorphism

$$\theta_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}.$$

Concretely, if $V \subset Y$ is an open subset containing $f(x)$ and if $s \in \mathcal{O}_Y(V)$, then θ_x sends the germ of s at $f(x)$ to the germ of $\theta_V(s)$ at x.

Exercice 2.8. Let $(f,\theta) : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ and $(g,\eta) : (Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z)$ be two morphisms of ringed spaces and let (h,μ) be their composite. Check that, for all $x \in X$, we have $\mu_x = \theta_x \circ \eta_{f(x)}$.

Let $\varphi : A \to B$ be a ring homomorphism. We have constructed a continuous map ${}^a\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. We want to show that it extends to a morphism of ringed spaces $({}^a\varphi, \varphi^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. It is easier to first construct a morphism of presheaves $\varphi^{\sharp,\prime} : \mathcal{O}'_{\mathrm{Spec}(A)} \to {}^a\varphi_*\mathcal{O}'_{\mathrm{Spec}(B)}$. For any open subset $U \subset \mathrm{Spec}(A)$, we need to construct a ring homomorphism

$$\varphi_U^{\sharp,\prime} : \mathcal{O}'_{\mathrm{Spec}(A)}(U) \to \mathcal{O}'_{\mathrm{Spec}(B)}(({}^a\varphi)^{-1}(U)),$$

i.e. a ring homomorphism $S_U^{-1}A \to S_{({}^a\varphi)^{-1}(U)}^{-1}B$. Note that $\varphi(S_U) \subset S_{({}^a\varphi)^{-1}(U)}$ so there is a unique ring homomorphism $S_U^{-1}A \to S_{({}^a\varphi)^{-1}(U)}^{-1}B$ extending φ. We choose this morphism as $\varphi_U^{\sharp,\prime}$. In order to prove that $\varphi^{\sharp,\prime}$ is a morphism of presheaves, it is sufficient to check that, for any inclusion of open subsets $V \subset U$, we have

$$\rho'_{({}^a\varphi)^{-1}(U),({}^a\varphi)^{-1}(V)} \circ \varphi_U^{\sharp,\prime} = \varphi_V^{\sharp,\prime} \circ \rho'_{U,V},$$

where the restriction map on the left is the one of $\mathcal{O}'_{\mathrm{Spec}(B)}$ and the one on the right is the one of $\mathcal{O}'_{\mathrm{Spec}(A)}$. This is left as an exercise.

Now we have a morphism of presheaves $\mathcal{O}'_{\mathrm{Spec}(A)} \to {}^a\varphi_*\mathcal{O}'_{\mathrm{Spec}(B)}$. If we apply the functor ${}^a\varphi_*$ to the natural morphism of presheaves $\mathcal{O}'_{\mathrm{Spec}(B)} \to \mathcal{O}_{\mathrm{Spec}(B)}$, we obtain a morphism of presheaves ${}^a\varphi_*\mathcal{O}'_{\mathrm{Spec}(B)} \to {}^a\varphi_*\mathcal{O}_{\mathrm{Spec}(B)}$ and by composition a morphism of presheaves $\mathcal{O}'_{\mathrm{Spec}(A)} \to {}^a\varphi_*\mathcal{O}_{\mathrm{Spec}(B)}$. As ${}^a\varphi_*\mathcal{O}_{\mathrm{Spec}(B)}$ is a sheaf, the universal property of the associated sheaf shows that this morphism extends to a morphism of sheaves $\mathcal{O}_{\mathrm{Spec}(A)} \to {}^a\varphi_*\mathcal{O}_{\mathrm{Spec}(B)}$ which is compatible with the ring structures. This is the morphism $\varphi^\sharp$.

Exercice 2.9. Show that the construction $A \mapsto \mathrm{Spec}(A)$ and $\varphi \mapsto ({}^a\varphi, \varphi^\sharp)$ induces a contravariant functor from the category of rings to the category of ringed spaces, i.e. that we have the following properties:

(i) for $\varphi : A \to B$ and $\psi : B \to C$, we have $({}^a(\psi \circ \varphi), (\psi \circ \varphi)^\sharp) = ({}^a\varphi, \varphi^\sharp) \circ ({}^a\psi, \psi^\sharp)$;

(ii) $({}^a\mathrm{Id}_A, \mathrm{Id}_A^\sharp) = (\mathrm{Id}, \mathrm{Id})$.

Exercice 2.10. Let $\varphi : A \to B$ be a ring homomorphism and let $({}^a\varphi, \varphi^\sharp) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the associated morphism of ringed spaces. Let $h \in A$. Show that $({}^a\varphi)^{-1}(D(h)) = D(\varphi(h))$ and that $\varphi^\sharp_{D(h)} : \mathcal{O}_{\mathrm{Spec}(A)}(D(h)) \to \mathcal{O}_{\mathrm{Spec}(B)}(D(\varphi(h)))$ is the ring homomorphism $A_h \to B_{\varphi(h)}$ extending φ.

