Pure mathematics

Néron models of algebraic tori

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Notation

In what follows K is a *local field*, i.e., K is a complete non-archimedean discretely-valued field. We will write $\mathscr{O}_K$ for its valuation ring and π_K is a uniformizer in its maximal ideal $\mathfrak{m}$. We will write $j_K\colon \operatorname{Spec} K \hookrightarrow \operatorname{Spec} \mathscr{O}_K$ for the canonical open immersion. Finally, let $k := \mathscr{O}_K/\mathfrak{m}$ be the residue field of $\mathscr{O}_K$ and let $i\colon \operatorname{Spec} k \hookrightarrow \operatorname{Spec} \mathscr{O}_K$ be the corresponding closed immersion. When k is not perfect, its characteristic $\operatorname{char} k$ is positive and is denoted by p.

We will write K^{nr} for the maximal unramified extension of K inside a fixed separable closure K^{sep} of K. Then K^{nr} is again a local field and its valuation ring $\mathscr{O}_{K^{\mathrm{nr}}}$ is a strict Henselianization of $\mathscr{O}_K$ which will be denoted by $\mathscr{O}_K^{\mathrm{sh}}$.

When we consider the stalks for abelian sheaves on the étale site over $\operatorname{Spec} \mathscr{O}_K$, we use the notations $\overline{\eta}$ for a geometric point over $\operatorname{Spec} K$ and $\overline{s}$ for a geometric point over $\operatorname{Spec} k$ and identify the limit over étale neighborhoods of $\overline{\eta}$ with $\operatorname{Spec} K^{\mathrm{sep}}$ and the limit over étale neighborhoods of $\overline{s}$ with $\operatorname{Spec} \mathscr{O}_K^{\mathrm{sh}}$.

For a finite separable extension L/K, which is always assumed to be a subextension of K^{sep}/K, L is again a local field in the above sense. We write $L_{\mathrm{nr}} = L \cap K^{\mathrm{nr}}$ for the maximal subextension of L which is unramified over K. Note that $L^{\mathrm{nr}} = LK^{\mathrm{nr}}$ is the maximal unramified extension of L inside K^{sep} and restriction-to-K^{nr} induces an isomorphism $\operatorname{Gal}(L^{\mathrm{nr}}/L) \cong \operatorname{Gal}(K^{\mathrm{nr}}/L_{\mathrm{nr}})$. We denote the inertia subgroup of $\operatorname{Gal}(L/K)$ by $I_{L/K}$. This is the kernel of the canonical map $\operatorname{Gal}(L/K) \longrightarrow \operatorname{Aut}_k(l)$. The inertia subgroup of $\operatorname{Gal}(K^{\mathrm{sep}}/K)$ is denoted by I_K. If there is no risk of confusion, we will just write I for I_K.

For a scheme S, we denote the fiber of an S-scheme T at a point $s \in S$ by T_s. A base change of a scheme T over an affine base $\operatorname{Spec} R$ will be written as a tensor product $T \otimes_R R'$. For a $\operatorname{Spec} \mathcal{O}_K$-scheme $\mathcal{T}$ we denote the generic and the special fiber as $\mathcal{T}_K$ and $\mathcal{T}_k$, respectively. Similarly, a base change from $\operatorname{Spec} K$ to $\operatorname{Spec} L$ is usually only indicated with the index $\cdot_L$ and we write $\mathfrak{R}_{R'/R}(\cdot)$ for the Weil restriction from $\operatorname{Spec} R'$ to $\operatorname{Spec} R$.

As usual, $\mathbb{G}_{m,U}$ denotes the multiplicative group over the scheme U. If U is affine, say $\operatorname{Spec} R$, then we just write $\mathbb{G}_{m,R}$. For an algebraic K-torus T we denote the character group $\operatorname{Hom}_{K^{\mathrm{sep}}-\mathrm{grp}}(T_{K^{\mathrm{sep}}}, \mathbb{G}_{m,K^{\mathrm{sep}}})$ by $X(T)$. This is a continuous $\operatorname{Gal}(K^{\mathrm{sep}}/K)$-module. If we view the character group as an abelian sheaf, then we write $\underline{X}(T)$ for it. This sheaf is more precisely the sheaf of rational characters.

In this work, sheaves are always abelian sheaves. By the étale site we mean the small étale site as defined in [M, II,§1]. The smooth site over a scheme X consists of the category of smooth X-schemes with surjective families of smooth morphisms as covers.

We will denote Néron models by calligraphic letters. So if S is a Dedekind scheme and η is the scheme of the generic fibers of S, then the Néron model of a smooth, separated algebraic η-group G_η is denoted by $\mathcal{G}$. For a point $s \in S$, the group of components of the smooth $s = \operatorname{Spec} k(s)$-scheme $\mathcal{G}_s$ is denoted by $\Phi(\mathcal{G}_s)$.

If G_η is, moreover, commutative, we can also consider the Néron model as an abelian sheaf on the smooth or étale site over S. This sheaf can canonically be identified with $j_* G_\eta$, where $j\colon \eta \longrightarrow S$ is the canonical open immersion.

However, for an algebraic K-torus T, we will abbreviate the group of components $\Phi(\mathcal{T}_k)$ of the special fiber of its Néron model by $\Phi(T)$. This should not lead to any confusion, since a K-torus is always connected as a scheme over K.

Introduction

Let K be a local field, by which we mean a discrete and non-Archimedean valued complete field [1]. In this work, we study (lft-)Néron models of algebraic K-tori over $\operatorname{Spec}\mathcal{O}_K$. In particular, their groups of components will be described. Since group of components are defined fiber-by-fiber and are invariant under completion, our descriptions also extend to global Néron models since we do not impose any restrictions on the residue field.

Our interest in this problem comes from the fact that Néron models of algebraic tori are among the basic building blocks of Néron models in general, because every commutative algebraic K-group scheme can be written as a successive extension of group schemes which are abelian varieties, unipotent group schemes or group schemes of multiplicative type. Further, algebraic tori appear in the rigid-analytic uniformization of abelian varieties, whence Néron models of algebraic tori can also be helpful in the description of Néron models of abelian varieties (see, e.g., [BX, §5]).

Now let T be an algebraic K-torus. A Néron model $\mathcal{T}$ of T over $\mathcal{O}_K$ always exists. Let Φ be the group of components of the special fiber $\mathcal{T}_k$. If k is perfect, Xavier Xarles was able to give a description of Φ in his paper [X]. In [X, Theorems 2.1 and 3.1] he defined natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z}) &\cong \mathrm{H}^0(I,X)\\ \operatorname{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z}) &\cong \mathrm{H}^1(I,X)\\ \Phi &\cong \operatorname{coker}\big(\operatorname{Hom}_{\mathbb{Z}}(X',\mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\big(M^I,\mathbb{Z}\big)\big)\end{aligned}$$

where X is the character group of T, I is the inertia group of $\operatorname{Gal}(K^{\mathrm{sep}}/K)$ and M and X' can be determined from an I-acyclic and torsion-free resolution of X (see loc. cit.).

Xarles proves this description, which generalizes the results of L. Bégueri [Be, Theorems 7.2.1 and 7.2.2], using cohomological methods. He interprets the Néron model as a sheaf j_*T on the étale and smooth sites over $\mathcal{O}_K$ and shows that R^1j_*T is trivial as an étale sheaf and also as a smooth sheaf if T has multiplicative reduction. In this way he obtains from a short exact sequence of algebraic tori a short exact sequence of their Néron models. He then applies the functor $\underline{\operatorname{Hom}}(\cdot,i_*\mathbb{Z})$ to the sequence of Néron models.

Now there exists a canonical isomorphism for $i=0$ in the étale topology and for $i=0,1$ in the smooth topology

$$\mathrm{R}^i\underline{\operatorname{Hom}}(\mathcal{T},i_*\mathbb{Z}) \cong \mathrm{R}^i\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z}).$$

Thus Xarles obtains a long exact sequence for the free parts $\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z})$ and the torsion parts $\operatorname{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z})$ of the groups of components of a given torus. Using such sequences he can reduce his proof of the general case to the cases where T has multiplicative reduction or has the form

[1] In algebraic number theory and arithmetic geometry, a local field is generally defined more restrictively : it is required that the residue field k of $\mathcal{O}_K$ be perfect or even finite. However, we are particularly interested in the case where k is not perfect.

$T = \mathfrak{R}_{L/K}(\mathbb{G}_{m,L})$.

In general, the above description is not valid in the case of an imperfect residue field. This is reflected in Xarles' proof via the fact that in the imperfect residue field case the étale sheaf $\mathrm{R}^1 j_* T$ is certainly not trivial.

We will establish the following statements in the general situation. The description in [X, Theorem 3.1] applies to algebraic tori which are split by a tamely ramified extension. Further, the validity of the description in [X, Theorem 3.1] is compatible with the formation of Weil restrictions along finite separable extensions of K. The description of the free part in [X, Theorem 2.1] remains valid for algebraic tori which are split by a non-residually ramified extension. For these tori, the description of the free part is also functorial, i.e. compatible with homomorphisms. The prime-to-p part of the group of components can generally be written as in [X, Theorem 3.1], i.e., the isomorphism given there holds true in the category of continuous $\mathbb{Z}\left[p^{-1}\right][G_k]$-modules.

One can construct algebraic tori that split only over a residually ramified extension and provide counterexamples to the claims of [X, Theorem 2.1]. In general, the free part can be described as an extension of a finite p-group by $X(T)^I$. The torsion part of the group of components is always annihilated by the order of the inertia group of a splitting extension. The same estimate also applies to $\mathrm{H}^1(I, X(T))$. For norm-one tori with respect to finite cyclic Galois extensions, the torsion part of the group of components is bounded.

Our investigations of the Néron models of algebraic tori are structured as follows.

In Chapter 0 we cover some basics. We explain local and global Néron models. We show that for a scheme S and a smooth and commutative S-group scheme G, the group of components of a fiber G_s with $s \in S$ can already be determined via S by those of G and of the identity component G^0 on the smooth or the étale site.

Further, we repeat definitions and properties of diagonalizable group schemes and of group schemes of multiplicative type. We consider Cartier duality, with which we can describe Weil restrictions of algebraic tori on the character groups as an induction of Galois modules. We also prove that Cartier duality converts short exact sequences of torsion-free character groups into short exact sequences with respect to the smooth or étale topology. Finally, we explain Xarles' proof [X] in greater detail.

In Chapter 1 we consider some specific algebraic tori for which we can determine their Néron models quite explicitly: the case of algebraic tori with multiplicative reduction can be reduced to the construction of the Néron model of $\mathbb{G}_{m,K}$ using Galois descent. For Weil restrictions of algebraic tori, we show that Weil restriction is compatible with the formation of the identity component of the Néron model, whence the same holds for the group of components. Finally, we construct the special fiber of the Néron model of a norm-one torus with respect to a cyclic Galois extension of prime degree. This generalizes a computation from [LL, §5] and yields the first counterexamples to a generalization of the description from [X].

In Chapter 2 we go back to the general situation and show that the group of components of a local Néron model $\mathcal{G}$ of a smooth and commutative algebraic K-group G_K is finitely generated. This answers a question of Lorenzini's [LL, §1.3]. To do this, we show that from a short exact sequence of Néron models (in the smooth topology over $\operatorname{Spec} \mathcal{O}_K$)

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3 \longrightarrow 0$$

we obtain a short exact sequence of groups of components

$$0 \longrightarrow \widetilde{\Phi} \longrightarrow \Phi((\mathcal{G}_2)_k) \longrightarrow \Phi((\mathcal{G}_3)_k) \longrightarrow 0,$$

where $\widetilde{\Phi}$ is a quotient of $\Phi((\mathcal{G}_1)_k)$. The finite generation statement is obtained by considering the short exact sequence defined by the embedding of the maximal torus with multiplicative reduction in G_K. This sequence induces a short exact sequence of the corresponding lft-Néron models. With the argument above, $\Phi(\mathcal{G}_k)$ is then an extension of a finite group by a finitely generated torsion-free group, i.e., it is finitely generated. With the construction of the Néron model of a subgroup we see that in the above situation $\widetilde{\Phi}$ must be a quotient of a finite subgroup of $\Phi((\mathcal{G}_1)_k)$.

In Chapter 3 we deal with integral models of algebraic tori. These are $\mathcal{O}_K$-models of K-tori which are flat and separated $\mathcal{O}_K$-groups. This class includes the ft-Néron model of Chai and Yu [ChYu], but also the standard model considered by Moroz, Voskresenskii, Kunyavskii and Popov. In order to make this literature useful, we include these models into the theory of Néron models.

Since for an lft-Néron model $\mathcal{G}$ of a smooth and commutative algebraic K-group the group $\Phi\left(\mathcal{G}_k\right)_{\mathrm{tors}}$ is finite, we can find a smooth open subgroup $\mathcal{G}^{\mathrm{ft}} \subseteq \mathcal{G}$ of finite type over $\mathcal{O}_K$ whose special fiber is exactly that which contains the connected components that induce the torsion part of the group of components.

We define this subgroup $\mathcal{G}^{\mathrm{ft}}$ as the ft-Néron model and show that it has a lifting property for certain étale points as well as mapping property similar to the Néron mapping property. Further, $\mathcal{G}^{\mathrm{ft}}$ is compatible with étale base changes and the formation of Weil restrictions. For algebraic tori, our definition is of course consistent with that of Chai and Yu.

The standard model of a torus T was introduced by Voskresenskii et al. and identified with the schematic closure of T under the embedding

$$T \hookrightarrow \mathfrak{R}_{L/K}(T_L) \cong \mathfrak{R}_{L/K}\left(\mathbb{G}_{m,L}^d\right) \hookrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\left(\mathbb{G}_{m,\mathcal{O}_L}^d\right)$$

for a splitting extension L/K of T [VKM, §5, Proposition 6]. Therefore, its smoothing is equal to the ft-Néron model. Using an idea from [Edi], we derive a criterion to decide when a monomorphism of algebraic tori induces a closed immersion of their Néron models. For an algebraic torus T we can identify the ft-Néron model with the étale sheaf $\underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})$, which gives us a deeper understanding of Xarles' description of tori with multiplicative reduction. In summary, we can say that the ft-Néron models already describe the identity component and the torsion of the group of components of the lft-Néron model and are therefore useful because they are, in principle, easier to determine.

After this digression, in Chapter 4 we analyze the sheaf $\mathrm{R}^1 j_*T$ in the étale and smooth topologies. This sheaf is always a $p = \mathrm{char}(k)$-primary torsion sheaf. If T splits over a tamely ramified extension, then $\mathrm{R}^1 j_*T = 0$. In addition, we describe the functors j_* and $\mathrm{R}^1 j_*$ for étale groups. In Chapter 5 we generalize approaches from [X], [BX] and [LL]. We show that, for $i = 0$ and 1, in the smooth topology we have

$$\mathrm{R}^i\underline{\mathrm{Hom}}(\mathcal{T}, i_*C) \cong \mathrm{R}^i \mathrm{Hom}_{\mathbb{Z}}(\Phi, C)$$

if C is a constant, torsion-free abelian sheaf. In analogy to [X], we examine the torsion-free part of the group of components, which we can only determine via an exact sequence

$$0 \longrightarrow X(T)^I \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z}) \longrightarrow E(T) \longrightarrow 0,$$

where $E(T)$ is a finitely generated p-primary torsion module, which we refer to as the defect term.

The defect term can be written as a group of components of a subset of $\mathrm{R}^1 j_*T'$ for a suitable torus T'. As abelian groups, $\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ and $X(T)^I$ remain isomorphic, but can support non-isomorphic Galois module structures. This means that the maps of the free parts of the group of components induced by homomorphisms of algebraic tori can only be described using the above sequences.

We also consider the possibilities of constructing exact sequences of group of components from an exact sequence of Néron models of algebraic tori. By refining the results from the second chapter, we generalize an idea contained in the proof of [X, Theorem 3.1] and can further describe the group of components of norm-one tori, showing that the defect terms for algebraic tori that split over a non-residually ramified extension are trivial.

In the last chapter we provide a description of the group of components as far as possible. For algebraic tori T which split over a tamely ramified extension, we can transfer the results from [X] to the smooth topology because $\mathrm{R}^1 j_* T = 0$. For algebraic tori that split after a non-residually ramified extension, the description of the free part is still valid. Since in this case the Néron model is no longer an exact functor, we can only see the finite part as an extension

$$0 \longrightarrow \mathrm{H}^1(I, X(T)) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\big(\Phi\big(\mathrm{R}^1 j_* T'\big), \mathbb{Z}\big) \longrightarrow 0$$

for a suitable K-torus T'.

Since $\mathrm{R}^1 j_* T$ and $E(T)$ are always p-primary torsion sheaves, it is reasonable to assume that that part of the group of components which consists of prime-to-p-torsion elements does not change. In fact, in general we can use the description from [X, Theorem 3.1] in the category of $\mathbb{Z}\big[p^{-1}\big][G_k]$-modules by applying Xarles' proof using the functor $\underline{\mathrm{Hom}}\big(\cdot, i_* \mathbb{Z}\big[p^{-1}\big]\big)$.

However, in the category of $\mathbb{Z}\big[p^{-1}\big][G_K]$-modules, not only the p-primary torsion of Φ is annihilated; also the isomorphism classes of the Galois structures become larger, since isomorphisms with coefficients in $\mathbb{Z}\big[p^{-1}\big]$ are now allowed.

Using norm-one tori, we give explicit examples of algebraic tori where the free part and $X(T)^I$ carry non-isomorphic Galois module structures.

For a complete description of the group of components, we still lack information about the p-primary torsion component. Similarly to the situation with Néron models of abelian varieties [ELL, Theorem 1], unfortunately we can only show that the p-primary torsion component of Φ is annihilated by the maximal power of p that divides the order of $I_{L/K}$, where L/K is a splitting extension of T. The same estimate also applies to $\mathrm{H}^1(I, X(T))$. In all the examples we know, the above estimate is not optimal.

The following conjecture seems plausible to the author: the torsion part of Φ is smaller, i.e., it is isomorphic to $\mathrm{H}^1(I, X(T))/E'$, where E' is some p-primary torsion module. In particular, counterexamples to the description in [X, Theorem 3.1] only occur in the presence of residual ramification.

This conjecture would be analogous to observations by Dino Lorenzini in the case of groups of components of Néron models of Jacobian varieties.

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Chapter 0

Basics

This chapter explains some terms and constructions needed to describe the Néron model of an algebraic torus and its group of components. We define Néron models and explain the connections between global and local Néron models. We also mention the most important existence statements for Néron models and lft-Néron models.