Let $\varphi : A \to B$ be a ring homomorphism, let $({}^a\varphi, \varphi^\sharp) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the associated morphism of ringed spaces and let $\mathfrak{p} \in \mathrm{Spec}(B)$. Recall that $\mathfrak{q} := {}^a\varphi(\mathfrak{p})$ is the prime ideal $\varphi^{-1}(\mathfrak{p})$ of A. The above construction, applied to the morphism $({}^a\varphi, \varphi^\sharp)$, provides a ring homomorphism

$$\varphi^\sharp_{\mathfrak{p}} : \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{q}} \longrightarrow \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{p}}.$$

By Proposition 2.26, we have canonical isomorphisms $A_{\mathfrak{q}} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{q}}$ and $B_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{p}}$, so that $\varphi^\sharp_{\mathfrak{p}}$ can be seen as a ring homomorphism $A_{\mathfrak{q}} \to B_{\mathfrak{p}}$.

Exercice 2.11. Check that $\varphi(A \smallsetminus \mathfrak{q}) \subset B \smallsetminus \mathfrak{p}$ and that the ring homomorphism $\varphi^\sharp_{\mathfrak{p}}$ is the unique ring homomorphism $A_{\mathfrak{q}} \to B_{\mathfrak{p}}$ extending φ.

Definition 2.27. *A* locally ringed space *is a ringed space $(X, \mathcal{O}_X)$ such that, for all points $x \in X$, the ring $\mathcal{O}_{X,x}$ is local. We denote usually by $\mathfrak{m}_x$ its maximal ideal.*

A morphism of locally ringed spaces *is a morphism $(f, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces such that for all $x \in X$ the ring homomorphism $\theta_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local morphism of rings, i.e. it sends $\mathfrak{m}_{f(x)}$ into $\mathfrak{m}_x$.*

The category of locally ringed spaces is the subcategory of the category of ringed spaces whose objects are the locally ringed spaces and whose morphisms are the morphisms of locally ringed spaces.

A consequence of Exercise 2.11 is that a morphism $({}^a\varphi, \varphi^\sharp)$ associated to a ring homomorphism φ is a morphism of locally ringed spaces.

Exercice 2.12. Let k be a field and let $A = k[\![X]\!]$ be the ring of power series with coefficients in k and one variable X.

a) Show that A is a local ring with unique maximal ideal (X).

b) Determine the ringed space $\mathrm{Spec}(A)$.

c) Construct a morphism of ringed spaces $\mathrm{Spec}(A) \to \mathrm{Spec}(A)$ which is not a morphism of locally ringed spaces and thus doesn't come from an endomorphism of A.

Theorem 2.28. *The construction $A \mapsto (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$, $\varphi \mapsto ({}^a\varphi, \varphi^\sharp)$ is a fully faithful (contravariant) functor from the category of rings to the category of locally ringed spaces.*

Proof. Let $\mathcal{R}$ be the category of rings and let $\mathcal{S}$ be the category of locally ringed spaces. In this proof we simply write $\mathrm{Spec}(A)$ for the locally ringed space $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. We need to prove that the map $\alpha : \mathrm{Hom}_{\mathcal{R}}(A, B) \to \mathrm{Hom}_{\mathcal{S}}(\mathrm{Spec}(B), \mathrm{Spec}(A))$ defined by $\varphi \mapsto ({}^a\varphi, \varphi^\sharp)$ is bijective. In order to do this we will construct a map $\beta : \mathrm{Hom}_{\mathcal{S}}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathcal{R}}(A, B)$. If $F = (f, \theta) \in \mathrm{Hom}_{\mathcal{S}}(\mathrm{Spec}(B), \mathrm{Spec}(A))$, we take

$$\beta(F) = \theta_{\mathrm{Spec}(A)} : A = \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \to B = \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)).$$

It is essentially the content of Exercise 2.10, applied with $h = 1$, that $\beta \circ \alpha = \mathrm{Id}$. We need to prove that $\alpha \circ \beta = \mathrm{Id}$. Let $F = (f, \theta)$, $\varphi = \beta(F)$ and $G = ({}^a\varphi, \varphi^\sharp) = \alpha(\varphi)$.

We first prove that ${}^a\varphi = f$. Let $\mathfrak{p} \in \mathrm{Spec}(B)$ and let $\mathfrak{q} = f(\mathfrak{p})$. By construction we have a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \lambda_B} \\ A_{\mathfrak{q}} & \xrightarrow{\theta_{\mathfrak{p}}} & B_{\mathfrak{p}} \end{array}$$

where λ_A and λ_B are the canonical maps. It is important to note that F is a morphism of locally ringed spaces so that $\theta_{\mathfrak{p}}$ sends the maximal ideal of $A_{\mathfrak{q}}$ into the maximal ideal of $B_{\mathfrak{p}}$. As $\theta_{\mathfrak{p}}^{-1}(\mathfrak{p}B_{\mathfrak{p}})$ is then a proper ideal of $A_{\mathfrak{q}}$ containing the maximal ideal $\mathfrak{q}A_{\mathfrak{q}}$, we even have

$$\theta_{\mathfrak{p}}^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = \mathfrak{q}A_{\mathfrak{q}}.$$

Let now $a \in A$. As the inverse image in B of $\mathfrak{p}B_{\mathfrak{p}}$ is $\mathfrak{p}$ and the inverse image in A of $\mathfrak{q}A_{\mathfrak{q}}$ is $\mathfrak{q}$, the commutativity of the diagram gives

$$\varphi(a) \in \mathfrak{p} \Leftrightarrow \lambda_B(\varphi(a)) \in \mathfrak{p}B_{\mathfrak{p}} \Leftrightarrow \theta_{\mathfrak{p}}(\lambda_A(a)) \in \mathfrak{p}B_{\mathfrak{p}} \Leftrightarrow \lambda_A(a) \in \mathfrak{q}A_{\mathfrak{q}} \Leftrightarrow a \in \mathfrak{q}.$$

Therefore ${}^a\varphi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) = \mathfrak{q} = f(\mathfrak{p})$, that is ${}^a\varphi = f$.

It remains to prove that $\varphi^\sharp = \theta$. As the distinguished open subsets form a basis of the topology of $\mathrm{Spec}(A)$ (Lemma 2.16), it is enough to prove that $\theta_{D(h)} = \varphi^\sharp_{D(h)}$ for all $h \in A$. Namely, if $U \subset \mathrm{Spec}(A)$ is an open subset and $s \in \mathcal{O}_{\mathrm{Spec}(A)}(U)$, we can write U as a union of distinguished open subsets $D(h_i)$; the sections $\theta_U(s)$ and $\varphi^\sharp_U(s)$ of $\mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(U))$ have then the same restriction to each $f^{-1}(D(h_i))$, and these open subsets cover $f^{-1}(U)$, so that $\theta_U(s) = \varphi^\sharp_U(s)$ as $\mathcal{O}_{\mathrm{Spec}(B)}$ is a sheaf.

So let $h \in A$. By Exercise 2.10 we have $f^{-1}(D(h)) = ({}^a\varphi)^{-1}(D(h)) = D(\varphi(h))$ and $\varphi^\sharp_{D(h)} : A_h \to B_{\varphi(h)}$ is the unique ring homomorphism extending φ. On the other hand, the compatibility of θ with the restriction maps gives a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_h & \xrightarrow{\theta_{D(h)}} & B_{\varphi(h)} \end{array}$$

where the vertical maps are the canonical ones and where we have used Corollary 2.23 to identify $\mathcal{O}_{\mathrm{Spec}(A)}(D(h))$ with A_h and $\mathcal{O}_{\mathrm{Spec}(B)}(D(\varphi(h)))$ with $B_{\varphi(h)}$. The top arrow is $\theta_{\mathrm{Spec}(A)} = \varphi$ by definition of φ. Therefore $\theta_{D(h)}$ is also a ring homomorphism $A_h \to B_{\varphi(h)}$ extending φ, and the universal property of the localization implies $\theta_{D(h)} = \varphi^\sharp_{D(h)}$. This proves that $\alpha \circ \beta = \mathrm{Id}$ and concludes the proof. $\square$

2.5 Irreducible subsets

Definition 2.29. *A topological space X is* irreducible *if it is non empty and if, for all decomposition $X = F \cup G$ with F and G closed, we have $F \subset G$ or $G \subset F$.*

Exercice 2.13. Show that for a non empty topological space X the following are equivalent:

(i) the topological space X is irreducible;

(ii) all non empty open subsets of X are dense in X;

(iii) the intersection of two non empty open subsets of X is non empty.

Exercice 2.14. If X is a topological space and Y is a subspace (with the induced topology), prove that Y is irreducible if and only if $\overline{Y}$ is irreducible.

Exercice 2.15. Show that if A is a domain, then $\mathrm{Spec}(A)$ is an irreducible topological space. Is the converse true?

Definition 2.30. *An* irreducible component *of a topological space X is a subset of X which is irreducible and maximal for the inclusion among irreducible subsets.*

Remark 2.31. An irreducible component is automatically closed, as follows from Exercise 2.14.

Lemma 2.32. *Let X be a topological space and let $x \in X$. Then x is contained in an irreducible component of X.*

Proof. This is a consequence of Zorn's lemma. Namely let $\mathcal{P}$ be the set of irreducible subsets of X containing x. It is non empty as it contains $\{x\}$. Moreover if $M \subset \mathcal{P}$ is a non empty totally ordered subset of $\mathcal{P}$, then M has an upper bound in $\mathcal{P}$: the union Y of the elements of M is irreducible, as M is totally ordered, so that $\overline{Y}$ is an irreducible subset of X containing x. Therefore $\mathcal{P}$ contains a maximal element Z. Finally Z is maximal among all irreducible subsets of X: an irreducible subset containing Z contains x, hence belongs to $\mathcal{P}$. □

Exercice 2.16. If a topological space X is a finite union $X_1 \cup \cdots \cup X_r$ of closed irreducible subsets such that, for all $i \neq j$, $X_i \not\subset X_j$, then $X_1, \ldots, X_r$ are exactly the irreducible components of X.