We outline the construction of the group of components of a smooth group scheme. We consider the group of components as a scheme and, in the case of a commutative group scheme, also as a smooth and as an étale sheaf.

We define group schemes of multiplicative type, in particular algebraic tori, and cite important properties of these group schemes. We consider here in particular the so-called Cartier duality: for a connected, normal and locally Noetherian scheme S with a geometric point $\bar{s}$, Cartier duality induces an antiequivalence between the category of algebraic S-tori and the category of continuous, finitely generated and torsion-free $\pi_1(S, \bar{s})$-modules.

In the case that the base S is the spectrum of a field, we show that Cartier duality transforms the Weil restriction functor into the induction of π_1-modules. Using Cartier duality, we construct exact sequences of algebraic tori in the smooth and étale topologies, which will be needed later for determining the groups of components.

Finally, we give an overview of the work [X], in which the group of components of the Néron model of an algebraic torus is described in the case of a local field with a perfect residue field.

0.1 Néron models

Let S be a Dedekind scheme, that is, a Noetherian normal scheme of dimension ≤ 1. The local rings of S are fields or discrete valuation rings. If S itself is a local scheme, i.e., the spectrum of a local ring, we speak of the local case. Otherwise we speak of the global case. The scheme S splits into a finite number of irreducible components S_i, whose generic points are denoted by η_i. We call $\eta := \mathrm{Spec}(\oplus k(\eta_i))$ the scheme of the generic points of S. By definition, we have an open immersion $j\colon \eta \longrightarrow S$. Using these notations, we can define an lft-Néron model:

Definition 0.1.1. Let G_η be a smooth and separated η-scheme of finite type. A Néron model of G_η is an S-model $\mathcal{G}$ of G_η that is smooth, separated and of finite type and has the following property, called the Néron mapping property:

For all smooth S-schemes Y and every η-morphism $\phi_\eta\colon Y_\eta \longrightarrow G_\eta$ there exists exactly one S-morphism $\phi\colon Y \longrightarrow \mathcal{G}$ which extends ϕ_η.

An S-model $\mathcal{G}$ of G_η that satisfies Néron's map property and is separated and smooth but only locally of finite type is called an lft-Néron model of G_η.

We will also refer to a Néron model in the local case as a local Néron model. Similarly, in the global case we speak of global Néron models. It follows from [BLR, Proposition 1.2.4] that global Néron models are composed of local Néron models. More precisely, for every closed point $s \in S$ the $\mathcal{O}_{S,s}$-scheme $\mathcal{G} \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is a Néron model of its generic fiber. On the other hand, [BLR, Proposition 1.4.1] states that a global Néron model exists if, and only if, a global Néron model exists over an open and dense subscheme $S' \subset S$ and the local Néron models exist at the finitely many closed points in $S - S'$. By glueing these models together we obtain the Néron model over S.

In the case of an η-group scheme G_η, the Néron model is an S-group scheme by the uniqueness of the lifting. A smooth and commutative group scheme G_η can be understood as a sheaf on the smooth and étale sites over η and its Néron model, if it exists, represents the sheaf j_*G_η on the smooth and étale sites over S. In the case of the smooth site, the Néron model as a scheme is clearly determined by this sheaf, since the Néron model is contained as a smooth scheme in the site.

In this work we will limit ourselves to local lft-Néron models. More precisely, we will consider the case where $\eta = \operatorname{Spec} K$ for a local field K and consider η as the generic fiber of $S \coloneqq \operatorname{Spec} \mathcal{O}_K$. In the local case, it is known under which conditions Néron models exist.

Theorem 0.1.2. [BLR, Theorem 1.3.1] *Let R be a discrete valuation ring with quotient field K and strict Henselianization R^{sh} and let K^{nr} be the quotient field of R^{sh}. Let G_K be a smooth K-group scheme of finite type. Then there exists a Néron model $\mathcal{G}$ of G_K over* $\operatorname{Spec} R$ *if, and only if, $G_K(K^{\mathrm{nr}})$ is bounded in G_K.*

Consequently, Néron models always exist for smooth K-group schemes that are proper, e.g., for abelian varieties. In the above theorem, the restriction to models of finite type is very important. For lft-Néron models, a full solution to the question of existence has so far only been achieved for commutative group schemes.

One can explicitly construct an lft-Néron model for the multiplicative group $\mathbb{G}_{m,K}$ [BLR, 10.1.5] and show that the additive group $\mathbb{G}_{a,K}$ cannot have a Néron model [BLR, 10.1.8]. Using Descent and an explicit consideration of anisotropic tori and wound unipotent groups, one can then show that a smooth and commutative K-group scheme G_K of finite type has an lft-Néron model over $\mathcal{O}_K$ if, and only if, $G_K \otimes_K K^{\mathrm{nr}}$ does not have a subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$ [BLR, 10.2.2]. This lft-Néron model is of finite type, i.e., a Néron model, if, in addition, $G_K \otimes_K K^{\mathrm{nr}}$ does not contain a subgroup of the form $\mathbb{G}_{m,K^{\mathrm{nr}}}$ [BLR, Theorem 10.2.1]. Here, a subgroup U of G_K is always a closed subgroup, because a subgroup, as a subscheme, is an open subscheme in a closed subscheme. Thus U is also an open subscheme in its closure $\bar{U}$. This is again a subgroup of G_K. But now an open subgroup over a field is already closed, so that $U = \bar{U}$ follows.

0.2 The group of components of a smooth scheme

In [SGA3, VIa and VIb] the identity component of a smooth group scheme is defined. To review this, let S be a scheme and let G be an S-group scheme. The identity component G^0 is defined in [SGA3, VIb, Definition 3.1] as a subgroup functor

$$U/S \quad \rightsquigarrow \quad G^0(U) \coloneqq \{ u \in G(U) \mid \forall s \in S, u_s(U_s) \subset G_s^0 \},$$

where G_s^0 is the identity component of $G_s \coloneqq G \times_S \operatorname{Spec} k(s)$ as a $k(s)$-group scheme [SGA3, VIa, §2].

If G is a smooth S-group scheme, it follows from [SGA3, VIb, Theorem 3.10] that G^0 is represented by an open and smooth subgroup. This is the union of the identity components of the fibers and is of finite type over S (see loc. cit. 3.4-3.6). The identity component is fiber-wise geometrically irreducible [SGA3, VIa, Proposition 2.4], i.e., all fibers G_s^0 are geometrically irreducible.

Now let $s \in S$ be a point. In the fiber above s we have the group of components of G_s, i.e., the quotient $\Phi(G_s) := G_s/G_s^0$. This quotient $\Phi(G_s)$ is represented by an étale $k(s)$-scheme [SGA3, VIa, 5.5] and the associated morphism $G_s \longrightarrow \Phi(G_s)$ is flat and surjective [SGA3, VIa, Theorem 3.2].

The above quotient can be described in the smooth and the étale topologies, provided we consider commutative group schemes G:

If G is commutative, one can form an exact sequence of abelian sheaves on the smooth site over S

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow \Phi(G) \longrightarrow 0. \tag{1}$$

For a point $i\colon s \hookrightarrow S$, i^* is an exact functor since difference kernels exist on the smooth site (cf. [M, II, 1.13 and 2.6]). By [M, II, 3.1(d)], i^*G^0 and i^*G are represented by G_s^0 and G_s, respectively, because G^0 and G are smooth group schemes. So we obtain an exact sequence

$$0 \longrightarrow G_s^0 \longrightarrow G_s \longrightarrow i^*\Phi(G) \longrightarrow 0.$$

In the fpqc topology, the quotient G_s/G_s^0 is represented by the étale group scheme $\Phi(G_s)$. The restriction from the fpqc topology to the smooth topology is left exact, so that $i^*\Phi(G)$ is a subsheaf of $\Phi(G_s)$. On the other hand, G_s^0 is a smooth scheme, so the morphism $G_s \longrightarrow \Phi(G_s)$ is surjective in the smooth topology. This means that $\Phi(G_s) \cong i^*\Phi(G)$.

We now look at sequence (1) over the étale site. Just as in the smooth case, i^* is exact. We factor i as the composition $s \xrightarrow{i_s} \operatorname{Spec} \mathcal{O}_{S,s} \xrightarrow{i_S} S$. By [M, II, 3.1(d)], we have a canonical map $i_S^*G \longrightarrow G_{\mathcal{O}_{S,s}}$ which is induced by the map

$$\begin{aligned} i_S^p G &\longrightarrow G_{\mathcal{O}_{S,s}} \\ (f,g) = (f \in G(U), g\colon U' \to U) &\longmapsto fg \in G(U') = G_{\mathcal{O}_{S,s}}(U'), \end{aligned}$$

where $U' \to \operatorname{Spec} \mathcal{O}_{S,s}$ and $U \to S$ are finite étale morphisms. By [EGA IV, 8.8.2] or [BLR, 1.2.5], for a map $f\colon U' \longrightarrow G_{\mathcal{O}_{S,s}}$ there exists an open neighborhood $S' \subseteq S$ of s on which a lifting $\tilde{f}\colon \widetilde{U} \longrightarrow G_{S'}$ of f exists with $\widetilde{U} \to S'$ étale. Conversely, for a pair $(f \in G(U), g\colon U' \longrightarrow U)$ with $fg = 0 \in G_{\mathcal{O}_{S,s}}(U')$ there exists an open neighborhood $S' \subseteq S$ of s such that $f|_{S'} = 0$. This shows that i_S^*G is represented by $G_{\mathcal{O}_{S,s}}$, and similarly $i_S^*G^0$ is represented by $G^0_{\mathcal{O}_{S,s}}$.

Using the canonical morphism from [M, II 3.1(d)], we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i_s^* G^0_{\mathcal{O}_{S,s}} & \longrightarrow & i_s^* G_{\mathcal{O}_{S,s}} & \longrightarrow & i^*\Phi(G) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \vdots & & \\ 0 & \longrightarrow & G_s^0 & \longrightarrow & G_s & \longrightarrow & \Phi(G_s) & \longrightarrow & 0. \end{array}$$

The bottom row is left-exact as a restriction of an fpqc-exact sequence. The surjectivity of $G_s \to \Phi(G_s)$ can be checked on the fiber over $\bar{s}$. By [M, II, 2.9(d)], for a sheaf represented by a $k(s)$-group scheme of finite type H, the stalk at $\bar{s}$ corresponds to $H(k(s)^{\text{sep}})$. The image of a point $x \in \Phi(G_s)(k(s)^{\text{sep}})$ in $\Phi(G_s)$ has an open preimage U_x in G_s which is not empty since $G_s \longrightarrow \Phi(G_s)$ is surjective. Since U_x is smooth over $k(s)$, $U_x(k(s)^{\text{sep}})$ is dense in U_x [BLR, 2.2.13] and therefore $U_x(k^{\text{sep}})$ is not empty. Since we can assume, after making a finite étale base change if necessary, that the image of x in G_s is geometrically connected, $U_x(k^{\mathrm{sep}})$ must contain a preimage of x, i.e., an element that maps to x.

Now [M, II, Theorem 3.2] yields the following for the fibers over $\bar{s}$:

$$G_{\bar{s}} = G\big(\mathcal{O}_{S,s}^{\mathrm{sh}}\big) = (i_s^* G)_{\bar{s}} \text{ and } G_{\bar{s}}^0 = G^0\big(\mathcal{O}_{S,s}^{\mathrm{sh}}\big) = \big(i_s^* G^0\big)_{\bar{s}}\,.$$

By [BLR, 2.3.5], the morphisms $G\big(\mathcal{O}_{S,s}^{\mathrm{sh}}\big) = G_{\mathcal{O}_{S,s}^{\mathrm{sh}}}\big(\mathcal{O}_{S,s}^{\mathrm{sh}}\big) \longrightarrow G_s(k(s)^{\mathrm{sep}})$ and $G^0\big(\mathcal{O}_{S,s}^{\mathrm{sh}}\big) = G^0_{\mathcal{O}_{S,s}^{\mathrm{sh}}}\big(\mathcal{O}_{S,s}^{\mathrm{sh}}\big) \longrightarrow G_s^0(k(s)^{\mathrm{sep}})$ are surjective, whence in the above diagram the maps α and β are surjective as well.

Thus there exists a map $i^*\Phi(G) \longrightarrow \Phi(G_s)$ and this map is surjective by the surjectivity of β and injective by the surjectivity of α. This means that $\Phi(G_s) \cong i^*\Phi(G)$.

Summarizing, we obtain

Proposition 0.2.1. *Let S be a Dedekind scheme and let G be a commutative smooth S-group scheme. Let $i\colon s \longrightarrow S$ be a point. Then the group scheme of components $\Phi(G_s)$ of G_s represents the sheaf $i^*\big(G/G^0\big)$ on the smooth and étale sites over k(s) and is uniquely determined by this sheaf on both of these sites.*

Proof. We have established the representability above. It remains to be shown that the group of components is uniquely determined by the sheaf.

In the smooth topology this is clear because $\Phi(G_s)$ is contained in the smooth site over $k(s)$. However, $\Phi(G_s)$ is not finite in general, whence $\Phi(G_s)$ does not belong to the étale site over $k(s)$. As an étale scheme, $\Phi(G_s)$ corresponds to the $\mathrm{Gal}(k(s)^{\mathrm{sep}}/k(s))$-module $\Phi(G_s)(k(s)^{\mathrm{sep}})$ which, by [M, II, Theorem 1.9], is uniquely determined by the étale sheaf represented by $\Phi(G_s)$. □

In this work we will usually regard a group of components as a Galois module. For our investigations we have to break up the group of components into a torsion part and a torsion-free part. Since we will see later that the group of components of Néron models are always finitely generated modules, the following considerations suffice :

Let Γ be a profinite topological group and let Φ be a finitely generated Γ-module. If one wants to decompose Φ into a torsion-free part and a torsion part, this must be done taking the Γ-module structure into account. Following Xarles, to do this we can dualize, i.e., apply the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$, and identify the torsion-free part with $\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ and the torsion part with $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z})$. Of course, $\mathbb{Z}$ has the trivial Γ-module structure and, for two Γ-modules A and B, $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ is equipped with the Γ module structure given by

$$\sigma \cdot f := \rho_B(\sigma) \circ f \circ \rho_A\big(\sigma^{-1}\big)\,,$$

where $\sigma \in \Gamma$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$ and $\rho_{(.)}$ denotes the Γ-action as a representation $\rho\colon \Gamma \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(\cdot)$.

The action on $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z})$ is explained similarly, since one can compute Ext using an injective resolution of $\mathbb{Z}$. Since Φ is finitely generated, the modules constructed in this way are again continuous and finitely generated.

As Xarles shows [X, Lemma 2.7], we can dualize these parts again and find an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\big(\mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z}), \mathbb{Z}\big) \longrightarrow \Phi \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0.$$

This sequence can also be understood by considering the torsion part Φ_{tors} of Φ as an abelian group. This is already a Γ-submodule, because the automorphisms with which Γ acts must restrict to automorphisms of the torsion part. If we define the torsion part in this way, then we can define the torsion-free part as the quotient of Φ by its torsion part.

0.3 Algebraic tori

Let S be a scheme and let G be an S-group functor from the category of schemes over S to the category of sets. Consider the contravariant functor

$D\colon (S-\text{ group functors}) \rightsquigarrow \quad (\text{commutative } S-\text{ group functors})$

$$G \quad \rightsquigarrow \quad D(G) := \underline{\mathrm{Hom}}_{S-\mathrm{grp}}(G, \mathbb{G}_{m,S})\,.$$

$D(G)$ is called the dual (or Cartier dual) of G and, by definition, for an S-scheme Y we have $D(G)(Y) := \mathrm{Hom}_{Y-\mathrm{grp}}(G_Y, \mathbb{G}_{m,Y})$. The dual is compatible with base changes under morphisms of schemes $S' \longrightarrow S$.

For an arbitrary group M and an arbitrary S-scheme X, let M_X be the constant X-group scheme associated to M. As a scheme, this is equal to $\amalg_{m\in M} X_m$, where $X_m = X$ for all $m \in M$. For an X-scheme Y, $\mathrm{Hom}_X(Y, M_X)$ is equal to the set of locally constant maps from Y to M. If M is an abelian group, then the functor $D(M_S)$ is represented by an S-group scheme.

Definition 0.3.1. [SGA3, VIII, Definition 1.1] An S-group scheme G is *diagonalizable* if there exists an abelian group M such that G is isomorphic to the scheme $D(M_S) = \underline{\mathrm{Hom}}_{S-\mathrm{grp}}(M_S, \mathbb{G}_{m,S})$.

For a diagonalizable scheme, the points with values in an S-scheme Y are computed as follows:

$$\begin{aligned} D(M_S)\,(Y) &= \mathrm{Hom}_S(Y, D(M_S)) = \mathrm{Hom}_S\big(Y, \underline{\mathrm{Hom}}_{S-\mathrm{grp}}(M_S, \mathbb{G}_{m,S})\big) \\ &= \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_S(Y, \mathbb{G}_{m,Y}))\,. \end{aligned}$$

Theorem 0.3.2. [SGA3, VIII, Theorem and Corollaries 1.2-1.4] *Let M be an abelian group and let S be a scheme. The canonical morphism $M_S \longrightarrow D(D(M_S))$ is an isomorphism and every character $\chi\colon D(M_S) \longrightarrow \mathbb{G}_{m,S}$ corresponds uniquely to a locally constant map $S \longrightarrow M$.*

If N is another abelian group, then there exists a natural isomorphism

$$\mathrm{Hom}_{S\text{-grp}}(D(M_S)\,, D(N_S)) \cong \mathrm{Hom}_{S\text{-grp}}(N_S, M_S)$$

If N is finitely generated, then the natural injection

$$(\mathrm{Hom}_{\mathbb{Z}}(N, M))_S \hookrightarrow \underline{\mathrm{Hom}}_{S-\mathrm{grp}}(N_S, M_S)$$

is an isomorphism and therefore

$$(\mathrm{Hom}_{\mathbb{Z}}(N, M))_S \cong \underline{\mathrm{Hom}}_{S-\mathrm{grp}}(D(N_S)\,, D(M_S))\,.$$

Proposition 0.3.3. (cf. [SGA3, VIII, Proposition 2.1]) *Let M be an abelian group. Then:*

1. *The scheme $D(M_S)$ is faithfully flat and affine over S. More precisely $D(M_S) = \mathrm{Spec}\,\mathcal{O}_S[M]$.*
2. *$D(M_S)$ is of finite presentation $\iff$ $D(M_S)$ is of finite type $\iff$ M is finitely generated.*
3. *$D(M_S)$ is finite $\iff$ M is finite.*
4. *$M = 0 \iff D(M_S)$ is the trivial group.*
5. *$D(M_S)$ is a smooth S-scheme $\iff$ M is finitely generated and the order of the torsion part of M is prime to the characteristic of the field $k(s)$ for every point $s \in S$.*

The functor $D(\cdot_S)$ transforms direct sums into fiber products over S. Further, we have $D(\mathbb{Z}_S) = \mathbb{G}_{m,S}$ and $D((\mathbb{Z}/n\mathbb{Z})_S) = \mu_{n,S}$, so every diagonalizable group of finite presentation is a fiber product of copies of the multiplicative group scheme and of groups of roots of unity.