Definition 2.33. *Let X be a topological space. We say that X is* noetherian *if any decreasing sequence of closed subsets of X is stationary.*

Exercice 2.17. Show that a topological space X is noetherian if and only if any non empty set of closed subsets of X has a minimal element for inclusion.

Exercice 2.18. Let A be a ring. Show that if A is noetherian, then $\operatorname{Spec}(A)$ is a noetherian topological space. Show that the converse is false.

Proposition 2.34. *Let X be a noetherian topological space. Then X has finitely many irreducible components.*

Proof. Using Exercise 2.16, it is sufficient to prove that X is a finite union of irreducible closed subsets. Let $\mathcal{E}$ be the set of closed subsets in X which are *not* finite unions of irreducible closed subsets. We want to prove that $\mathcal{E}$ is empty. Assume that it is non empty. By Exercise 2.17, $\mathcal{E}$ has a minimal element F. Note that F is not irreducible, as otherwise F would be a union of one irreducible closed subset. We can therefore write $F = F_1 \cup F_2$ with F_1 and F_2 closed, $F_1 \not\subset F_2$ and $F_2 \not\subset F_1$. In particular $F_1 \subsetneq F$ and $F_2 \subsetneq F$, so that $F_1, F_2 \notin \mathcal{E}$ by minimality of F and thus F_1 and F_2 are finite unions of closed irreducible subsets. Then F itself is a finite union of closed irreducible subsets and $F \notin \mathcal{E}$, which is absurd. □

Let X be a topological space and let $x \in X$. Then $\{x\}$ is irreducible and so is $\overline{\{x\}}$. More generally if F is a closed subset of X, we say that a point $x \in F$ is a *generic point* of F if $\overline{\{x\}} = F$.

Proposition 2.35. *Let A be a ring and let I be an ideal of A. Then $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal in A.*

Proof. Assume that $\sqrt{I}$ is prime. In particular $\sqrt{I} \neq A$, so that $V(I)$ is non empty. Let $V(I) = V(I_1) \cup V(I_2)$. By Remark 2.9, we have $V(I) = V(I_1 I_2)$. Moreover it follows from Corollary 2.18 that $I_1 I_2 \subset \sqrt{I_1 I_2} = \sqrt{I}$. As $\sqrt{I}$ is prime, we have $I_1 \subset \sqrt{I}$ or $I_2 \subset \sqrt{I}$, that is $V(I) \subset V(I_1)$ or $V(I) \subset V(I_2)$ so that $V(I)$ is irreducible.

Conversely assume that $V(I)$ is irreducible. As $V(I)$ is non empty, we have $\sqrt{I} \neq A$. If $\sqrt{I}$ is not prime, there exist $f_1, f_2 \notin \sqrt{I}$ such that $f_1 f_2 \in \sqrt{I}$. Let $J_i = (f_i) + \sqrt{I}$ for $i = 1, 2$. We have $J_1 J_2 \subset \sqrt{I}$, so that $V(\sqrt{I}) \subset V(J_1 J_2) = V(J_1) \cup V(J_2)$, and $\sqrt{I} \subset J_i$, so that $V(J_i) \subset V(\sqrt{I})$. Hence $V(I) = V(\sqrt{I}) = V(J_1) \cup V(J_2)$. However, as $f_i \notin \sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$, there exists $\mathfrak{p} \in V(I)$ such that $f_i \notin \mathfrak{p}$, that is $\mathfrak{p} \notin V(J_i)$. Therefore $V(J_i) \subsetneq V(I)$ for $i = 1, 2$, which contradicts the irreducibility of $V(I)$. $\square$

Corollary 2.36. *Let A be a ring. A closed irreducible subset of* $\operatorname{Spec}(A)$ *has a unique generic point.*

Proof. Let $F \subset \operatorname{Spec}(A)$ be a closed irreducible subset. We have $F = V(I)$ for some ideal $I \subset A$. By Proposition 2.35, $\mathfrak{p} = \sqrt{I}$ is a prime ideal, and by Proposition 2.12 1 we have $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = V(\sqrt{I}) = V(I) = F$, so that $\mathfrak{p}$ is a generic point of F. The uniqueness is left as an exercise. $\square$

Corollary 2.37. *Let A be a noetherian ring. Then A has finitely many minimal prime ideals.*

Proof. By Exercise 2.18 the topological space $\operatorname{Spec}(A)$ is noetherian, so that it has finitely many irreducible components by Proposition 2.34. Moreover it follows from Corollary 2.36 that the closed irreducible subsets of $\operatorname{Spec}(A)$ are exactly the $V(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec}(A)$, and we have $V(\mathfrak{p}) \subset V(\mathfrak{q})$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Therefore the irreducible components of $\operatorname{Spec}(A)$ are exactly the $V(\mathfrak{p})$ for $\mathfrak{p}$ a minimal prime ideal of A. $\square$

3 Schemes

3.1 Definitions

Definition 3.1. *An* affine scheme *is a locally ringed space isomorphic to* $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ *for some ring* A.