Theorem 0.3.4. [SGA3, VIII, Proposition 3.1] *Let S be a scheme and let*

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

be an exact sequence of abelian groups. Then the dual of this sequence

$$0 \longrightarrow D(M''_S) \xrightarrow{v'} D(M_S) \xrightarrow{u'} D(M'_S) \longrightarrow 0$$

is exact, i.e., u' is flat and quasi-compact and v' induces an isomorphism of $D(M''_S)$ with the kernel of u'.

Using Descent Theory, one can enlarge the category of diagonalizable group schemes to the category of group schemes of multiplicative type. The latter are the group schemes that arise from a diagonalizable group scheme through a flat and quasi-compact descent.

Definition 0.3.5. (see [SGA3, IX, 1.1]) Let S be a scheme and let G be an S-group scheme. Then G is called a *group scheme of multiplicative type* if G is locally diagonalizable in the faithfully flat and quasi-compact topology. That is, for every $s \in S$ there exists an open neighborhood U of s in S and a faithfully flat and quasi-compact S-morphism $U' \longrightarrow U$ such that $G_{U'}$ is a diagonalizable U'-group scheme.

The group G is called quasi-isotrivial if one can further require $U' \longrightarrow U$ to be étale and surjective. If there exists an étale, surjective and *finite* morphism $S' \longrightarrow S$ such that $G_{S'}$ is diagonalizable, then G is said to be *isotrivial* of multiplicative type.

Definition 0.3.6. (cf. [SGA3, IX, 1.3]) Let S be a scheme. An S-torus T is an S-group scheme that is locally isomorphic, in the faithfully flat and quasi-compact topology, to the group scheme $\mathbb{G}^r_{m,S}$ for some integer $r \geq 0$.

By an S-torus T we will always understand an isotrivial (!) S-torus of finite type. Since we will only consider fields or discrete valuation rings as basis S, isotriviality is not an actual restriction:

Proposition 0.3.7. [SGA3, X, 5.16] *Let S be a normal and locally noetherian scheme. Then every group of multiplicative type and of finite type over S is isotrivial.*

The isotrivial group schemes of multiplicative type can be described using the theory of Galois descent. To do this, we first Galois morphisms.

Definition 0.3.8. [M, I, §5] Let G be a finite group and let Y and X be connected schemes. Let G_Y denote the constant Y-group scheme G. A morphism of schemes $Y \longrightarrow X$ is called *Galois*, with Galois group G, if it is finite and faithfully flat and G acts on $Y \longrightarrow X$ in such a way that G acts trivially on X and the induced morphism

$$\begin{aligned} \psi : G_Y = \amalg_{\sigma \in G} Y_\sigma &\longrightarrow Y \times_X Y \\ y \in Y_\sigma &\longmapsto (y, \sigma y) \end{aligned}$$

is an isomorphism. In other words, Y is a G-torsor over X and the morphism $Y \longrightarrow X$ is necessarily étale.

For a connected scheme S with a geometric point $\bar{s} \longrightarrow S$, one can construct the fundamental group $\pi_1(S, \bar{s})$. This is a compact topological group which is a projective limit of finite discrete groups. The fundamental group can be characterized by the property that it induces an equivalence between the category of finite and étale S-schemes and the category of finite continuous $\pi_1(S, \bar{s})$-modules [M, I, §5, particularly Theorem 5.3].

Proposition 0.3.9. [SGA3, X, 1.2] *Let S be a connected scheme and let $\bar{s} \longrightarrow S$ be a geometric point of S. Then the category of isotrivial S-group schemes of multiplicative type is anti-equivalent to the category of continuous $\pi_1(S, \bar{s})$-modules.*

Via this anti-equivalence, an isotrivial group scheme of multiplicative type corresponds to its *character group*:

Definition 0.3.10. (Character group) Let S be a connected scheme and let $\bar{s}$ be a geometric point of S. Let T be an isotrivial S-group scheme of multiplicative type and of finite type. Then the $\pi_1(S, \bar{s})$-module $X_{\bar{s}}(T) := \mathrm{Hom}_{\bar{s}\text{-grp}}(T_{\bar{s}}, \mathbb{G}_{m,\bar{s}})$ is called the character group of T.

For a morphism of schemes $S' \longrightarrow S$, we call $\mathrm{Hom}_{S'\text{-grp}}(T_{S'}, \mathbb{G}_{m,S'})$ the group of S'-rational characters of T and $\underline{X}(T) := \underline{\mathrm{Hom}}_{S\text{-grp}}(T, \mathbb{G}_{m,S})$ the sheaf of rational characters.

The character group of an algebraic torus is a finitely generated, torsion-free and continuous $\pi_1(S, \bar{s})$-module. Since the group structure on T is also defined by Galois descent, $\underline{X}(T)(S') = X_{\bar{s}}(T)^{\pi_1(S', \bar{s}')}$. Thus, in particular, the sheaf of rational characters is actually a sheaf on the étale site over S.

If T is an isotrivial S-group scheme of multiplicative type, then a Galois extension $S' \longrightarrow S$ over which T is diagonalizable is called a *splitting extension* of T.

We now want to use the character group to construct sequences of algebraic tori, which can be seen as exact sequences of sheaves on the étale or smooth site.

Proposition 0.3.11. *Let S be a connected scheme with a geometric point $\bar{s}$ and let $(T_i)_{i=1,2,3}$ be isotrivial S-group schemes of multiplicative type. If there exists a short exact sequence of $\pi_1(S, \bar{s})$-modules*

$$0 \longrightarrow X_{\bar{s}}(T_3) \longrightarrow X_{\bar{s}}(T_3) \longrightarrow X_{\bar{s}}(T_3) \longrightarrow 0 \tag{2}$$

and $X_{\bar{s}}(T_3)$ is finitely generated and torsion-free, then the induced sequence

$$0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$$

is a short exact sequence of abelian sheaves on both the smooth and étale sites over S.

Proof. By the anti-equivalence mentioned above, the homomorphisms of character groups induce homomorphisms of group schemes $T_1 \longrightarrow T_2$ and $T_2 \longrightarrow T_3$. These, in turn, induce morphisms of the corresponding étale (respectively, smooth) sheaves.

By isotriviality, there exists a finite, étale and surjective morphism $S' \longrightarrow S$ such that all $T_i \times_S S'$ are diagonalizable group schemes. Without loss of generality, we may assume that S' is connected.

After the indicated base change, the maps between character groups (viewed as $\pi_1(S', \bar{s}')$-modules with trivial action) induce maps $T_i \times_S S' \longrightarrow T_{i+1} \times_S S'$. It follows from [SGA3, VIII, Theorem 3.1] that the sequence

$$0 \longrightarrow T_1 \times_S S' \longrightarrow T_2 \times_S S' \longrightarrow T_3 \times_S S' \longrightarrow 0$$

is exact in the fpqc topology over S'. It is therefore certainly left-exact in the smooth and étale topologies. By the assumption that $X_{\bar{s}}(T_1)$ is finitely generated and torsion-free, T_1 is a smooth S' scheme, whence the map $T_2 \longrightarrow T_3$ is surjective in the smooth topology.

In the étale topology we can check surjectivity on the stalks. So let $s' \in S'$ be a point and let $\bar{s}'$ be a geometric point that lies over s'. Now let $\mathcal{O}_{S',\bar{s}'} = \mathcal{O}^{\mathrm{sh}}_{S',s'}$ be the limit of all étale neighborhoods of $\bar{s}'$. Then, by [M, II, 2.9(d)], the sequence of stalks in $\bar{s}'$ is isomorphic to the sequence

$$0 \longrightarrow T_1(\mathcal{O}_{S',\bar{s}'}) \longrightarrow T_2(\mathcal{O}_{S',\bar{s}'}) \longrightarrow T_3(\mathcal{O}_{S',\bar{s}'}),$$

and this is isomorphic by Cartier duality to the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(X_{\bar{s}}(T_1), \mathcal{O}^*_{S',\bar{s}'}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(X_{\bar{s}}(T_2), \mathcal{O}^*_{S',\bar{s}'}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(X_{\bar{s}}(T_3), \mathcal{O}^*_{S',\bar{s}'}\big).$$

Since (2) is exact and $X_{\bar{s}'}(T_1)$ is torsion-free, we have $\mathrm{Ext}^1_{\mathbb{Z}}\big(X_{\bar{s}'}(T_1), \mathcal{O}^*_{S',\bar{s}'}\big) = 0$. Consequently, the sequence of stalks is surjective at $T_3(\mathcal{O}_{S',\bar{s}'})$.

Since $S' \longrightarrow S$ is a covering in the étale and smooth topologies, the sequence

$$0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$$

is also exact on the corresponding sites over S. □

Proposition 0.3.12. *Let S be a connected scheme with a geometric point $\bar{s}$. Let T be an isotrivial S-torus of finite type. Then on the étale site over S we have:*

$$\mathrm{Hom}_S(\cdot, T) \cong \underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,S}),$$

where $\underline{X}(T)$ is the sheaf of rational characters of T.

Proof. Without loss of generality, we may assume that $T \times_S S'$ is diagonalizable, where $S' \longrightarrow S$ is a Galois morphism. We can verify the isomorphism of the statement locally, so we may assume that $U = \mathrm{Spec}\, A$ is affine and $U \longrightarrow S$ is finite and étale. Further, let U' be a connected component of $U \times_S S'$. We may assume that $U' = \mathrm{Spec}\, B$ is affine and Galois over U. Now T is diagonalizable over U', whence $T_{U'} = \mathrm{Spec}\, B[X_{\bar{s}}(T)]$ and T_U comes from $T_{U'}$ via Galois descent with respect to the finite group $\mathrm{Gal}(U'/U)$. More precisely, $\mathrm{Gal}(U'/U)$ acts on the algebra $B[X_{\bar{s}}(T)]$ via the canonical action on B and the induced action on $X_{\bar{s}}(T)$. Note that $\mathrm{Gal}(U'/U)$ is a quotient of $\mathrm{Gal}(S'/S)$.

Thus

$$\begin{aligned} T(U) &= T_U(U) = T_{U'}(U')^{\mathrm{Gal}(U'/U)} \\ &= \mathrm{Hom}_B(B[X_{\bar{s}}(T)], B)^{\mathrm{Gal}(U'/U)} = \mathrm{Hom}_{\mathbb{Z}}(X_{\bar{s}}(T), B^*)^{\mathrm{Gal}(U'/U)}. \end{aligned}$$

Conversely, $\underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,S})(U') = \mathrm{Hom}_{\mathbb{Z}}(X_{\bar{s}}(T), B^*)$, since for each sheaf $\mathcal{F}$ the identity $\underline{\mathrm{Hom}}(\mathbb{Z}, \mathcal{F}) = \mathcal{F}$ holds, where $\mathbb{Z}$ stands for the constant sheaf with value $\mathbb{Z}$. Since $U' \longrightarrow U$ is a Galois cover, it follows from the sheaf condition, i.e., the exactness of the sequence $\underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,S})(U) \longrightarrow \underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,S})(U') \rightrightarrows \underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,S})(U' \times_U U')$, that we have an isomorphism $\underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,S})(U) \cong \mathrm{Hom}_{\mathbb{Z}}(X_{\bar{s}}(T), B^*)^{\mathrm{Gal}(U'/U)}$.

According to the definition of diagonalizable group schemes, these isomorphisms are functorial in U' and therefore also in U. □

Incidentally, note that the reverse duality does not hold, i.e., the étale sheaf $\underline{\mathrm{Hom}}(T, \mathbb{G}_{m,S})$ is not isomorphic to $\mathrm{Hom}_S(\cdot, \underline{X}(T))$ in general. For example, consider a perfect local field K of characteristic p. By [M, III, 1.7(c)], the sheaf $\underline{\mathrm{Hom}}(\mathbb{G}_{m,K}, \mathbb{G}_{m,K})$ is represented on the étale site over K by the module

$$M := \bigcup_H \mathrm{Hom}_H((K^{\mathrm{sep}})^*, (K^{\mathrm{sep}})^*),$$

where H runs through all open normal subgroups of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. If the reverse duality were valid, then we should have $M = \mathbb{Z}$. By the perfectness of K, for every $x \in (K^{\mathrm{sep}})^*$ there is exactly one p-th root in K^{sep} and the map $x \longmapsto x^{1/p}$ is an isomorphism. Thus for all open normal subgroups $H \subset \operatorname{Gal}(K^{\text{sep}}/K)$ we have

$$\begin{aligned} \mathbb{Z}\left[p^{-1}\right] &\longmapsto \operatorname{Hom}_H((K^{\text{sep}})^*, (K^{\text{sep}})^*) \\ n/p^r &\longmapsto \left(x \longmapsto x^{n/p^r}\right), \end{aligned}$$

whence $\mathbb{Z}\left[p^{-1}\right] \subset M$.

0.4 Weil restrictions of algebraic tori

Next we want to describe the Weil restriction of algebraic tori. For simplicity, we limit ourselves to tori over local fields.

If we work with a fixed separable closure of K, we do not explicitly specify a geometric point $\bar{s}$ in the fundamental group, since this must factor through the morphism $\operatorname{Spec} K^{\text{sep}} \longrightarrow \operatorname{Spec} K$. For any field K, we have $\pi_1(\operatorname{Spec} K) = \operatorname{Gal}(K^{\text{sep}}/K)$ [M, I, 5.2(a)].

We briefly recall the definition of the Weil restriction of a scheme:

Definition 0.4.1. Let $S' \longrightarrow S$ be a morphism of schemes and

$$X' \colon (\text{Schemes}/S') \longrightarrow (\text{Sets})$$

a contravariant functor. Then the contravariant functor

$$\begin{aligned} \mathfrak{R}_{S'/S}(X) \colon\ (\text{Schemes}/S) &\longrightarrow (\text{Sets}) \\ Y &\longmapsto X'(Y \times_S S') \end{aligned}$$

is called the Weil restriction of X' with respect to $S' \longrightarrow S$.

If X' is a representable functor, we also denote the representing S'-scheme by X'. If $S' \longrightarrow S$ is a finite, locally free and faithfully flat morphism, then the Weil restriction of a representable functor is again representable [BLR, Theorem 7.6.4]. In this case, $\mathfrak{R}_{S'/S}(X')$ also denotes the representing S-scheme. For further properties of the Weil restriction please see [BLR, 7.6].

To describe the character group of the Weil restriction of a torus, we need the concept of induction.

Definition 0.4.2. Let G be a group and let H be a subgroup. Let M be an H-module. Then the G-module $\operatorname{Ind}_G^H M := M \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$ is called the *induction of M with respect to $G \supset H$*. A module of the form $\mathbb{Z}[G] \otimes_{\mathbb{Z}} M$ is called an induced G-module.

The G-module $\operatorname{Coind}_G^H M := \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$ is called the *coinduction of M with respect to $G \supset H$*. A module of the form $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M)$ is called a coinduced module.

By Shapiro's lemma [Br, Proposition III.6.2], we have $\mathrm{H}^n\left(G, \operatorname{Coind}_G^H M\right) = \mathrm{H}^n(H, M)$ and $\mathrm{H}_n\left(G, \operatorname{Ind}_G^H M\right) = \mathrm{H}_n(H, M)$. If $[G : H]$ is finite, then induction and coinduction are isomorphic to each other [S, VII, §1, p. 110]. If $H \subset G$ is a subgroup, then an induced G-module is also an induced H-module. If G is a finite group, then an induced G-module M is cohomologically trivial, i.e., for all $k \in \mathbb{Z}$ and every subgroup H of G, the Tate cohomology group $\check{\mathrm{H}}^k(H, M)$ is trivial.

Proposition 0.4.3. *Let L/K be a finite separable extension of local fields of degree $n = [L:K]$ and let T' be a torus over L. Let $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $G_L := \mathrm{Gal}(K^{\mathrm{sep}}/L)$. Then the Weil restriction $\mathfrak{R}_{L/K}(T')$ is a torus over K with character group*

$$X\big(\mathfrak{R}_{L/K}(T')\big) = \mathrm{Ind}_{G_K}^{G_L} X(T')\,.$$

Proof. We show the claim using the defining property of the Weil restriction for the torus of the specified character group. So let M/L be a finite, Galois extension such that T' splits over M. Thus $T' \otimes_L M \cong \mathrm{Spec}\, M[X(T')]$ and by Galois descent we have $T' \cong \mathrm{Spec}\, M\,[X(T')]^{\mathrm{Gal}(M/L)}$. For an affine K-scheme $Y = \mathrm{Spec}\, B$ we have

$$\begin{aligned}\mathfrak{R}_{L/K}(T')\,(Y) = T'(Y \otimes_K L) = \mathrm{Hom}_L(Y \otimes_K L, T') &= \mathrm{Hom}_M(Y \otimes_K M, T' \otimes_L M)^{\mathrm{Gal}(M/L)}\\ &= \mathrm{Hom}_{M\text{-alg}}(M[X(T')]\,, B \otimes_K M)^{\mathrm{Gal}(M/L)}\\ &= \mathrm{Hom}_{\mathbb{Z}}\big(X(T')\,,(B \otimes_K M)^*\big)^{\mathrm{Gal}(M/L)}\,.\end{aligned}$$

Let d be the rank of $X(T')$ and let $e_1, \dots, e_d$ be a $\mathbb{Z}$-basis of $X(T')$. Let the elements of $\mathrm{Gal}(M/L)$ be $(\tau_k)_{k=1,\dots,m}$, where $m := [M\colon L]$, and let the action of τ_k on $X(T')$ be represented by the matrix $(t(k)_{i,j}) \in \mathrm{GL}(d, \mathbb{Z})$.

Then an element from $\mathrm{Hom}_{\mathbb{Z}}\big(X(T')\,, (B \otimes_K M)^*\big)^{\mathrm{Gal}(M/L)}$ is determined by specifying elements $b_j \in (B \otimes_K M)^*$ for $j = 1, \dots, d$ such that

$$\tau_k\,(b_j) = \prod_{i=1}^{d} b_i^{t(k)_{i,j}},$$

where τ_k acts on $B \otimes_K M$ via its canonical action on M.

Now let R be the K-torus with character group $\mathrm{Ind}_{G_K}^{G_L} X(T')$. Then R splits over the finite Galois extension M/K. In particular, we can redefine G_L as $\mathrm{Gal}(M/L)$ and G_K as $\mathrm{Gal}(M/K)$ without changing the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-action on $\mathrm{Ind}_{G_K}^{G_L} X(T')$.

Let $\sigma_1, \dots, \sigma_n \in \mathrm{Gal}(M/K)$ represent the G_L-coordinate classes in G_K. This gives us a $\mathbb{Z}$-basis for $\mathrm{Ind}_{G_K}^{G_L}(e_{j,l})$ with $j = 1, \dots, d$ and $l = 1, \dots, n$.

An element $\xi \in G_K$ permutes the G_L-coclasses, i.e., there is a permutation ψ_ξ of $\{1, .., n\}$ such that $\xi\sigma_l G_L = \sigma_{\psi_\xi(l)} G_L$. Further, for such ξ and every index l there is an index $k(\xi, l) \in \{1, .., m\}$ so that

$$\xi\sigma_l = \sigma_{\psi_\xi(l)}\tau_{k(\xi,l)},$$

where $\tau_{k(\xi,l)} \in G_L$ is as above. This allows us to compute the G_K-action on the basis $e_{j,l}$

$$\xi e_{j,l} = \tau_{k(\xi,l)} e_{j,\psi_\xi(l)} = \sum_{i=1}^{d} t(k(\xi, l))_{i,j} e_{i,\psi_\xi(l)},$$

where $t(\cdot)_{i,j}$ is defined as above.

Now we have $R \cong \mathrm{Spec}\, M\left[\mathrm{Ind}_{G_K}^{G_L} X(T')\right]^{\mathrm{Gal}(M/K)}$ and for an affine K-scheme $Y = \mathrm{Spec}\, B$ Galois descent yields

$$\begin{aligned}\mathrm{Hom}_K(Y, R) &= \mathrm{Hom}_M(Y \otimes_K M, R \otimes_K M)^{\mathrm{Gal}(M/K)}\\ &= \mathrm{Hom}_M\Big(Y \otimes_K M, \mathrm{Spec}\, M\Big[\mathrm{Ind}_{G_K}^{G_L} X(T')\Big]\Big)^{\mathrm{Gal}(M/K)}\\ &= \mathrm{Hom}_{M-\mathrm{alg}}\Big(M\Big[\mathrm{Ind}_{G_K}^{G_L} X(T')\Big]\,, B \otimes_K M\Big)^{\mathrm{Gal}(M/K)}\\ &= \mathrm{Hom}_{\mathbb{Z}}\Big(\mathrm{Ind}_{G_K}^{G_L} X(T')\,,(B \otimes_K M)^*\Big)^{\mathrm{Gal}(M/K)}\,.\end{aligned}$$

A morphism $\beta \in \mathrm{Hom}_K(Y,R)$ corresponds to the specification of elements $b_{j,l} \in (B \otimes_K M)^*$ such that for all $\xi \in G_K$ we have

$$\xi(b_{j,l}) = \prod_{i=1}^{d} \left(b_{i,\psi_\xi(l)}\right)^{t(k(\xi,l)_{i,j}} . \tag{0.4.3.1}$$

Without loss of generality, we may assume that the representative σ_1 is the identity element of G_K. It follows that

$$\tau_k(b_{j,1}) = \prod_{i=1}^{d} (b_{i,1})^{t(k)_{i,j}} .$$

So a morphism β yields an element from $T'(Y \otimes_K L)$ represented by the given elements $(b_{j,1})_{j=1,\dots,d}$ from $(B \otimes_K M)^*$. This assignment is clearly functorial in Y and compatible with the group laws on R and T'.

Conversely, we obtain a point $(b_{j,1}) \in T'(Y \otimes_K L)$, by setting $b_{j,l} := \sigma_l(b_{j,1})$. Now let $j \in \{1,..,d\}, \xi \in G_K$ and $l \in \{1,..,n\}$ be arbitrary. Then the following holds

$$\begin{aligned}\xi(b_{j,l}) &= \xi\sigma_l(b_{j,1}) = \sigma_{\psi_\xi(l)}\tau_{k(\xi,l)}(b_{j,1}) \\ &= \sigma_{\psi_\xi(l)}\left(\prod_{i=1}^{d} (b_{i,1})^{t(k(\xi,j)_{i,j}}\right) = \prod_{i=1}^{d} \left(b_{i,\psi_\xi(l)}\right)^{t(k(\xi,l)_{i,j}}\end{aligned}$$

So all relations of the form 0.4.3.1 hold. Summarizing, we have an isomorphism $R(Y) \cong T'(Y \otimes_K L) = \mathfrak{R}_{L/K}(T')\,(Y)$ for affine K-schemes Y. By construction, this isomorphism is functorial in Y, so that $R = \mathfrak{R}_{L/K}(T')$. □

Using the above explicit description of the character group of a Weil restriction, we can obtain interesting statements.

Proposition 0.4.4. *Let K be a local field and let T be an algebraic K-torus. Further, let L/K be a finite separable extension. Then there exists an exact sequence of algebraic K-tori in both the smooth and the étale topology*

$$0 \longrightarrow T' \longrightarrow \mathfrak{R}_{L/K}(T_L) \longrightarrow T \longrightarrow 0. \tag{3}$$

Proof. Let d be the rank of $X(T)$. Then $X(T)$, as a $\mathbb{Z}$ module, has a base $(e_j)_{j=1,\dots,d}$. Let M/L be a finite Galois extension of degree $m := [M : L]$ such that T trivializes over M and $\sigma_1, \dots, \sigma_m \in \mathrm{Gal}(M/K)$ are representatives of the $G_L := \mathrm{Gal}(M/L)$ cosets of $G_K := \mathrm{Gal}(M/K)$. Further, let the elements of G_L be uniquely denoted by $\tau_1, \dots, \tau_m$. Below we recall some of the notation from the proof of Proposition 0.4.3 for the L-torus T_L.

We define a $\mathbb{Z}$-linear map

$$\begin{aligned}\iota\colon X(T) &\longrightarrow \mathrm{Ind}_{G_K}^{G_L} \sigma_i X(T_L) \\ v &\longmapsto \left(\sigma_1^{-1}(v), .., \sigma_l^{-1}(v), .., \sigma_m^{-1}(v)\right) .\end{aligned}$$

This map is obviously injective. It is also compatible with the action of G_K. Indeed, as in the previous proof, let $\psi_\xi(l)$ and $k(\xi,l)$ be represented by the equations $\xi\sigma_l = \sigma_{\psi_\xi(l)}\tau_{k(\xi,l)}$ in G_K. Then for any $v \in X(T)$ we have

$$\sigma_{\psi_\xi(l)}^{-1}\xi v = \tau_{k(\xi,l)}\sigma_l^{-1} v.$$

Now let $\phi_\xi := \psi_\xi^{-1}$ be the inverse function. It follows that

$$\begin{aligned}\iota(\xi v) &= \left(\sigma_1^{-1}(\xi v), .., \sigma_l^{-1}(\xi v), .., \sigma_m^{-1}(\xi v)\right) \\ &= \left(\tau_{k(\xi,\phi_\xi(1))}\sigma_{\phi_\xi(1)}^{-1}(v), \dots, \tau_{k(\xi,\phi_\xi(l))}\sigma_{\phi_\xi(1)}^{-1}(\xi v), ., \tau_{k(\xi,\phi_\xi(m))}\sigma_{\phi_\xi(1)}^{-1}(\xi v)\right) .\end{aligned}$$

This is exactly $\xi\iota(v)$ since the $l = \psi_\xi(\phi_\xi(l))$-th component of $\xi\iota(v)$ is $\tau_{k(\xi,\phi_\xi(l))}$ relative to the $\phi_\xi(l)$-th component of $\iota(v)$.

The image of ι is saturated in $\mathbb{Z}^d[\mathrm{Gal}(L/K)]$. Thus when we add an $r \in \mathbb{N}$ and compare

$$rv_i = \sigma_i^{-1}(v)$$

for $i = 1, \dots, n$ with $v \in X(T)$ and $v_i \in \mathbb{Z}^d$, then v must already be in $rX(T)$ because the σ_i are isomorphisms.

So we have a short exact sequence of continuous and torsion-free $\mathrm{Gal}(L/K)$-modules

$$0 \longrightarrow X(T) \longrightarrow \mathbb{Z}^d[\mathrm{Gal}(L/K)] \longrightarrow X(T') \longrightarrow 0. \tag{4}$$

Thus the assertion follows from Proposition 0.3.11. □

By the universal property of the Weil restriction, there exists a map $T \longrightarrow \mathfrak{R}_{L/K}(T_L)$ which corresponds to the identity of T_L. This map is a closed immersion, because it is a homomorphism of group schemes and T is separated. Using the notations as above, on character groups this corresponds to the map

$$\mathrm{Ind}_{G_K}^{G_L} X(T_L) \longrightarrow X(T)$$
$$(v_1, \dots, v_n) \longmapsto \sum_{i=1}^{n} \sigma_i(v_i)\,.$$

Proposition 0.4.5. *Let L/K be a finite Galois extension of local fields with Galois group G. Then every torsion-free and finitely generated G module X admits a G-acyclic resolution*

$$X \longrightarrow J \longrightarrow J' \longrightarrow \cdots$$

by torsion-free and finitely generated G-modules.

Proof. Let d be the rank of X and consider the embedding $\iota\colon X \longrightarrow \mathbb{Z}^d[G]$ for X constructed in the proof of Proposition 0.4.4. Since the group G is finite, the induced G-module $\mathbb{Z}^d[G]$ is coinduced, hence (cohomologically) G-acyclic. So set $J \coloneqq \mathbb{Z}^d[G]$ with the embedding $\iota\colon X \longrightarrow J$.

The quotient $J/\iota(X)$ is, as shown above, torsion-free and finitely generated as the quotient of a finitely generated module. So, using the same construction as above, we can embed the quotient in a torsion-free and finitely generated G-acyclic module J'. The assertion yields an inductive continuation of this construction. □

We now want to define the so-called *norm-one tori*.

Definition 0.4.6. Let L/K be a finite separable extension of local fields. Then the *norm-one torus* associated to the extension L/K is the K-torus T_N defined by the exact sequence (3) corresponding to the choice $T = \mathbb{G}_{m,K}$:

$$0 \longrightarrow T_N \longrightarrow \mathfrak{R}_{L/K}(\mathbb{G}_{m,L}) \longrightarrow \mathbb{G}_{m,K} \longrightarrow 0. \tag{5}$$

By Proposition 0.4.4, existence is clear and we see that

$$X(T_N) = \mathrm{Coker}\Big(\mathbb{Z} \longrightarrow \mathrm{Ind}_{G_K}^{G_L} \mathbb{Z}\Big)\,,$$

where the map $\mathbb{Z} \longrightarrow \mathrm{Ind}_{G_K}^{G_L} \mathbb{Z}$ corresponds to the diagonal embedding $m \longmapsto (m, \dots, m)$.

The name *norm-one torus* is justified by the following observation.

Proposition 0.4.7. *Let $Y = \operatorname{Spec} B$ be affine and let L/K be a finite separable extension of local fields. Further, set $G \coloneqq \operatorname{Hom}_K(L, K^{\mathrm{sep}})$. Then the map*

$$\mathfrak{R}_{L/K}(\mathbb{G}_{m,L})(Y) \longrightarrow \mathbb{G}_{m,K}(Y)$$

in the definition of the norm-one torus corresponds to the norm map

$$\begin{aligned}(B \otimes_K L)^* &\longrightarrow B^* \\ x &\longmapsto \prod_{\sigma \in G} (\mathrm{id} \otimes \sigma)(X) \in (B \otimes_K L)^G \cong B.\end{aligned}$$

Proof. Let M/L be a finite Galois extension and set $n \coloneqq [L : K]$. Let $\sigma_1, \dots, \sigma_n$ represent the $G_L \coloneqq \operatorname{Gal}(M/L)$ co-classes of $G_K \coloneqq \operatorname{Gal}(M/K)$. An element $\beta \in \mathfrak{R}_{L/K}(\mathbb{G}_{m,L})$ corresponds to giving elements $(b_j)_{j=1,\dots,n}$ in $B \otimes_K M$ such that, for all $\xi \in \operatorname{Gal}(M/K)$, the relations

$$\xi(b_j) = \tau_{k(\xi,j)}\big(b_{\psi_\xi(j)}\big)$$

hold, where the elements are represented by the equation $\xi\sigma_j = \sigma_{\psi_\xi(j)}\tau_{k(\xi,j)}$ with $\tau_{k(\xi,j)} \in G_L$ uniquely determined.

The relations imply that G_L acts trivially on the b_j, so the b_j already come from $B \otimes_K L$. Further, $b_j = \sigma_j(b_1)$ if we assume (without loss of generality) that $\sigma_1 = e$. The map of the character group is the diagonal embedding, so that the point (b_j) is mapped to the product $\prod_{j=1}^n b_j \in B^*$. □

0.5 Néron models of algebraic tori

Let S be a Dedekind scheme and let η be the scheme of generic fibers of S. It is shown in [BLR, 10.1.6] that every algebraic η-torus has an lft-Néron model over S. Further, [BLR, Theorem 1.2.4] also applies to lft-Néron models, i.e., a global lft-Néron model is obtained by glueing local lft-Néron models. Since the group of components is defined fiber by fiber, it suffices to examine the group of components in the local case. Algebraic tori are connected, so one is interested only in the group of components of the special fiber. Since Néron models are compatible with completions and a completion is an isomorphism on the special fiber, it suffices to consider the case of a local field.

From now on we only consider algebraic tori over a local field K. The lft-Néron model $\mathcal{G}$ of $\mathbb{G}_{m,K}$ can be constructed explicitly (see. [BLR, 10.1.5]) and can be described by an exact sequence

$$0 \longrightarrow \mathbb{G}_{m,\mathcal{O}_K} \longrightarrow \mathcal{G} \longrightarrow i_*\mathbb{Z} \longrightarrow 0.$$

See also [SGA7, VIII, §6].

A finite unramified Galois extension L/K of local fields induces an étale and faithfully-flat (even Galois) extension of the associated discrete valuation rings. So the Néron model of an algebraic K-torus T which splits over L can be derived from the Néron model of $T_L \cong \mathbb{G}_{m,L}^d$.

Such tori are called *algebraic tori with multiplicative reduction* and have the following properties :

Definition 0.5.1. [NX, 1.2] Let K be a local field. An algebraic K-torus T has multiplicative reduction if one of the following equivalent conditions is satisfied:

1. $X(T)^I = X(T)$, i.e., the inertia group acts trivially on the character group $X(T)$.

2. T splits over an unramified extension of K.
3. There is a torus $\mathcal{T}^0$ over Spec $\mathcal{O}_K$ such that $\mathcal{T}^0_K = T$.
4. The identity component of the Néron model of T is a torus over Spec $\mathcal{O}_K$.
5. The reduction $\mathcal{T}^0_k$ of the identity component of the Néron model is a k-torus.

We now use Proposition 0.3.11 to define certain specific exact sequences of K-tori.

Proposition 0.5.2. *Let K be a local field and let T be an algebraic K-torus. Then there exists a canonical maximal quotient T^I of T which is a torus with multiplicative reduction. We have a short exact sequence of algebraic K-tori in the smooth and the étale topologies*

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0. \tag{6}$$

A homomorphism of K-tori $\phi\colon T_1 \longrightarrow T_2$ induces a commutative diagram of algebraic K-tori

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{T_1} & \longrightarrow & T_1 & \longrightarrow & T_1^I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{T_2} & \longrightarrow & T_2 & \longrightarrow & T_2^I & \longrightarrow & 0 \end{array}$$

Proof. Let $X(T)$ be the character group of T. A quotient T^I of T which is a torus corresponds one-to-one to a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-submodule of $X(T)$. By the Definition 0.5.1, there exists a maximal quotient with multiplicative reduction and it corresponds to the torus with the character group $X(T)^I$. This submodule is saturated, i.e., the quotient $X(T)/X(T)^I$ is a continuous and torsion-free $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module $X(\widetilde{T})$. This gives us an exact sequence of continuous and torsion-free $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules

$$0 \longrightarrow X(T)^I \longrightarrow X(T) \longrightarrow X(\widetilde{T}) \longrightarrow 0. \tag{7}$$

A morphism $\phi\colon T_1 \longrightarrow T_2$ corresponds to a homomorphism of Galois modules $D(\phi)\colon X(T_2) \longrightarrow X(T_1)$ and, clearly, there exists an induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X(T_2)^I & \longrightarrow & X(T_2) & \longrightarrow & X(\widetilde{T_2}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D(\phi)|_{X(T_2)^I}} & & \downarrow{\scriptstyle D(\phi)} & & \downarrow & & \\ 0 & \longrightarrow & X(T_1)^I & \longrightarrow & X(T_1) & \longrightarrow & X(\widetilde{T_1}) & \longrightarrow & 0 \end{array}$$

Thus the assertion in the statement follows from Proposition 0.3.11. □

Proposition 0.5.3. (cf. [X, 2.13]) *Let K be a local field and let T be an algebraic K-torus. Then there exists an exact sequence of algebraic K-tori in the smooth and étale topologies*

$$0 \longrightarrow M \longrightarrow Q \longrightarrow T \longrightarrow 0, \tag{8}$$

where M has multiplicative reduction and Q is such that $H^1(I, X(Q)) = 0$, where I is the inertia subgroup of $\mathrm{Gal}\,(K^{\mathrm{sep}}/K)$.