A quasi-affine scheme *is a locally ringed space isomorphic to* $(U, \mathcal{O}_{\mathrm{Spec}(A)|U})$ *where* U *is an open subset of a spectrum* $\mathrm{Spec}(A)$.

Note that, for $f \in A$, we have an isomorphism of locally ringed spaces $(D(f), \mathcal{O}_{\mathrm{Spec}(A)|D(f)}) \simeq (\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)})$ so that a quasi-affine scheme is locally isomorphic to an affine scheme.

Definition 3.2. *A* scheme *is a locally ringed space which is locally isomorphic to a quasi-affine scheme. In other words, this is a locally ringed space* $(X, \mathcal{O}_X)$ *such that, for all* $x \in X$*, there exists an open neighbourhood* U *of* x *in* X *such that* $(U, \mathcal{O}_{X|U})$ *is isomorphic to a quasi-affine scheme.*

A morphism of schemes *is a morphism of locally ringed spaces.*

Remark 3.3. Recall from Lemma 2.16 that the distinguished open subsets form a basis of the topology of $\mathrm{Spec}(A)$, and that $(D(f), \mathcal{O}_{\mathrm{Spec}(A)|D(f)}) \simeq \mathrm{Spec}(A_f)$ is affine. This implies in particular that any quasi-affine subscheme can be covered by open affine subschemes which are affine. It follows that any scheme X has an open cover by affine open subschemes.

From now on we will change the notation for a scheme. If we denote a scheme by the letter X, we will denote by $|X|$ the underlying topological space and by $\mathcal{O}_X$ the structure sheaf over $|X|$. In particular if A is a ring, we will use the notation $\mathrm{Spec}(A)$ for the locally ringed space $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ so that $|\mathrm{Spec}(A)|$ is what was earlier denoted $\mathrm{Spec}(A)$. Similarly if X and Y are schemes and $f : X \to Y$ is a morphism of schemes we will denote by $|f|$ the underlying continuous map $|X| \to |Y|$ and by $f^\sharp$ the morphism of sheaves of rings $\mathcal{O}_Y \to |f|_*\mathcal{O}_X$.

If A is a ring and $n \geqslant 0$ an integer we denote by $\mathbb{A}^n_A$ the scheme $\mathrm{Spec}(A[X_1, \dots, X_n])$.

3.2 Subschemes

Definition 3.4. *Let* X *be a scheme. An* open subscheme *of* X *is a scheme* U *such that* $|U|$ *is an open subset of* $|X|$ *and* $\mathcal{O}_U$ *is the restriction of* $\mathcal{O}_X$ *to* $|U|$.

A morphism of schemes $j : Y \to X$ is an open immersion *if $|j|$ is a homeomorphism from $|Y|$ onto some open subset U of $|X|$ and if $j^\sharp$ induces an isomorphism from $\mathcal{O}_{X|U}$ onto $(|j|_*\mathcal{O}_Y)_{|U}$.*

Remark 3.5. The open subschemes of a scheme $(X, \mathcal{O}_X)$ correspond to the open subsets of $|X|$.

Definition 3.6. *A* closed immersion *is a morphism of schemes $i : Y \to X$ such that $|i|$ induces a homeomorphism from $|Y|$ onto a closed subset of $|X|$ and such that the morphism $i^\sharp : \mathcal{O}_X \to |i|_*\mathcal{O}_Y$ is surjective. We say that two closed immersions $i : Y \to X$ and $i' : Y' \to X$ are* equivalent *if there exists an isomorphism $\eta : Y' \to Y$ such that $i' = i \circ \eta$.*

A closed subscheme *of a scheme X is an equivalence class of closed immersions.*

A closed subscheme of a scheme X is thus the data of a closed subset $F \subset |X|$ and a sheaf of rings $\mathcal{O}_Y$ over F making $(F, \mathcal{O}_Y)$ a scheme, together with a surjective map $\theta : \mathcal{O}_X \to i_{F*}\mathcal{O}_Y$ such that (i_F, θ) is a morphism of schemes, where i_F denotes the inclusion of F in $|X|$.

Example 3.7. Let A be a ring and let X be the affine scheme $\mathrm{Spec}(A)$. Let I be an ideal of A. Then the map $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is a closed immersion. We identify $\mathrm{Spec}(A/I)$ with a closed subscheme of $\mathrm{Spec}(A)$. Namely, as the $D(f)$ form a basis of the topology of $|X|$, it is sufficient to check that, for any $f \in A$, the map $\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) \to \mathcal{O}_{\mathrm{Spec}(A/I)}(D(f) \cap V(I))$ is surjective, but this is the map $A_f \to (A/I)_{\overline{f}} \simeq A_f/IA_f$ obtained by localization.