Proof. It suffices to construct the corresponding sequence of character groups. To do this, we start with the sequence (4)

$$0 \longrightarrow X(T) \longrightarrow \mathbb{Z}^d[\mathrm{Gal}(L/K)] \longrightarrow X(R) \longrightarrow 0.$$

and consider the preimage $X(Q)$ of $X(R)^I$ in $\mathbb{Z}^d[\mathrm{Gal}(L/K)]$. Then $X(Q)$ is a torsion-free and saturated $\mathrm{Gal}(L/K)$-submodule and we have a sequence

$$0 \longrightarrow X(T) \longrightarrow X(Q) \longrightarrow X(R)^I \longrightarrow 0.$$

By definition, this sequence is exact except, perhaps, at $X(Q)$. All that remains to be shown is that each element from the kernel of the map $X(Q) \longrightarrow X(R)^I$ comes from $X(T)$. But this is clear because the initial sequence was exact. By definition we have an exact sequence

$$0 \longrightarrow X(Q) \longrightarrow \mathbb{Z}^d[\mathrm{Gal}(L/K)] \longrightarrow X(R)/X(R)^I \longrightarrow 0. \tag{9}$$

Since the Galois action on $X(Q)$ factors through $\mathrm{Gal}(L/K)$ and $X(Q)$ is torsion-free, we have $\mathrm{H}^1(G_L, X(Q)) = 0$. By [S, VII, §6 Proposition 5] and the exactness of the direct limit functor, we conclude that $\mathrm{H}^1(I, X(Q)) = \mathrm{H}^1\big(I_{L/K}, X(Q)\big)$, where $I_{L/K}$ is the inertia group of the extension L/K. Now, since $\mathrm{H}^0\big(I_{L/K}, X(R)/X(R)^I\big) = \mathrm{H}^1\big(I_{L/K}, \mathbb{Z}^d[\mathrm{Gal}(L/K)]\big) = 0$, the long exact $I_{L/K}$-cohomology sequence induced by (9) yields $\mathrm{H}^1\big(I_{L/K}, X(Q)\big) = 0$. □

Now we can explain the description of the group of components of the Néron model of an algebraic torus given by Xavier Xarles in [X]. This description assumes that the residue field is perfect.

So let K be a local field with a perfect residue field and let T be an algebraic K-torus with character group $X(T)$. Let $\mathcal{T}$ be the Néron model of T over $\mathrm{Spec}\,\mathcal{O}_K$ and let $\Phi := \Phi(\mathcal{T}_s)$ be the group of components of the special fiber of the Néron model. This is always interpreted as a $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$-module. Finally, let $I := \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})$ be the inertia group of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. As we saw in Proposition 0.2.1, one can determine the group of components of the special fiber of the Néron model of an algebraic torus T in the étale topology. Xarles also takes this approach. In [X, Theorem 1.1] it is stated that $\Phi \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ if T has multiplicative reduction. Xarles proves this by explicitly determining the sequence

$$0 \longrightarrow \mathcal{T}^0 \longrightarrow \mathcal{T} \longrightarrow i_*\Phi \longrightarrow 0.$$

In [X, Theorem 2.1], Xarles shows that for any K-torus T there are natural isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z}) \cong X(T)^I = \mathrm{H}^0(I, X(T)) \tag{0.5.3.1}$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z}) \cong \mathrm{H}^1(I, X(T)) \tag{0.5.3.2}$$

To prove these two statements, Xarles uses two basic tools. On the one hand, he uses that in the étale topology the formation of the Néron model for algebraic tori is exact, whence $\mathrm{R}^1 j_* T = 0$ [X, Lemma 2.3]. This means that Xarles obtains short exact sequences of their Néron models from short exact sequences of algebraic tori.

On the other hand, Xarles identifies the sheaves

$$\underline{\mathrm{Hom}}(\mathcal{T}, i_*\mathbb{Z}) \cong \underline{\mathrm{Hom}}(i_*\Phi, i_*\mathbb{Z}) \cong i_* \mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$$

in the smooth and étale topologies and, in the smooth topology, the sheaves

$$\underline{\mathrm{Ext}}^1(\mathcal{T}, i_*\mathbb{Z}) \cong \underline{\mathrm{Ext}}^1(i_*\Phi, i_*\mathbb{Z}) \cong i_* \mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z}).$$

These identifications are given in [X, Lemmas 2.2 and 2.12 and proof of Proposition 2.14]. With these tools, he gets the statement 0.5.3.1 from the sequence (6)

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0$$

by applying the functors j_* and $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ in the étale topology.

On the other hand, he gets the statement 0.5.3.2 from the sequence (8)

$$0 \longrightarrow M \longrightarrow Q \longrightarrow T \longrightarrow 0$$

by applying the functors j_* and $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ in the smooth topology. For this he needs the auxiliary results that $\mathrm{R}^1 j_* T$ vanishes in the smooth topology for tori with multiplicative reduction [X, Lemma 2.11], and that algebraic tori T with $\mathrm{H}^1(I, X(T)) = 0$ have a torsion-free group of components [X, Proposition 2.7], whence $\underline{\mathrm{Ext}}^1(j_*R, i_*\mathbb{Z}) = 0$. The statement [X, Proposition 2.7] is essentially based on properties of the Weil restriction [X, Proposition 2.6].

The main result of Xarles is [X, Theorem 3.1]. Here Xarles chooses an I-acyclic resolution

$$X(T) \longrightarrow J' \longrightarrow J'' \longrightarrow \cdots$$

of the character group with $\mathbb{Z}$-free continuous $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules and defines

$$X' := \ker(J' \longrightarrow J'')\,.$$

This gives him the statement

Theorem 0.5.4. [X, Theorem 3.1] *There exists an exact sequence of G_k-modules*

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big((X')^I, \mathbb{Z}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(M^I, \mathbb{Z}\big) \longrightarrow \Phi \longrightarrow 0.$$

Xarles proves this result by using the short exact sequence

$$0 \longrightarrow X(T) \longrightarrow J \longrightarrow X' \longrightarrow 0$$

and Cartier duality to obtain a short exact sequence of algebraic tori

$$0 \longrightarrow T' \longrightarrow T_J \longrightarrow T \longrightarrow 0$$

and shows that the associated short exact sequence of Néron models induces a short exact sequence of groups of components

$$0 \longrightarrow (\Phi')^{\vee\vee} \longrightarrow \Phi_J \longrightarrow \Phi \longrightarrow 0.$$

Here $(\cdot)^\vee = \mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$.

To understand the proofs in [X] some remarks are necessary: Xarles uses the fact that the isomorphism $\Phi \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ from [X, 1.1] is compatible with homomorphisms of algebraic tori without proving this explicitly. This compatibility implies that the isomorphisms from [X, 2.1] are also natural, i.e. compatible with homomorphisms. This is also implicitly required when computing Φ in [X, Theorem 3.1].

Further, Xarles formulates his results without specifying a finite splitting extension of T. This leads to a problem with [X, Theorem 3.1]: the I-acyclic module J cannot be finitely generated [Br, VI, Theorem 8.7(v)], since I is an infinite profinite group, and thus the Cartier dual of J is not an algebraic torus.

To get around this problem, it is advisable to formulate the descriptions relative to a finite Galois splitting extension L/K of T. To do this, we specify our notations: I_K is the inertia group of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, I_L is the inertia group of $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ and $I_{L/K}$ is the inertia group of $\mathrm{Gal}(L/K)$. Since L/K is Galois, I_L is a normal subgroup of I_K and $I_{L/K} \cong I_K/I_L$. Since the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ factors through the quotient $\mathrm{Gal}(L/K)$, it follows that $X(T)^{I_K} =$

$X(T)^{I_{L/K}}$. Using the canonical inflation-restriction exact sequence [S, VII, §6 Proposition 5] and the exactness of the direct limit functor, we obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^1\big(I_{L/K}, X(T)\big) \longrightarrow \mathrm{H}^1(I_K, X(T)) \longrightarrow \mathrm{H}^1(I_L, X(T))\,.$$

Since the profinite group I_L acts trivially on the torsion-free group $X(T)$, we have $\mathrm{H}^1(I_L, X(T)) = 0$. Thus we can also compute H^1 relatively. In particular, the Galois structure induced by $\mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow \mathrm{Gal}(l/k)$ on the group $\mathrm{H}^i\big(I_{L/K}, X(T)\big)$ equals the canonical Galois structure on the group $\mathrm{H}^i(I_K, X(T))$ (for $i = 0, 1$).

Similarly, an I-acyclic resolution can also be understood as an $I_{L/K}$-acyclic resolution by continuous $\mathrm{Gal}(L/K)$-modules. These are again transformed into continuous $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules by the projection $\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$.

As already mentioned in the Introduction, the results [X, 2.1 and 3.1] cannot be extended to arbitrary local fields. The reason for this is that the formation of the Néron model is no longer exact in general, since the Brauer group of K^{nr} is no longer trivial when k is not perfect.

From a more general perspective, new interpretations of the proofs in [X] arise: the sequence used by Xarles in the proof of [X, Theorem 1.1] can be defined for arbitrary tori and then has the form

$$0 \longrightarrow \mathcal{T}^{\mathrm{ft}} \longrightarrow \mathcal{T} \longrightarrow i_*\mathrm{H}^0(I, X(T)) \longrightarrow \cdots,$$

where $\mathcal{T}^{\mathrm{ft}}$ is the ft-Néron model of T defined in Theorem 3.1.3.

The considerations from the proof of [X, Theorem 3.1] can be generalized. From any short exact sequence of Néron models

$$0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0$$

we can obtain a short exact sequence of groups of components

$$0 \longrightarrow \widetilde{\Phi}(\mathcal{N}_1) \longrightarrow \Phi(\mathcal{N}_2) \longrightarrow \Phi(\mathcal{N}_3) \longrightarrow 0,$$

where $\widetilde{\Phi}(\mathcal{N}_1)$ is a quotient of $\Phi(\mathcal{N}_1)$ by a suitable torsion subgroup. With this result one can derive [X, 3.1] from the description 0.5.3.1 using [X, 2.7]. The description 0.5.3.2 then follows as a corollary to [X, Theorem 3.1]. We will follow this method of proof in Theorem 6.1.1, for example.

Chapter 1

Néron models of some specific algebraic tori

Let K be a local field and let T be an algebraic K-torus. The lft-Néron model $\mathcal{T}$ of T exists and we denote the group of components of the special fiber of this model by $\Phi(T) := \Phi(\mathcal{T}_k)$. Since we are considering algebraic tori, Néron models in this chapter are always lft-Néron models.

First, we consider an algebraic torus T with multiplicative reduction. Starting from the explicit construction of the Néron model of $\mathbb{G}_{m,K}$, we can describe $\mathcal{T}$ in this case via Galois descent. We can identify the identity component $\mathcal{T}^0$ and the $\mathcal{O}_K$-torus $\underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})$. This gives us an isomorphism $\Phi(T) \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$, which is compatible with homomorphisms of tori with multiplicative reduction .

Next we consider the case where $T = \mathfrak{R}_{L/K}(T')$, where L/K is a finite separable extension of local fields and T' is an L-torus. If $\mathcal{T}'$ is the Néron model of T', then $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{T}')$ is the Néron model of T and we will show that its identity component is equal to $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)$. By the exactness of the Weil restriction functor in the étale topology, it follows that $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(i_*\Phi(T')) = i_*\Phi(T)$.

Conversely, the Weil restriction on the character groups corresponds to the induction of Galois modules. Thus [X, Theorem 3.1] holds for $\Phi(T)$ if, and only if, it holds for $\Phi(T')$.

After these direct generalizations of some of the results from [X], we want to provide a first family of counterexamples. To do this, we generalize the calculation of the reduction of the Néron model for norm-one tori with respect to a cyclic and totally ramified extension L/K of degree $p = \mathrm{char}(k)$ from [LL, §5].

For these tori, [X, Theorem 3.1] predicts a group of components of the form $\mathbb{Z}/p\mathbb{Z}$. If L/K induces a trivial extension of the residue fields, this remains valid. However, our calculations provide counterexamples if there is residual ramification. Specifically, in our examples the group of components is trivial.

These examples also provide a counterexample to a generalization of [NX, Proposition 3.2], since in the case of residual ramification we find that, instead of $\mathbb{G}_{a,k}^{p-1}$, we obtain a k-wound unipotent group as the reduction of the identity component.

1.1 Néron models of tori with multiplicative reduction

Let T be an algebraic K-torus with multiplicative reduction and let $\mathcal{T}$ be the Néron model of T. Then there exists a finite, unramified and Galois extension L of K such that the torus T

trivializes over L, i.e. $T_L \cong \mathbb{G}_{m,L}^d \cong \operatorname{Spec} L[X(T)]$, where $d = \dim(T)$. Since the formation of Néron models is compatible with an unramified base change [BLR, 10.1.3], one can obtain the Néron model $\mathcal{T}$ via Galois descent from the Néron model of $\mathbb{G}_{m,L}^d$ over $\mathcal{O}_L$.

The Néron model $\mathcal{G}_{\mathcal{O}_L}^d$ of $\mathbb{G}_{m,L}^d$ over $\operatorname{Spec} \mathcal{O}_L$ is constructed by gluing copies of

$$\pi_L^{\nu_1} \mathbb{G}_{m,\mathcal{O}_L} \times_{\mathcal{O}_L} \pi_L^{\nu_2} \mathbb{G}_{m,\mathcal{O}_L} \times_{\mathcal{O}_L} \cdots \times_{\mathcal{O}_L} \pi_L^{\nu_d} \mathbb{G}_{m,\mathcal{O}_L} \cong \mathbb{G}_{m,\mathcal{O}_L}^d,$$

where $\nu_1, \dots, \nu_d \in \mathbb{Z}$, along the generic fibers.

On the generic fiber, the trivialization $\mathbb{G}_{m,L}^d \cong \operatorname{Spec} L[X(T)]$ yields an effective descent datum in the form of an action of the Galois group $\operatorname{Gal}(L/K)$ on $\operatorname{Spec} L[X(T)]$: this action is defined on the algebra $L[X(T)]$ by the simultaneous canonical action on the scalars from L and on the characters (as $\operatorname{Gal}(L/K)$-module, since by hypothesis $\operatorname{Gal}(K^{\text{sep}}/L)$ acts trivially on $X(T)$). The effectiveness is clear because the operating group is finite.

By the Néron mapping property, the action extends to an action on the Néron model $\mathcal{G}_{\mathcal{O}_L}^d$ and also yields an effective descent datum.

The isomophism $\mathbb{G}_{m,L}^d \cong \operatorname{Spec} L[X(T)]$ extends to an isomorphism

$$\left(\mathcal{G}_{\mathcal{O}_L}^d\right)^0 \cong \mathbb{G}_{m,\mathcal{O}_L}^d \cong \operatorname{Spec} \mathcal{O}_L[X(T)].$$

The Galois group $\operatorname{Gal}(L/K)$ acts on $L[X(T)]$ via K-automorphisms and these clearly are limited to $\mathcal{O}_K$-automorphisms of $\mathcal{O}_L[X(T)]$. This means that the identity component of $\mathcal{G}_{\mathcal{O}_L}^d$ is stable under the descent datum and maps into the $\mathcal{O}_K$-torus $T_{\mathcal{O}} := \operatorname{Spec} \mathcal{O}_L[X(T)]^{\operatorname{Gal}(\mathcal{O}_L/\mathcal{O}_K)}$ (defined by $X(T)$ as $\operatorname{Gal}(\mathcal{O}_K^{\text{sh}}/\mathcal{O}_K)$-module). So $\mathcal{T}^0 = T_{\mathcal{O}}$.

We now want to determine the short exact sequence

$$0 \longrightarrow \mathcal{T}^0 \longrightarrow \mathcal{T} \longrightarrow i_*\Phi(T) \longrightarrow 0.$$

We repeatedly use the decomposition theorem [M, II, Example 3.12, p. 75], which states that there exists an equivalence of categories between the category of abelian sheaves on the étale site over $\operatorname{Spec} \mathcal{O}_K$ and the category of triples (M_K, N_k, ϕ), where M_K is a continuous $\operatorname{Gal}(K^{\text{sep}}/K)$-module, N_k is a continuous $\operatorname{Gal}(k^{\text{sep}}/k)$-module and $\phi\colon : N_k \longrightarrow M^I$ is a $\operatorname{Gal}(k^{\text{sep}}/k)$-module homomorphism. For an étale sheaf $\mathcal{F}$, under the equivalence M_K is the representing module of $j^*\mathcal{F}$ on the étale site over $\operatorname{Spec} K$, N_k is the representing module of $i^*\mathcal{F}$ on the étale site over $\operatorname{Spec} k$ and ϕ corresponds to the morphism $i^*\mathcal{F} \longrightarrow i^*j_*j^*\mathcal{F}$. Morphisms of sheaves correspond to pairs of continuous homomorphisms of the Galois modules which commute with the maps ϕ.

Proposition 1.1.1. *Let T be an algebraic K-torus with multiplicative reduction and let $\mathcal{T}$ be its Néron model over $\mathcal{O}_K$. Then there exists a commutative diagram in the étale topology*

$$\begin{array}{ccc} \mathcal{T}^0 & \hookrightarrow & \mathcal{T} \\ \cong\downarrow & & \cong\downarrow \\ \underline{\operatorname{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) & \hookrightarrow & \underline{\operatorname{Hom}}(j_*\underline{X}(T), j_*\mathbb{G}_{m,K}) \end{array}$$

where the inclusion on the top row comes from the canonical open immersion of the identity component and the map on the bottom row is induced by the inclusion $\iota\colon \mathbb{G}_{m,\mathcal{O}_K} \longrightarrow \mathcal{G}_K$ coming from the short exact sequence for the Néron model of $\mathbb{G}_{m,K}$.

Proof. We first establish the isomorphisms. By Cartier duality, we have $T \cong \underline{\operatorname{Hom}}(\underline{X}(T), \mathbb{G}_{m,K})$. It follows that

$$\mathcal{T} = j_*T = j_*\underline{\operatorname{Hom}}(\underline{X}(T), \mathbb{G}_{m,K}) = \underline{\operatorname{Hom}}(j_*\underline{X}(T), j_*\mathbb{G}_{m,K}),$$

since $\underline{X}(T) \cong j^* j_* \underline{X}(T)$.

Since T has multiplicative reduction, $\underline{X}(T)$ can be viewed as a sheaf over $\mathcal{O}_K$ by identifying $\underline{X}(T)$ with the triple $(X(T), X(T), \mathrm{id}) = j_* \underline{X}(T)$. By Cartier duality, it follows that $\mathcal{T}^0 = T_{\mathcal{O}} = \underline{\mathrm{Hom}}(\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})$. To check commutativity, it suffices to check it on test schemes of the following forms: $U = \operatorname{Spec} L'$, where L'/K is a finite and separable field extension, and $U = \operatorname{Spec} \mathcal{O}_{L'}$ for a finite and unramified field extension L'/K. For schemes of the first form, on the top row we have $\mathcal{T}^0(\operatorname{Spec} L') = T(L')$, as well as $\mathcal{T}(\operatorname{Spec} L') = T(L')$, and the inclusion $\mathcal{T}^0 \hookrightarrow \mathcal{T}$ corresponds to the identity map $\mathrm{id}\colon T \longrightarrow T$ on the generic fiber. For the bottom row we find

$$\underline{\mathrm{Hom}}(j_* \underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})\,(U) = \mathrm{Hom}_U(\underline{X}(T)|_U, \mathbb{G}_{m,U}) \quad \text{and}$$
$$\underline{\mathrm{Hom}}(j_* \underline{X}(T), j_* \mathbb{G}_{m,K})\,(U) = \mathrm{Hom}_U\big(\underline{X}(T)|_U, j_* \mathbb{G}_{m,K}|_U\big).$$

Since the pullback over U of the map $\mathbb{G}_{m,\mathcal{O}_K} \longrightarrow j_* \mathbb{G}_{m,K}$ is the identity on $\mathbb{G}_{m,U}$, both rows of the diagram are isomorphic via Cartier duality.

We now consider test schemes of the second form, $U = \operatorname{Spec} \mathcal{O}_{L'}$, where L'/K is a finite unramified extension. Without loss of generality, we may assume that the splitting extension L was chosen so that $L \supset L'$.

On the top row we can use the $U = \operatorname{Spec} \mathcal{O}_{L'}$-valued points as the $\mathrm{Gal}(L/L')$-invariant $\operatorname{Spec} \mathcal{O}_L$-valued points, whereby the Galois action is derived from the trivialization

$$\mathcal{T}^0(U) = \mathrm{Hom}_{\mathcal{O}_L}\big(\operatorname{Spec} \mathcal{O}_L, \mathbb{G}^d_{m,\mathcal{O}_L}\big)^{\mathrm{Gal}(L/L')} = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathcal{O}_L^*)^{\mathrm{Gal}(L/L')}$$
$$\mathcal{T}(U) = \mathrm{Hom}_{\mathcal{O}_L}\big(\operatorname{Spec} \mathcal{O}_L, \mathcal{G}^d_{\mathcal{O}_L}\big)^{\mathrm{Gal}(L/L')} = \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)^{\mathrm{Gal}(L/L')}$$

Now the trivialisation $T_L = \operatorname{Spec} L[X(T)]$ induces the trivialisation $T_{\mathcal{O}_L} = \operatorname{Spec} \mathcal{O}_L[X(T)]$. Thus the map $\mathcal{O}_L^* \hookrightarrow L^*$ also induces the inclusion

$$\mathcal{T}^0(U) = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathcal{O}_L^*)^{\mathrm{Gal}(L/L')} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)^{\mathrm{Gal}(L/L')} = \mathcal{T}(U).$$

Let us now look at the bottom row :

A $\psi \in \mathrm{Hom}_U(j_* \underline{X}(T)|_U, \mathbb{G}_{m,U}) = \underline{\mathrm{Hom}}(j_* \underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})\,(U)$ corresponds, according to the decomposition theorem, to a pair (ψ_η, ψ_s) with a $\mathrm{Gal}(K^{\mathrm{sep}}/L')$-module homomorphism $\psi_\eta\colon X(T) \longrightarrow (K^{\mathrm{sep}})^*$ together with a $\mathrm{Gal}(K^{\mathrm{nr}}/L')$ - module homomorphism $\psi_s\colon X(T) \longrightarrow (\mathcal{O}_K^{\mathrm{sh}})^*$ and a compatibility condition, namely that ψ_η on the I-invariants matches ψ_s. Since $X(T)^I = X(T)$, ψ_η and ψ_s must already be equal. This means that ψ is already uniquely determined by ψ_s. We obtain

$$\begin{aligned}\mathrm{Hom}_U(j_* \underline{X}(T)|_U, \mathbb{G}_{m,U}) &= \mathrm{Hom}_{\mathrm{Gal}(K^{\mathrm{nr}}/L')}\big(X(T), (\mathcal{O}_K^{\mathrm{sh}})^*\big)\\ &= \mathrm{Hom}_{\mathbb{Z}}\big(X(T), (\mathcal{O}_K^{\mathrm{sh}})^*\big)^{\mathrm{Gal}(K^{\mathrm{nr}}/L')}\end{aligned}$$

and similarly, using the formula $(j_* \mathbb{G}_{m,K})_{\bar{s}} = (K^{\mathrm{nr}})^*$,

$$\underline{\mathrm{Hom}}\,(j_* \underline{X}(T), j_* \mathbb{G}_{m,K})\,(U) = \mathrm{Hom}_U\big(j_* \underline{X}(T)|_U, j_* \mathbb{G}_{m,K}|_U\big)$$
$$= \mathrm{Hom}_{\mathrm{Gal}(K^{\mathrm{nr}}/L')}(X(T), (K^{\mathrm{nr}})^*) = \mathrm{Hom}_{\mathbb{Z}}\,(X(T), (K^{\mathrm{nr}})^*)^{\mathrm{Gal}(K^{\mathrm{nr}}/L')}.$$

Since T trivializes over L, in these descriptions we can always replace Gal (K^{nr}/L') by Gal (L/L'). On the stalks above $\bar{s}$, the map $\mathbb{G}_{m,\mathcal{O}_K} \longrightarrow j_* \mathbb{G}_{m,K}$ corresponds to the canonical inclusion $(\mathcal{O}_K^{\mathrm{sh}})^* \longrightarrow (K^{\mathrm{nr}})^*$, which shows commutativity. □

This gives us two important results:

Theorem 1.1.2. (See [X, 1.1]). *Let K be a local field and let T be a K-torus with multiplicative reduction and character group $X(T)$. Then the sequence*

$$0 \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_m, \mathcal{O}_K) \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathcal{G}) \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), i_*\mathbb{Z}) \longrightarrow 0,$$

which results by applying $\underline{\mathrm{Hom}}(j_*\underline{X}(T), \cdot)$ to the short exact sequence of the Néron model of $\mathbb{G}_{m,K}$, is exact and isomorphic to the sequence

$$0 \longrightarrow j_*T^0 \longrightarrow j_*T \longrightarrow i_*\Phi(T) \longrightarrow 0.$$

In particular, $\Phi(T) \cong \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ for the group of components as a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-module.

Proof. Proposition 1.1.1 yields the isomorphism of the sequences if the first sequence is exact, for which we have to show that $\underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) = 0$.

Since T has multiplicative reduction, there exists a finite unramified and Galois extension L/K such that T trivializes over L. This means that $j_*\underline{X}(T)|_{\operatorname{Spec}\mathcal{O}_L}$ is equal to the constant sheaf $\mathbb{Z}^d$, where $d = \dim T$.

Thus, for an étale morphism $U \longrightarrow \operatorname{Spec}\mathcal{O}_L$, we have

$$\mathrm{Ext}^1_U\left(j_*\underline{X}(T)|_U\,,\, \mathbb{G}_{m,\mathcal{O}_K}|_U\right) = \mathrm{H}^1\left(U, \mathbb{G}_{m,U}\right)^d$$

Since these cohomology groups vanish locally, $\underline{\mathrm{Ext}}^1\left(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}\right)$ must vanish since it is the sheafification of the presheaf $V \mapsto \mathrm{Ext}^1_V\left(j_*\underline{X}(T)|_V\,,\, \mathbb{G}_{m,\mathcal{O}_K}|_V\right)$.

Finally, the sheaf $\underline{\mathrm{Hom}}\left(j_*\underline{X}(T), i_*\mathbb{Z}\right)$ is a skyscraper sheaf and its preimage in the étale site over $\operatorname{Spec} k$ is given by $\mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ (as a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-module) (cf. [M, Example III.1.7(c)]). □

Secondly, this description is even functorial, which more precisely means the following:

Theorem 1.1.3. *Let $\phi\colon T_1 \longrightarrow T_2$ be a morphism of algebraic K-tori with multiplicative reduction and let $D(\phi)\colon X(T_2) \longrightarrow X(T_1)$ be the associated map of character groups. Then the map $j_*\phi\colon \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between the Néron models induces a map $\Phi(T_1) \longrightarrow \Phi(T_2)$ of the group of components, which via the above identification is equal to $D(\phi)^\vee\colon \mathrm{Hom}_{\mathbb{Z}}(X(T_2)\,,\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X(T_1)\,,\mathbb{Z})$.*

Proof. The map $D(\phi)$ induces a sheaf homomorphism $j_*\underline{X}(T_2) \longrightarrow j_*\underline{X}(T_1)$ and thus a morphism of functors $\underline{\mathrm{Hom}}(j_*\underline{X}(T_1)\,,\cdot) \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T_2)\,,\cdot)$. This morphism of functors induces a commutative diagram of étale sheaves

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(j_*\underline{X}(T_1), \mathbb{G}_{m,\mathcal{O}_K}) & \hookrightarrow & \underline{\mathrm{Hom}}(j_*\underline{X}(T_1), \mathcal{G}) & \twoheadrightarrow & \underline{\mathrm{Hom}}(j_*\underline{X}(T_1), i_*\mathbb{Z}) \\
\downarrow{\scriptstyle \psi^0} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar{\psi}} \\
\underline{\mathrm{Hom}}(j_*\underline{X}(T_2), \mathbb{G}_{m,\mathcal{O}_K}) & \hookrightarrow & \underline{\mathrm{Hom}}(j_*\underline{X}(T_2), \mathcal{G}) & \twoheadrightarrow & \underline{\mathrm{Hom}}(j_*\underline{X}(T_2), i_*\mathbb{Z}).
\end{array}$$

We claim that the above diagram is isomorphic to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (j_*T_1)^0 & \longrightarrow & j_*T_1 & \longrightarrow & i_*\Phi(T_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_*\phi^0} & & \downarrow{\scriptstyle j_*\phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \\
0 & \longrightarrow & (j_*T_2)^0 & \longrightarrow & j_*T_2 & \longrightarrow & i_*\Phi(T_2) & \longrightarrow & 0
\end{array}$$

Note that the identity component $\mathcal{T}_1^0$ maps under the group homomorphism $j_*\phi$ into the identity component $\mathcal{T}_2^0$, making the last diagram well-defined.

It is clearly enough to show the equality of the first two vertical maps. Using the argument from Proposition 1.1.1, one sees that the maps ψ and ψ^0 are clearly determined by their “generic fibers" ψ_η or ψ_η^0. These correspond, via Cartier duality over K, to the morphism $\phi\colon T_1 \longrightarrow T_2$.

Conversely, the morphisms $j_*\phi$ and $j_*\phi^0$ are also uniquely determined by their generic fibers, by the Néron mapping property and Cartier duality, respectively. □

1.2 Néron models of Weil restrictions

Theorem 1.2.1. *Let K be a local field, let T be an algebraic K-torus and let L/K be a finite separable extension. Assume, in addition, that there exists an L-torus T' such that $T \cong \mathfrak{R}_{L/K}(T')$.*

Then the description from [X, Theorem 3.1] *holds for $\Phi(T')$ if, and only if, it holds for $\Phi(T)$.*

To establish this theorem, we must first prove two lemmas. First we generalize [NX, Proposition 2.4]:

Lemma 1.2.2. *Let L/K be a finite separable extension of local fields and let $\mathcal{T}'$ be an affine smooth* Spec $\mathcal{O}_L$*-group scheme with connected fibers, i.e. $T' := \mathcal{T}' \otimes_{\mathcal{O}_L} L$ and $T'_l := \mathcal{T}' \otimes_{\mathcal{O}_L} l$ are connected. Then $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{T}')$ also has connected fibers.*

Proof. On the generic fiber we obtain

$$\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{T}') \otimes_{\mathcal{O}_K} K \cong \mathfrak{R}_{L/K}(\mathcal{T}' \otimes_{\mathcal{O}_L} L) = \mathfrak{R}_{L/K}(T')\,.$$

It can be shown that a Weil restriction along a separable extension of an affine, smooth and connected group scheme over a field is again connected. Let $\bar{K}$ be an algebraic closure of K and let $G_{L/K} := \mathrm{Hom}_K(L, \bar{K})$ be the group of K-embeddings of L into this closure. After tensoring with $\bar{K}$ we obtain:

$$\begin{aligned}\mathfrak{R}_{L/K}(T') \otimes_K \bar{K} &\cong \mathfrak{R}_{L\otimes_K \bar{K}/\bar{K}}(T' \otimes_L L \otimes_K \bar{K}) \\ &\cong \mathfrak{R}_{\prod_{G_{L/K}} \bar{K}/\bar{K}}\left(\coprod_{G_{L/K}} T'_{\bar{K}}\right) \cong \prod_{G_{L/K}} \mathfrak{R}_{\bar{K}/\bar{K}}(T'_{\bar{K}}) \\ &\cong \prod_{G_{L/K}} T'_{\bar{K}}.\end{aligned}$$

Since $T'_{\bar{K}}$ is connected, $\prod_{G_{L/K}} T'_{\bar{K}}$ is connected as well by [SGA3, VIa, Lemma 2.1.2]. This means that the generic fiber is geometrically connected, i.e., connected a fortiori.

Since $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{T}') \otimes_{\mathcal{O}_K} k \cong \mathfrak{R}_{\mathcal{O}_L\otimes_{\mathcal{O}_K} k/k}(\mathcal{T}' \otimes_{\mathcal{O}_L} (\mathcal{O}_L \otimes_{\mathcal{O}_K} k))$, we first need to determine $\mathcal{O}_L \otimes_{\mathcal{O}_K} k$. Since the Weil restriction functor is compatible with subextensions, it suffices to consider the following particular cases: L/K is unramified and L/K is totally ramified.

In the first case $\mathcal{O}_L \otimes_{\mathcal{O}_K} k \cong l$ and l/k is a separable field extension. This is entirely similar to the situation on the generic fiber. So assume that L/K is totally ramified. Then $\mathcal{O}_L \otimes_{\mathcal{O}_K} k \cong l[X]/(X^e)$ is a radicial extension of k. By [SGA3, VIa, Lemma 2.1.2]

$$\mathcal{T}' \otimes_{\mathcal{O}_L} (\mathcal{O}_L \otimes_{\mathcal{O}_K} k) \cong \mathcal{T}'_l \otimes_l l[X]/(X^e)$$

is a smooth, affine and connected k-group scheme. It now follows from [SGA3, XVII, Appendix III, Proposition 5.1] that the Weil restriction is connected since $l[X]/(X^e)$ is a radicial extension of k. □

Lemma 1.2.3. *Let L/K be a finite separable extension of local fields and let $\tilde{L}/L$ be a finite Galois extension. Let $X(T')$ be a finitely generated continuous $\mathrm{Gal}(K^{\mathrm{sep}}/L)$-module on which $\mathrm{Gal}\left(K^{\mathrm{sep}}/\tilde{L}\right)$ acts trivially. Let I_L be the inertia group of $G_L := \mathrm{Gal}(\tilde{L}/L)$ and similarly let I_K be the inertia group of $\mathrm{Gal}(\tilde{L}/K)$. Now let*

$$0 \longrightarrow X(T') \longrightarrow J_L^0 \longrightarrow J_L^1 \longrightarrow J_L^2 \longrightarrow \dots$$

be a resolution of $X(T')$ by finitely generated continuous $\mathrm{Gal}(K^{\mathrm{sep}}/L)$-modules, which are torsion-free and I_L-acyclic. Further, let $X' := \ker\left[J_L^1 \longrightarrow J_L^2\right]$ and set (as in [X, Theorem 3.1])

$$\Phi_L := \mathrm{coker}\left[\mathrm{Hom}_{\mathbb{Z}}\left((X')^{I_L}, \mathbb{Z}\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left((J_L^0)^{I_L}, \mathbb{Z}\right)\right].$$

Then, for $X := \mathrm{Ind}_{G_K}^{G_L} X(T')$ and $J^i := \mathrm{Ind}_{G_K}^{G_L} J_L^i$ with $G_K := \mathrm{Gal}(\tilde{L}/K)$,

$$0 \longrightarrow X \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots$$

is a resolution of X by finitely generated continuous $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules on which $\mathrm{Gal}\left(K^{\mathrm{sep}}/\tilde{L}\right)$ acts trivially. Further, the J^i are torsion-free and I_K-acyclic and we have

$$\Phi := \mathrm{coker}\left[\mathrm{Hom}_{\mathbb{Z}}\left(X_0^{I_K}, \mathbb{Z}\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left((J^0)^{I_K}, \mathbb{Z}\right)\right] \cong \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \Phi_L,$$

where $X_0 := \ker\left[J^1 \longrightarrow J^2\right]$, L_{nr} denotes the maximal unramified extension of K in L and $G_{L_{\mathrm{nr}}} := \mathrm{Gal}\left(\tilde{L}/L_{\mathrm{nr}}\right)$.

Proof. It is clear that the J^i are torsion-free. Since L/K is a finite extension, induction and coinduction are isomorphic with respect to the inclusion $G_L \subset G_K$. We can now decompose the field extension L/K into a chain $L \supset L_{\mathrm{nr}} \supset K$. Then, for a finitely generated continuous G_L-module N, we have

$$\mathrm{Ind}_{G_K}^{G_L} N \cong \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \mathrm{Ind}_{G_{L_{\mathrm{nr}}}}^{G_L} N.$$

Obviously I_L is a subgroup of finite index in I_K and, via restriction to the category of I_K-modules, the induction $\mathrm{Ind}_{G_{L_{\mathrm{nr}}}}^{G_L} N$ is isomorphic to $\mathrm{Ind}_{I_K}^{I_L} N$. Thus, for any $j \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{H}^j\left(I_K, \mathrm{Ind}_{G_K}^{G_L} N\right) &= \mathrm{H}^j\left(I_K, \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \mathrm{Ind}_{G_{L_{\mathrm{nr}}}}^{G_L} N\right) \\ &\cong \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \mathrm{H}^j\left(I_K, \mathrm{Ind}_{I_K}^{I_L} N\right) = \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \mathrm{H}^j(I_L, N), \end{aligned}$$

where Shapiro's lemma was applied in the last step. This makes it clear that the J^i are again I_K-acyclic.