Example 3.8. Let k be a field. Let $I = (X^2, XY) \subset A = k[X, Y]$ and let $Z = \mathrm{Spec}(A/I)$. Then Z is a closed subscheme of $\mathbb{A}^2_k$, whose underlying topological space is $V(I) = V(X)$. The only point $z \in Z$ such that $\mathcal{O}_{Z,z}$ is non reduced is the point $(0, 0)$ corresponding to (X, Y).

Example 3.9. Let X be a scheme. We define a new sheaf of rings over $|X|$ as the sheaf $\mathcal{O}_X^{\mathrm{red}}$ associated to the presheaf $U \mapsto \mathcal{O}_X(U)^{\mathrm{red}}$, where $\mathcal{O}_X(U)^{\mathrm{red}}$ denotes the reduced quotient of $\mathcal{O}_X(U)$. Let X^{red} be the scheme $(|X|, \mathcal{O}_X^{\mathrm{red}})$. The natural map $X^{\mathrm{red}} \to X$ is a closed immersion and X^{red} is called the *reduced closed subscheme* of X.

Definition 3.10. *A scheme X is* reduced *if its structure sheaf $\mathcal{O}_X$ is a sheaf of reduced rings, equivalently if $\mathcal{O}_X(U)$ is reduced for every open subset U.*

A scheme X is integral *if it is reduced and $|X|$ is irreducible.*

Exercice 3.1. Show that a scheme X is reduced if and only if $\mathcal{O}_{X,x}$ is a reduced ring for all $x \in |X|$.

Exercice 3.2. Show that a scheme X is integral if and only if for every non empty open subset U, the ring $\mathcal{O}_X(U)$ is a domain.

3.3 Construction of schemes by glueing

We call a *glueing data* the data $(\mathcal{X},\mathcal{U},\mathcal{F})$ where $\mathcal{X} = (X_i)_{i\in I}$ is a family of schemes, $\mathcal{U} = (U_{i,j} \subset X_i)_{(i,j)\in I^2}$ is a family of open subschemes and $\mathcal{F} = (\varphi_{i,j})_{(i,j)\in I^2}$ is a family of isomorphisms $\varphi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i}$ of schemes such that:

(i) for $i = j$, $U_{i,i} = X_i$ and $\varphi_{i,i} = \mathrm{Id}_{X_i}$;

(ii) for $i, j, k \in I$, we have $\varphi_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$ and $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ on $U_{i,j} \cap U_{i,k}$.

We define a *glueing* of $(\mathcal{X},\mathcal{U},\mathcal{F})$ to be a pair $(X,\mathcal{I})$ where X is a scheme and $\mathcal{I} = (\psi_i)_{i\in I}$ is a family of open immersions $\psi_i : X_i \to X$ such that $|X| = \bigcup_{i\in I}|\psi_i|(|X_i|)$, such that $|\psi_i|(|U_{i,j}|) = |\psi_i|(|X_i|) \cap |\psi_j|(|X_j|)$ for all $(i,j) \in I^2$ and such that $\psi_j \circ \varphi_{i,j} = \psi_{i|U_{i,j}}$ for all $(i,j) \in I^2$. An *isomorphism* between two glueings $(X,(\psi_i))$ and $(X',(\psi'_i))$ is an isomorphism of schemes $f : X \to X'$ such that $\psi'_i = f \circ \psi_i$ for all $i \in I$.

Theorem 3.11. *Let $(\mathcal{X},\mathcal{U},\mathcal{F})$ be a glueing data. Then there exists a glueing of $(\mathcal{X},\mathcal{U},\mathcal{F})$, unique up to unique isomorphism.*

Proof. We need to construct the topological space $|X|$ and the sheaf $\mathcal{O}_X$. The topological space $|X|$ is the quotient of the disjoint union of the $|X_i|$ by the equivalence relation for which $x \in |X_i|$ and $y \in |X_j|$ satisfy $x \sim y$ if and only if $x \in |U_{i,j}|$ and $y = \varphi_{i,j}(x)$. This is a topological space with open immersions $|\psi_i| : |X_i| \hookrightarrow |X|$. If U is an open subset of $|X|$, we define

$$\mathcal{O}_X(U) = \left\{(s_i) \in \prod_i \mathcal{O}_{X_i}(|\psi_i|^{-1}(U)) \mid \forall (i,j) \in I^2,\ \varphi^\sharp_{i,j}(s_{j|U_{j,i}}) = s_{i|U_{i,j}}\right\},$$

where the restrictions are taken to $U_{j,i}\cap|\psi_j|^{-1}(U)$ and $U_{i,j}\cap|\psi_i|^{-1}(U)$ respectively. Then $\mathcal{O}_X$ is a sheaf and there is an isomorphism of sheaves $\mathcal{O}_{X||\psi_i|(|X_i|)} \simeq \mathcal{O}_{X_i}$ so that $X = (|X|, \mathcal{O}_X)$ is a scheme. We leave as an exercise that it satisfies the expected properties.

The uniqueness of the glueing is a consequence of Proposition 3.13 below. □

Lemma 3.12. *Let X and Y be schemes and let $(V_i)_{i\in I}$ be an open cover of $|X|$, each V_i being seen as an open subscheme of X. For $i \in I$, let $g_i : V_i \to Y$ be a morphism of schemes and assume that $g_{i|V_i\cap V_j} = g_{j|V_i\cap V_j}$ for all $(i,j) \in I^2$. Then there exists a unique morphism of schemes $g : X \to Y$ such that $g_{|V_i} = g_i$ for all $i \in I$.*

Proof. The continuous maps $|g_i|$ coincide on the intersections $V_i \cap V_j$, so that there is a unique continuous map $|g| : |X| \to |Y|$ such that $|g|_{|V_i} = |g_i|$ for all $i \in I$. Let $W \subset |Y|$ be an open subset and let $s \in \mathcal{O}_Y(W)$. The open subsets $|g|^{-1}(W) \cap V_i = |g_i|^{-1}(W)$ cover $|g|^{-1}(W)$ and the sections $g^\sharp_{i,W}(s) \in \mathcal{O}_X(|g_i|^{-1}(W))$ coincide on the intersections, by the assumption on the g_i. As $\mathcal{O}_X$ is a sheaf, there is a unique section $g^\sharp_W(s) \in \mathcal{O}_X(|g|^{-1}(W))$ whose restriction to $|g_i|^{-1}(W)$ is $g^\sharp_{i,W}(s)$ for all $i \in I$. One checks that $g^\sharp_W$ is a ring homomorphism and that $g^\sharp$ is compatible with the restriction maps, so that $(|g|, g^\sharp)$ is a morphism of ringed spaces. Finally, if $x \in V_i$, the ring homomorphism $g^\sharp_x$ coincides with $(g_i)^\sharp_x$, as a stalk only depends on the restriction of the sheaves to V_i; it is therefore a local homomorphism and g is a morphism of schemes. The uniqueness is clear, as $|g|$ and the sections $g^\sharp_W(s)$ are determined by their restrictions to the members of an open cover. □

Proposition 3.13. *Let $((X_i)_{i\in I}, (U_{i,j}), (\varphi_{i,j})_{(i,j)\in I^2})$ be a glueing data and let $(X, (\psi_i))$ be a glueing of this data. For any scheme Y the map $f \mapsto (f \circ \psi_i)_i$ induces a bijection from the set* $\mathrm{Hom}(X, Y)$ *to the set*

$$\{(f_i)_{i\in I} \in \prod_i \mathrm{Hom}(X_i, Y) \mid \forall (i,j) \in I^2,\ f_{i|U_{i,j}} = f_{j|U_{j,i}} \circ \varphi_{i,j}\}.$$

Proof. Let us first check that the map is well defined. Let $f \in \mathrm{Hom}(X, Y)$ and let $f_i = f \circ \psi_i$. By definition of a glueing we have $\psi_j \circ \varphi_{i,j} = \psi_{i|U_{i,j}}$, so that

$$f_{j|U_{j,i}} \circ \varphi_{i,j} = f \circ \psi_j \circ \varphi_{i,j} = f \circ \psi_{i|U_{i,j}} = f_{i|U_{i,j}}$$

for all $(i,j) \in I^2$, and the family $(f_i)_{i\in I}$ indeed belongs to the set on the right hand side.

Let $V_i = |\psi_i|(|X_i|)$, seen as an open subscheme of X. As ψ_i is an open immersion, it induces an isomorphism of schemes $X_i \xrightarrow{\sim} V_i$, which we still denote ψ_i. By definition of a glueing, the V_i form an open cover of $|X|$ and $\psi_i(U_{i,j}) = V_i \cap V_j$; moreover the relation $\psi_j \circ \varphi_{i,j} = \psi_{i|U_{i,j}}$ can be rewritten as $\varphi_{i,j} = \psi_j^{-1} \circ \psi_i$ on $U_{i,j}$.

Let us prove that the map is injective. If $f, g \in \mathrm{Hom}(X, Y)$ satisfy $f \circ \psi_i = g \circ \psi_i$ for all $i \in I$, then $f_{|V_i} = g_{|V_i}$ for all $i \in I$, so that $f = g$ by the uniqueness in Lemma 3.12.