Since the induction is an exact functor, it follows that

$$X_0 = \ker\left(\mathrm{Ind}_{G_K}^{G_L} J_L^1 \longrightarrow \mathrm{Ind}_{G_K}^{G_L} J_L^2\right) = \mathrm{Ind}_{G_K}^{G_L} X'$$

and thus, after decomposing the induction and applying Shapiro's lemma, $X_0^{I_K} = \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} (X')^{I_L}$ and $(J^0)^{I_K} = \mathrm{Ind}_{G_{L_{\mathrm{nr}}}}^{G_K} (J_L^0)^{I_L}$. Next, the following isomorphism is valid for any finitely generated continuous $G_{L_{\mathrm{nr}}}$-module N:

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} N, \mathbb{Z}\right) \cong \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}).$$

Since the induction is an exact functor, it follows that $\Phi \cong \mathrm{Ind}_{G_K}^{G_{L_{\mathrm{nr}}}} \Phi_L$. □

Proof of Theorem 1.2.1. The formation of Néron models is compatible with Weil restriction: if $\mathcal{T}'$ is the Néron model of T' over $\operatorname{Spec}\mathcal{O}_L$, then $\mathcal{T} := \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{T}')$ is the Néron model of $T = \mathfrak{R}_{L/K}(T')$. Further, the Weil restriction with respect to a finite morphism is an exact functor on the étale site. Therefore the exact sequence

$$0 \longrightarrow (\mathcal{T}')^0 \longrightarrow \mathcal{T}' \longrightarrow i_*\Phi(T') \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big) \longrightarrow \mathcal{T} \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(i_*\Phi(T')) \longrightarrow 0.$$

We will show that $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)$ is the identity component of $\mathcal{T}$. The identity component $(\mathcal{T}')^0$ is a smooth, affine and open subgroup of the Néron model $\mathcal{T}'$. Since the Weil restriction of a group scheme is again a group scheme and the Weil restriction is also compatible with open immersions (see, e.g. [BLR, 7.6]), we see that $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)$ is a smooth and open subgroup of the Néron Model $\mathcal{T}$.

By [SGA3, VIb, Lemma 3.10.1], we have $\mathcal{T}^0 \subseteq \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)$ and necessarily $\mathcal{T}^0 = \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)^0$. Since $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)$ has connected fibers by Lemma 1.2.2, we must have $\mathcal{T}^0 = \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{T}')^0\big)$.

Summarizing, $\Phi := \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(i_*\Phi(T'))$ is the group of components of $\mathcal{T}$. Since the Weil restriction can be computed successively via subextensions, we first decompose the extension L/K into a chain $L \supset L_{\mathrm{nr}} \supset K$, where L/L_{nr} is totally ramified and L_{nr}/K is unramified. The totally ramified extension is solvable, so it can be broken up into subextensions which are either totally ramified with trivial residue field extension or totally residually ramified. Since we only have to determine the Weil restriction as an étale sheaf, it suffices to determine the $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-module

$$\Phi(k^{\mathrm{sep}}) = \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(i_*\Phi(T'))(k^{\mathrm{sep}}) = \Phi(T')(k^{\mathrm{sep}} \otimes_k (\mathcal{O}_L \otimes_{\mathcal{O}_K} k))\,.$$

We consider first the case where L/K is unramified, i.e., the residual extension l/k is separable. Set $G_{l/k} := \operatorname{Hom}_k(l, k^{\mathrm{sep}})$. Then

$$k^{\mathrm{sep}} \otimes_k (\mathcal{O}_L \otimes_{\mathcal{O}_K} k) = k^{\mathrm{sep}} \otimes_k l \cong \prod_{G_{l/k}} k^{\mathrm{sep}}$$

regarded as an l-algebra.

Let us now understand the effect of a $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k) = \mathrm{Aut}_k(k^{\mathrm{sep}})$ on $\Phi(k^{\mathrm{sep}})$. To do this, let $x \in l$ be a primitive element for l/k and write $l = k[X]/(f(X))$, where $f(X)$ is the minimal polynomial of x. Then $k^{\mathrm{sep}} \otimes_k l \cong k^{\mathrm{sep}}[X]/(f(X))$ and therefore $f(X)$ factors through k^{sep} as $f(X) = \Pi(X - \tau_j(X))$ for suitable representatives $\tau_j \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$ of $G_{l/k}$.

A Galois morphism $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$ now induces a permutation of the zeros $\tau_j(X)$, more precisely the component associated to $\tau_j(X)$ is mapped to the component of the zero $\sigma(\tau_j(X)) =: \tau_{j'}(X)$ while leaving l fixed, i.e., via the l-morphism $\tau_{j'}^{-1} \circ \sigma \circ \tau_j$. This yields

$$\Phi\left(k^{\mathrm{sep}}\right) \cong \prod_{G_{l/k}} \Phi\left(T'\right)\left(k^{\mathrm{sep}}\right) = \operatorname{Ind}_{\mathrm{Gal}(k^{\mathrm{sep}}/k)}^{\mathrm{Gal}(k^{\mathrm{sep}}/l)} \Phi\left(T'\right)\left(k^{\mathrm{sep}}\right).$$

The case where L/K is totally ramified with trivial residue field extension is covered in [X, §2.6]. In this case the Galois module is $\Phi \cong \Phi(T')$. If L/K is totally ramified and induces a purely inseparable residue field extension l/k, then we have an isomorphism $\mathrm{Gal}(k^{\mathrm{sep}}/k) \cong \mathrm{Gal}(l^{\mathrm{sep}}/l)$, which yields $l^{\mathrm{sep}} \cong k^{\mathrm{sep}} \otimes_k l$.

Further, without loss of generality, $[L\colon K] = [l : k]$, so that $\mathcal{O}_L \otimes_{\mathcal{O}_K} k \cong l$. It follows that $\Phi(k^{\mathrm{sep}}) \cong \Phi(T')(l^{\mathrm{sep}})$.

Thus, for an arbitrary finite and separable extension of local fields L/K, it follows that

$$\Phi \cong \mathrm{Ind}_{\mathrm{Gal}(K^{\mathrm{nr}}/K)}^{\mathrm{Gal}(L^{\mathrm{nr}}/L)} \Phi(T')$$

Now the isomorphism $\mathrm{Gal}(L^{\mathrm{nr}}/L) \cong \mathrm{Gal}(K^{\mathrm{nr}}/L_{\mathrm{nr}})$ together with Lemma 1.2.3 yields the equivalence of the validity of [X, Theorem 3.1] for the two groups of components. □

1.3 The norm-one torus of a cyclic extension of prime degree

If K is a local field with algebraically closed residue field, the article [LL] describes the norm-one torus associated to a cyclic and totally ramified Galois extension L/K of degree $p = \mathrm{char}(k)$. In analogy to the work of Liu and Lorenzini, in this section we assume that $p > 0$ and $k = k^{\mathrm{sep}}$ is separably closed. In [LL, 5.5 and 5.6], the reduction of the Néron model of such a norm-one torus is described explicitly. We will show that this description is also valid in the imperfect residue field case provided there is no residual ramification.

Proposition 1.3.1. *Let L/K be a totally ramified Galois extension of local fields of degree $p = \mathrm{char}(k)$ such that the corresponding residue field extension is separable and therefore necessarily trivial.*

Then the norm-one torus T_N of L/K has a Néron model that is isomorphic to

$$\mathrm{Spec}\, \mathcal{O}_K[X_0, .., X_{p-1}] \,/\, (G(X_0, .., X_{p-1}))\,,$$

where the polynomial $G(X_0, .., X_{p-1}) \in \mathcal{O}_K[X_0, .., X_{p-1}]$ is congruent modulo π to $X_m^p - uX_m$ for some $u \in \mathcal{O}_K^*$ and $m \in \{0, 1, .., p-1\}$.

Proof. The proof in [LL] covers [op.cit., Lemmas 5.3 and 5.4 and Theorems 5.5 and 5.6]. The following statements about the extension L/K are assumed (see [LL, 5.2]):

1. The extension is of Eisenstein type, i.e., it has the form

$$L = K[t]/\big(t^p - s_1 t^{p-1} + \ldots + (-1)^p s_p\big)\,,$$

   where the s_i are elements in $\pi_K \mathcal{O}_K$ and s_p has valuation $\nu_K(s_p) = 1$. Further, the class of t in L is a uniformizing element in $\mathcal{O}_L$.

2. The classes of $1, t, \ldots, t^{p-1}$ in L constitute a complete basis of $\mathcal{O}_L$ over $\mathcal{O}_K$.

3. The different is computed as

$$\nu_L\big(\mathcal{D}_{L/K}\big) = \min_{0 \le i \le p-1} \{p\nu_K(s_i) + p - 1 - i\} = (p-1)\nu_L(\sigma(T) - t),$$

   where $s_0 := p$ and σ is an arbitrary (fixed) generator of $\mathrm{Gal}(L/K)$.

In the imperfect residue field case, the facts that L/K is totally ramified and l/k is trivial still imply that the extension is of Eisenstein type. See [S, I, Proposition 18]. As an Eisenstein extension, it is therefore clear that $\big(1, t, .., t^{p-1}\big)$ is a complete basis.

It follows from [S, III, Corollary 2 to Proposition 11] that, for every monogenic extension $B = A[X]/f(X)$ of a complete discrete valuation ring A, the different may be computed as

$\nu_L\big(\mathcal{D}_{L/K}\big) = \nu(f'(X))$ with x equal to the class of X in B. The class of t in B remains a uniformizing element and $\nu_L(X) = p\nu_K(X)$ for all $x \in K$. In this case it follows that

$$\begin{aligned}\nu_L\big(\mathcal{D}_{L/K}\big) &= \min_{1\leq i\leq p-1}\big\{\nu_L\big(pt^{p-1}\big)\,, \nu_L\big((p-i)s_{p-i}t^{p-1-i}\big)\big\}\\ &= \min_{0\leq i\leq p-1}\{p\nu_K(s_{p-i}) + p - 1 - i\}\end{aligned}$$

(with $s_0 := p$). This leads to the first formula for the different. The second formula follows from the same corollary if we note that

$$f(X) = \prod_{\sigma\in \mathrm{Gal}(L/K)} X - \sigma(T) \quad \text{so that } f'(T) = \prod_{\tau\in \mathrm{Gal}(L/K)\setminus\{\mathrm{id}\}} \tau(T) - t$$

and exploit that the order $[L\colon K]$ is prime, whence $\nu_L(\tau(T) - t) = \nu_L(\sigma(T) - t)$ for every $\tau \neq \mathrm{id}$.

Thus, in the proof, the hypothesis of an algebraically closed residue field can be replaced by the hypothesis of a trivial residue field extension (with a separably closed residue field) and the rest of the proof remains valid without changes. □

In the case of a totally ramified extension L/K with a non-trivial inseparable residue field extension, one can proceed in a similar manner to [LL]. Thus, let L/K be a Galois extension of local fields of degree p which induces a purely inseparable extension of degree p of the residue fields. We have

$$L = K[t]/\big(t^p - s_1t^{p-1} + s_2t^{p-2} + .. + (-1)^p s_p\big)$$

with suitable $s_i \in K$. Since the associated extension of discrete valuation rings is monogenic, we can write without loss of generality

$$\mathcal{O}_L = \mathcal{O}_K[t]/\big(t^p - s_1t^{p-1} + s_2t^{p-2} + .. + (-1)^p s_p\big)\,,$$

Thus we may assume that $s_i \in \mathcal{O}_K$. The extension of the residue fields must have the form $l = k[t]/(t^p - \overline{s_p})$, that is, $s_p \in \mathcal{O}_K^*$ but $s_i \in \pi_K\mathcal{O}_K$ for $i = 1, .., p-1$.

The extension has ramification index one, i.e., $\nu_L(X) = \nu_K(X)$ for all $x \in K$. We further identify t with its image in L. In contrast to the above, t is now an element of $\mathcal{O}_L^*$. Due to the form of the extension, one can determine the different similarly to the minimal polynomial f of t:

$$\nu_L\big(\mathcal{D}_{L/K}\big) = \nu_L(f'(T)) = \nu\left(\sum_{i=0}^{p-1}(p-i)t^{p-1-i}s_i\right) = \min_{i=1,..,p-1}\{\nu(p), \nu_K(s_i)\}$$

Note that the residue classes of T^I form a basis of l/k, so that an arbitrary sum is $\sum_{i=0}^{p-1}\alpha_i t^i$ with $\alpha_i \in \mathcal{O}_K^* \cup \{0\}$ in $\mathcal{O}_L^*$, provided that there is at least one $\alpha_i \neq 0$. Therefore $\nu_L\big(\mathcal{D}_{L/K}\big)$ cannot be smaller than the minimum of $\nu_L(\alpha_i)$.

Now $\nu_L\big(\mathcal{D}_{L/K}\big)$ must again be equal to $(p-1)\nu_L(\sigma(T) - t)$. So we set $\nu_m := \nu_L\big(\mathcal{D}_{L/K}\big)$ and $r := \nu_m/p - 1$. Clearly $r \geq 1$ in all cases.

Proposition 1.3.2. *Let L/K be a totally ramified Galois extension of local fields of degree p with a non-trivial purely inseparable extension of residue fields. Then the Néron model of T_N has the form*

$$\operatorname{Spec}\mathcal{O}_K[X_0, .., X_{p-1}]\,/(G(X_0, .., X_{p-1}))$$

with a polynomial $G(X_0, .., X_{p-1}) \in \mathcal{O}_K[X_0, .., X_{p-1}]$ that is congruent modulo π_K to

$$\sum_{i=0}^{p-1} s_p^i X_i^p + \sum_{\{\,j\,|\,j\geq 0,\,\nu(s_j)=\nu_m\}} \frac{\mathrm{Tr}_{L/K}\big(t^j\big)}{\pi^{\nu_m}} X_j,$$

where $s_0 := p$.

Proof. As in [LL], then the following holds

Lemma 1.3.3. [LL, 5.3]. *Let $A = \mathbb{Z}[s_1, .., s_p, y_0, .., y_{p-1}]$ be a polynomial ring in $2p$ variables and set $B = A[u]/\big(u^p - s_1u^{p-1} + .. + (-1)^p s_p\big)$. Let t be the image of u in B and set $N := N_{B/A}\big(y_0 + y_1t + .. + y_{p-1}t^{p-1}\big)$. The following holds:*

(1) *N is homogeneous of degree p in the variables $y_0, y_1, \dots, y_{p-1}$.*

(2) *Let $0 \le j \le p-1$. Then the coefficient of y_j^p in N is equal to s_p^j and for $j \neq 0$ the coefficient of $y_0^{p-1}y_j$ equals $\mathrm{Tr}_{L/K}\big(t^j\big)$.*

(3) *The coefficients of $y_0^{\lambda_0} \dots y_{p-1}^{\lambda_{p-1}}$ in N lie in the ideal $(ps_p, s_1, \dots, s_{p-1})$, provided that $\lambda_0 \le p-2$.*

First, we will show

Lemma 1.3.4 (analogous to [LL, 5.4])**.** *Let*

$$b = (1 + a_0) + a_1t + .. + a_{p-1}t^{p-1} \in L$$

with $a_i \in K$ and $N_{L/K}(b) = 1$. Then, for $0 \le i \le p-1$, we have

$$\nu(a_i) \ge r$$

Proof. Since the norm of b is in $\mathcal{O}_K$, we have $b \in \mathcal{O}_L$. Since the powers of t form a full basis, all a_i lie in $\mathcal{O}_K$. Thus Lemma 1.3.3 applies

$$1 = N_{L/K}(b) = (1 + a_0)^p + s_pa_1^p + .. + s_p^{p-1}a_{p-1}^p + \text{ term from } IJ$$

with the ideals $I := (p, s_1, .., s_{p-1})$ and $J := (a_1, .., a_{p-1})$. Thus it follows that

$$\nu\big((1 + a_0)^p - 1 + s_pa_1^p + .. + s_p^{p-1}a_{p-1}^p\big) \ge \min_{1\le i\le p-1}\{\nu(s_i)\,, \nu(p)\} + \min_{1\le j\le p-1}\{\nu(a_j)\}\,.$$

By definition, the first minimum equals ν_m with the choice $1 \le j_0 \le p-1$, so that the second minimum equals $\nu(a_{j_0})$. It now follows that

$$p\nu(a_{j_0}) \ge \nu\big((1 + a_0)^p - 1 + s_pa_1^p + .. + s_p^{p-1}a_{p-1}^p\big) \ge \nu_m + \nu(a_{j_0})\,,$$

because the residue classes of $1, s_p, .., s_p^{p-1}$ form a k-basis of l. Hence $\nu(a_{j_0}) \ge \nu_m/(p-1) = r$. Thus, if $1 \le j \le p-1$, then the following holds in general:

$$p\nu(a_j) \ge \nu\big((1 + a_0)^p - 1 + s_pa_1^p + . + s_p^{p-1}a_{p-1}^p\big) \ge \nu_m + \nu(a_{j_0}) \ge \nu_m + r.$$

Thus $\nu(a_j) \ge (\nu_m + r)/p = [(p-1)r + r]/p = r$.

It remains to consider $\nu(a_0)$. Let $e' := \nu(p)/(p-1)$. According to the definition of ν_m, we have $e' \ge r = \nu_m/(p-1)$. Thus, in the case $\nu(a_0) \ge e'$, there is nothing to prove.

Otherwise $(p-1)\nu(a_0) < \nu(p)$ and since $p \mid \binom{p}{k}$ we conclude that

$$\nu(a_0^p) = p\nu(a_0) < \nu\left(\binom{p}{k}a_0^k\right) = \nu(p) + k\nu(a_0)\,.$$

This yields $\nu((1 + a_0)^p - 1) = \nu\left(a_0^p + \sum_{k=1}^{p-1}\binom{p}{k}a_0^k\right) = p\nu(a_0)$. It follows similarly as above that

$$p\nu(a_0) \ge \nu\big((1 + a_0)^p - 1 + s_pa_1^p + .. + s_p^{p-1}a_{p-1}^p\big) \ge \nu_m + r,$$

whence $\nu(a_0) \ge r$ also. ∎

In the representation $T_N = \operatorname{Spec} K[x_0,..,x_{p-1}] / \Big(N_{L/K}\Big(1+\sum_{i=0}^{p-1} T^I x_i\Big) - 1\Big)$ we substitute $x_j := \pi^r X_j$ and obtain

$$F(X_0,..,X_{p-1}) = N_{L/K}\left(1+\sum_{j=0}^{p-1} \pi^r t^j X_j\right) - 1.$$

Using $y_0 := 1+x_0 = 1+\pi^r X_0$ as well as $y_i := x_i = \pi^r X_i$, Lemma 1.3.3 yields

$$\begin{aligned} F(X_0,..,X_{p-1}) =& 1+\sum_{k=1}^{p-1}\binom{p}{k}\pi^{kr}X_0^k + \pi^{pr}X_0^p - 1 + \sum_{i=1}^{p-1} s_p^i \pi^{pr} X_i^p \\ &+ \sum_{i=1}^{p-1} \operatorname{Tr}_{L/K}\big(T^I\big)(1+\pi^r X_0)^{p-1}\,\pi^r X_i \\ &+ \sum a_{\lambda_0,..,\lambda_{p-1}}(1+\pi^r X_0)^{\lambda_0}\prod_{i=1}^{p-1}\pi^{r\lambda_i}, \end{aligned}$$

where in the last sum the indices $\lambda_i \geq 0$ satisfy $\lambda_0 \leq p-2$ and $\sum_{i=0,\dots,p-1}\lambda_i = p$ and the coefficients $a_{\lambda_0,\dots,\lambda_{p-1}}$ are chosen appropriately. If one looks closely, one can see that the coefficients of the monomials of F always have a valuation $\geq pr$. Indeed:

Since $\nu(s_p) = 0$, we have $\nu\big(s_p^i \pi^{pr}\big) = pr$ and therefore each of the monomials of the form X_i^p has a coefficient with valuation pr.