Let us finally prove that the map is surjective. Let $(f_i)_{i\in I}$ be an element of the set on the right hand side and let $g_i = f_i \circ \psi_i^{-1} : V_i \to Y$. Let $(i,j) \in I^2$. We have $\psi_i^{-1}(V_i \cap V_j) = U_{i,j}$ and, on $V_i \cap V_j$,

$$g_j = f_j \circ \psi_j^{-1} = f_j \circ \varphi_{i,j} \circ \psi_i^{-1} = f_i \circ \psi_i^{-1} = g_i,$$

where we have used the relation $\varphi_{i,j} = \psi_j^{-1} \circ \psi_i$ and the compatibility of the family $(f_i)_{i\in I}$. By Lemma 3.12 there exists a morphism of schemes $f : X \to Y$ such that $f_{|V_i} = g_i$ for all $i \in I$, and then $f \circ \psi_i = g_i \circ \psi_i = f_i$. □

Example 3.14. Take $X_1 = X_2 = \mathbb{A}^1_k$ and $U_{1,2} = U_{2,1} = \mathbb{A}^1_k \smallsetminus V(T)$. Choose $\varphi_{1,2} = \mathrm{Id}$. Then we obtain a new scheme X which is the affine line with a double point at the origin. This is not a very interesting scheme.

Keeping the same choices for $X_1, X_2, U_{1,2}, U_{2,1}$ but taking for $\varphi_{1,2}$ the isomorphism of $U = \mathrm{Spec}(k[T, T^{-1}])$ induced by $T \mapsto T^{-1}$, the glued scheme X is called the projective line and denoted $\mathbb{P}^1_k$. This is a much more interesting scheme which will be studied later. We have $|\mathbb{P}^1_k| = |\mathbb{A}^1_k| \cup \{\infty\}$ obtained by adding a point at ∞.

Exercice 3.3. Show that $\mathcal{O}_{\mathbb{P}^1_k}(|\mathbb{P}^1_k|) = k$.

Exercice 3.4. What should be $\mathbb{P}^2_k$?

Proposition 3.15. *Let X be a scheme and let $Y = \mathrm{Spec}(A)$ be an affine scheme. Then the map $f \mapsto f^\sharp_{|Y|}$ from $\mathrm{Hom}(X, Y)$ to $\mathrm{Hom}_{rings}(A, \mathcal{O}_X(|X|))$ is bijective.*

Proof. Note first that if $V \subset |X|$ is an open subset and if $f : X \to Y$ is a morphism of schemes, then $(f_{|V})^\sharp_{|Y|}$ is the composite of $f^\sharp_{|Y|}$ with the restriction map $\mathcal{O}_X(|X|) \to \mathcal{O}_X(V)$. This follows directly from the definitions.

Assume first that X is affine, say $X \simeq \mathrm{Spec}(B)$. Then $\mathcal{O}_X(|X|) \simeq B$ by Proposition 2.22 and the map of the statement is the map β of the proof of Theorem 2.28, which is bijective.

We now treat the general case. By Remark 3.3 we can choose an open cover $(V_i)_{i\in I}$ of $|X|$ by affine open subschemes.

Let us prove that the map is injective. Let $f, g \in \mathrm{Hom}(X, Y)$ be such that $f^\sharp_{|Y|} = g^\sharp_{|Y|}$. By the above we have $(f_{|V_i})^\sharp_{|Y|} = (g_{|V_i})^\sharp_{|Y|}$ for all $i \in I$, so that $f_{|V_i} = g_{|V_i}$ by the affine case. By the uniqueness in Lemma 3.12 we conclude that $f = g$.

Let us prove that the map is surjective. Let $\varphi : A \to \mathcal{O}_X(|X|)$ be a ring homomorphism and, for $i \in I$, let φ_i be the composite of φ with the restriction map $\mathcal{O}_X(|X|) \to \mathcal{O}_X(V_i)$. By the affine case there exists a unique morphism of schemes $f_i : V_i \to Y$ such that $(f_i)^\sharp_{|Y|} = \varphi_i$. Let $(i, j) \in I^2$ and let $W \subset V_i \cap V_j$ be an affine open subscheme. Then $(f_{i|W})^\sharp_{|Y|}$ and $(f_{j|W})^\sharp_{|Y|}$ are both equal to the composite of φ with the restriction map $\mathcal{O}_X(|X|) \to \mathcal{O}_X(W)$, so that $f_{i|W} = f_{j|W}$ by the affine case. As $V_i \cap V_j$ is covered by such open subschemes W, by Remark 3.3 applied to the scheme $V_i \cap V_j$, Lemma 3.12 gives $f_{i|V_i\cap V_j} = f_{j|V_i\cap V_j}$. Using Lemma 3.12 again, there exists a morphism of schemes $f : X \to Y$ such that $f_{|V_i} = f_i$ for all $i \in I$. Finally $f^\sharp_{|Y|}$ and φ are two ring homomorphisms $A \to \mathcal{O}_X(|X|)$ whose composites with the restriction map $\mathcal{O}_X(|X|) \to \mathcal{O}_X(V_i)$ are both equal to φ_i. As $(V_i)_{i\in I}$ is an open cover of $|X|$ and $\mathcal{O}_X$ is a sheaf, we conclude that $f^\sharp_{|Y|} = \varphi$. $\square$