Further, we have $\nu(p) \geq \nu_m = (p-1)r$, whence $\nu\big(\binom{p}{k}\pi^{kr}\big) \geq pr + (k-1)r$. Therefore, the middle terms in the first row have a valuation greater than pr, except for $p\pi^r X_0$ in the case that $\nu(p) = \nu_m$.

The following identity holds for $1 \leq j \leq p-1$ (see proof of [LL, 5.5]).

$$\operatorname{Tr}_{L/K}\big(t^j\big) + (-1)^j j s_j = \sum_{1\leq l\leq j-1} (-1)^{l+1} s_l \operatorname{Tr}_{L/K}\big(t^{j-l}\big)\,.$$

This shows that $\nu\big(\operatorname{Tr}_{L/K}\big(T^I\big)\big) \geq \nu_m$ always. More precisely, the equality holds if, and only if, $\nu(s_i) = \nu_m$. This also applies in the case $i = 0$ if one sets $s_0 := p = \operatorname{Tr}_{L/K}(1)$.

After multiplying, we obtain terms of the following form on the third line

$$\binom{p-1}{k}\operatorname{Tr}_{L/K}\big(T^I\big)\,\pi^{r+kr}X_0^k X_i.$$

Thus the coefficients of these terms have minimum valuation for $k = 0$ and $\nu(s_i) = \nu_m$ and this valuation is $\nu_m + r = pr$.

The terms on the last line have coefficients

$$a_{\lambda_0,\dots,\lambda_{p-1}}\binom{\lambda_0}{k}\pi^{kr}\pi^{r\sum_{1\leq i\leq p-1}\lambda_i},$$

where $0 \leq k \leq \lambda_0$ and $\sum_{1\leq i<p-1}\lambda_i$ is at least 2. Further, according to Lemma 1.3.3, the factors $a_{\lambda_0,\dots,\lambda_{p-1}}$ belong to the ideal $(p, s_1, \dots, s_{p-1})$, so they have valuation at least ν_m. Consequently, only coefficients with a valuation greater than or equal to $\nu_m + 2r = pr + r > pr$ appear.

This means $G(X_0,..,X_{p-1}) := \pi^{-pr}F(X_0,..,X_{p-1}) \in \mathcal{O}_K[X_0,..,X_{p-1}]$ is congruent modulo π to

$$\sum_{i=0}^{p-1} s_p^i X_i^p + \sum_{\{j|\nu(s_j)=\nu_m\}} \frac{\operatorname{Tr}_{L/K}\big(t^j\big)}{\pi^{\nu_m}} X_j,$$

where above we set $s_0 = p$.

This substitution can be interpreted as follows: If we pass from T_N to the $\mathcal{O}_K$-model

$$\mathcal{T} := \operatorname{Spec} \mathcal{O}_K[x_0, .., x_{p-1}] \,/ \bigl(N_{L/K}\bigl(1 + x_0 + tx_1 + .. + t^{p-1}x_{p-1}\bigr) - 1\bigr)$$

and blow-up the zero section ($x_0 = 1, x_1 = .. = x_{p-1} = 0$) of the special fiber r times, then we obtain as dilatation the scheme

$$\mathcal{T}^{\mathrm{sm}} := \operatorname{Spec} \mathcal{O}_K[X_0, .., X_{p-1}] \,/ (G(X_0, .., X_{p-1}))\,.$$

Note that $\mathcal{T}$ is an $\mathcal{O}_K$-group scheme, because on T_N one obtains the multiplication map. Indeed, in the polynomial ring

$$L\Big[X_0^{(2)}, .., X_{p-1}^{(2)}\Big] \otimes_L L\Big[X_0^{(3)}, .., X_{p-1}^{(3)}\Big]$$

one can certainly write the product

$$M := \Big(X_0^{(2)} + tX_1^{(2)} + .. + t^{p-1}X_{p-1}^{(2)}\Big)\Big(X_0^{(3)} + tX_1^{(2)} + \ldots + t^{p-1}X_{p-1}^{(3)}\Big)$$

in the form

$$M = \sum_{i=0}^{p-1} f_i\Big(X_0^{(2)}, .., X_{p-1}^{(2)}, X_0^{(3)}, .., X_{p-1}^{(3)}\Big)\, T^I$$

with polynomials f_i en $K\Big[X_0^{(2)}, .., X_{p-1}^{(2)}, X_0^{(3)}, .., X_{p-1}^{(3)}\Big]$. Then the multiplication

$$\begin{array}{c} K[X_0^{(1)}, .., X_{p-1}^{(1)}]/(N(X_0^{(1)}, .., X_{p-1}^{(1)})) \\ \Big\downarrow{\scriptstyle \mu} \\ K[X_0^{(2)}, .., X_{p-1}^{(2)}]/(N(X_0^{(2)}, .., X_{p-1}^{(2)})) \otimes_K K[X_0^{(3)}, .., X_{p-1}^{(3)}]/(N(X_0^{(3)}, .., X_{p-1}^{(3)})) \end{array}$$

is given by $X_l^{(1)} \mapsto f_i\Big(X_0^{(2)}, .., X_{p-1}^{(2)}, X_0^{(3)}, .., X_{p-1}^{(3)}\Big)$. Since the minimal polynomial of t only has coefficients in $\mathcal{O}_K$, the polynomials f_i only have coefficients in $\mathcal{O}_K$ and thus one can extend the group law on T_N to a group law on $\mathcal{T}$. Similarly, the zero section and the formation of inverses can also be extended to $\mathcal{T}$, since these are defined over $\mathcal{O}_K$; for the latter, note that

$$\bigl(X_0 + tX_1 + .. + t^{p-1}X_{p-1}\bigr)^{-1} = \prod_{\tau \in \mathrm{Gal}(L/K)\setminus\{\mathrm{id}\}} \tau\bigl(X_0 + tX_1 + .. + t^{p-1}X_{p-1}\bigr)\,.$$

Thus, by [BLR, 3.2.2d], $\mathcal{T}^{\mathrm{sm}} := \operatorname{Spec} \mathcal{O}_K[X_0, .., X_{p-1}] \,/ (G(X_0, .., X_{p-1}))$ is a group scheme. It is even an integral model of T_N because it is separated and flat. The latter holds because π is not a divisor of $G(X_0, .., X_{p-1})$.

By the hypothesis $\mathcal{O}_K = \mathcal{O}_K^{\mathrm{sh}}$, Lemma 1.3.4 shows that the canonical map $\mathcal{T}^{\mathrm{sm}}\bigl(\mathcal{O}_K^{\mathrm{sh}}\bigr) \longrightarrow T_N(K^{\mathrm{nr}})$ is surjective. It now follows from [BLR, 7.1.1] that $\mathcal{T}^{\mathrm{sm}}$ is the Néron model of T_N if it is smooth. The latter follows from the Jacobi criterion because

$$dG = \sum_{\{j | \nu(s_j) = \nu_m\}} \frac{\mathrm{Tr}_{L/K}\bigl(t^j\bigr)}{\pi^{\nu_m}} dX_j$$

with $\frac{\mathrm{Tr}_{L/K}(t^j)}{\pi^{\nu_m}} \in \mathcal{O}_K^*$ for all j with $\nu(s_j) = \nu_m$. $\square$

Using the above representation, we obtain the following

Corollary 1.3.5. *Let L/K be a cyclic totally ramified extension of local fields of degree $p = \mathrm{char}(k)$. Assume that the corresponding extension of residue fields is purely inseparable of degree p and let T_N be the norm torus with respect to L/K.*

Then the group of components of the Néron model of T_N is trivial and the reduction of its identity component is a k-wound unipotent group.

Proof. The special fiber of the Néron model has the form

$$\mathcal{T}_k^{\mathrm{sm}} = \mathrm{Spec}\, k\,[X_0, .., X_{p-1}]\,/\,\left(\bar{G}\,(X_0, .., X_{p-1})\right),$$

where

$$\bar{G} = \sum_{i=0}^{p-1} \overline{s_p^i} X_i^p + \sum_{\substack{i=0 \\ \nu(s_i)=\nu_m}}^{p-1} \overline{\frac{\mathrm{Tr}_{L/K}(T^I)}{\pi^{\nu_m}}} X_i^{p^0}$$

is obviously a p-polynomial.

After blowups, the group law over $\mathcal{O}_K$ results from the following calculation:

$$\left(1 + \pi^r X_0^{(2)} + t\pi^r X_1^{(2)} + .. + t^{p-1}\pi^r X_{p-1}^{(2)}\right) \otimes \left(1 + \pi^r X_0^{(3)} + t\pi^r X_1^{(3)} + .. + t^{p-1}\pi^r X_{p-1}^{(3)}\right)$$
$$= 1 + \pi^r \sum_{i=0}^{p-1} T^I \left(1 \otimes X_l^{(3)} + X_l^{(2)} \otimes 1\right) + \pi^{2r}\Gamma\left(X_0^{(2)}, .., X_{p-1}^{(3)}\right),$$

for some appropriate $\Gamma\left(X_0^{(2)}, \dots, X_{p-1}^{(3)}\right) \in \mathcal{O}_K\left[X_0^{(2)}, \dots, X_{p-1}^{(2)}\right] \otimes \mathcal{O}_K\left[X_0^{(3)}, \dots, X_{p-1}^{(3)}\right]$. After splitting into powers of t and dividing by π^r, we obtain modulo π the law $X_l^{(2)} \mapsto 1 \otimes X_l^{(3)} + X_l^{(2)} \otimes 1$ for $0 \le i \le p-1$.

Thus the special fiber is a subgroup of $\mathbb{G}_{a,k}^p$.

The principal part of G has no non-trivial rational zero: after choosing the extension L/K we have $s_p = t^p$ in l.

A root of the principal part now corresponds to an equation

$$s_p^0 a_0^p + .. + s_p^{p-1} a_{p-1}^p = 0 \text{ with } a_i \in k$$

This equation can also be read in l as

$$\left(t^0 a_0\right)^p + .. + \left(t^{p-1} a_{p-1}\right)^p = \left(t^0 a_0 + .. + t^{p-1} a_{p-1}\right)^p = 0.$$

Since l is a field, this already means that $a_0 t^0 + .. + a_{p-1} t^{p-1} = 0$ and therefore $a_i = 0$ for every i, because the powers of t are a basis of l/k. So the only rational zero is the trivial one.

The claim now follows from Proposition A.1, provided that for an $i_0 \in \{0, \dots, n\}$ the linear term to X_{i_0} is trivial. Thus let $L = K[t]$ with $\nu(p) = \nu(s_1) = \dots = \nu(s_{p-1})$ so that all linear terms appear. In particular, $\mathrm{char}(K) = 0$. But since T_N does not depend on the particular choice of $t \in L$, we can replace t with the (also generating) element $T' := t - (s_1/p)$. Note that $s_1/p \in \mathcal{O}_K^*$, so that T' is integral. Now if $\chi(X) = X^p - s_1 X^{p-1} + \,..\, + (-1)^p s_p$ is the minimal polynomial of t, then the minimal polynomial of $t - \frac{s_1}{p}$ equals

$$\chi\left(X + \frac{s_1}{p}\right) = \left(X + \frac{s_1}{p}\right)^p - s_1\left(X + \frac{s_1}{p}\right)^{p-1} + .. + (-1)^p s_p$$
$$= X^p + X^{p-1}\left(p\frac{s_1}{p} - s_1\right) + \text{ further terms }.$$

Thus if we carry out the construction with T', the result is a polynomial G' that does not contain a linear term containing X_1. □

Finally, we sketch the (well-known) tamely ramified case: let L/K be a finite totally and tamely ramified extension of local fields of degree q. The extension is also an Eisenstein extension $L = K[t]/\sum_{i=0}^{q} s_i T^I$ with $s_q = 1, s_0, .., s_{q-1} \in (\pi_K)$ and $\nu_K(s_0) = 1$. Since we assumed that $k = k^{\text{sep}}$, we may assume by Hensel's lemma that there exists a uniformizing element $\pi_L \in \mathcal{O}_L$ such that $\pi_L^q \in \mathcal{O}_K$. It follows that we may assume the Eisenstein equation to be of the form $t^q - \pi_K$.

This means that for $a_0, .., a_{q-1} \in \mathcal{O}_K$ the equation is

$$N_{L/K}\left(\sum_{i=0}^{q-1} a_i T^I\right) \equiv a_0^q \bmod (\pi_K)$$

and we find the smooth $\mathcal{O}_K$-model for T_N

$$\operatorname{Spec} \mathcal{O}_K[X_0, .., X_{q-1}] \,/ \left(N_{L/K}\left(\sum_{i=0}^{q-1} X_i T^I\right) - 1\right)$$

with special fiber $\operatorname{Spec} k[X_0, .., X_{q-1}] \,/ (X_0^q - 1)$. Thus, the group of components is equal to $\mathbb{Z}/q\mathbb{Z}$.

Chapter 2

Groups of components of Néron models

In this chapter we will study the group of components of the special fiber of a (local) lft-Néron model of a smooth and commutative algebraic K-group. Our first main result is that the group of components is a finitely generated module (see Theorem 2.3.2). This answers a question of Lorenzini's [LL, Remark 1.3].

To do this we show that, given a smooth and commutative algebraic K-group G_K, there exists an exact sequence of smooth and commutative algebraic K-groups

$$0 \longrightarrow T_I \longrightarrow G_K \longrightarrow G' \longrightarrow 0,$$

where T_I is a torus with multiplicative reduction and $G' \otimes_K K^{\mathrm{nr}}$ does not contain a subgroup of the form $\mathbb{G}_{m,K^{\mathrm{nr}}}$. Further, $G_K \otimes_K K^{\mathrm{nr}}$ does not contain a subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$ if, and only if, this is the case for $G' \otimes_K K^{\mathrm{nr}}$. Now, if G_K has an lft-Néron model, we obtain a short exact sequence of the associated lft-Néron models in the smooth topology

$$0 \longrightarrow \mathcal{T}_I \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow 0.$$

Now, as in [BX, §4], we define the functor that assigns to a smooth sheaf over $\operatorname{Spec} \mathcal{O}_K$ its identity component. More precisely, after restriction to the special fiber, this functor is represented by the identity component there. As in the formal setting, this functor is right-exact, so we have a short exact sequence of groups of components

$$0 \longrightarrow \widetilde{\Phi} \longrightarrow \Phi(\mathcal{G}_k) \longrightarrow \Phi(\mathcal{G}'_k) \longrightarrow 0,$$

where $\widetilde{\Phi}$ denotes a suitable quotient of $\Phi((\mathcal{T}_I)_k)$. This then yields our description of the Néron models of algebraic tori with multiplicative reduction and [BLR, Theorem 10.2.1] is our first main result.

Our second main result is that a homomorphism $G_1 \longrightarrow G_2$ of smooth and commutative algebraic K-groups which is a closed immersion induces a homomorphism $\Phi((\mathcal{G}_1)_k) \longrightarrow \Phi((\mathcal{G}_2)_k)$ between groups of components of their lft-Néron models (if these models exist) with a finite kernel.

This is based on the fact that the induced homomorphism $\mathcal{G}_1 \longrightarrow \mathcal{G}_2$ of the lft-Néron models is quasi-compact because a dilatation is a quasi-compact morphism. Later, using finiteness of the kernel, we will describe the exact sequence of group of components in the case of lft-Néron models of algebraic tori in more concrete terms.

2.1 The maximal subtorus with multiplicative reduction

Let S be a scheme and let G be an S-group scheme of finite type. Then we can define a maximal (sub-)torus of G as follows:

Definition 2.1.1. [SGA3, XII, Definition 1.3] Let S be a scheme and let G be an S-group scheme of finite type. A subgroup scheme T of G is called a *maximal torus of* G if the following conditions hold:

1. T is a torus.

2. If s is any point of S and $\bar{s}$ denotes the spectrum of an algebraic closure of $k(s)$, then $T_{\bar{s}}$ is a maximal torus of $G_{\bar{s}}$, i.e., an algebraic subgroup which is a torus and is maximal with respect to this property.

Maximal tori exist for smooth algebraic groups over a field:

Theorem 2.1.2. [SGA3, XIV, Theorem 1.1] *Let K be a field and let G be a smooth algebraic K-group. Then G has a maximal torus T and therefore a Cartan subgroup $C = C_G(T)$.*

By [SGA3, XII, Corollary 1.15], a commutative group scheme can have at most one maximal torus.

Lemma 2.1.3. *Let K be a local field and let T be a K-torus with character group $X(T)$. Then there exists in T a maximal subtorus $T_I \hookrightarrow T$ with multiplicative reduction. The formation of T_I is compatible with unramified extensions.*

Proof. Let L/K be a finite Galois extension such that T splits over L. Let $I_{L/K}$ be the inertia group of L/K. We now look for a maximal torsion-free quotient of $X(T)$ with trivial $I_{L/K}$ action. To do this, consider the map

$$\mathrm{Tr}_I \colon X(T) \longrightarrow X(T)^I = X(T)^{I_{L/K}},\ x \mapsto \sum_{\tau \in I_{L/K}} \tau x.$$

The above map is well-defined since $I_{L/K}$ is a normal subgroup of $\mathrm{Gal}(L/K)$. Further, it is a homomorphism of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules. We obtain a short exact sequence

$$0 \longrightarrow \ker(\mathrm{Tr}_I) \longrightarrow X(T) \longrightarrow \mathrm{Im}(\mathrm{Tr}_I) := X(T_I) \longrightarrow 0$$

of continuous and finitely generated $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules. As submodules of torsion-free modules, $\ker(\mathrm{Tr}_I)$ and $\mathrm{im}(\mathrm{Tr}_I)$ are torsion-free. By definition, $I_{L/K}$ acts trivially on $X(T_I)$. We now show that $X(T_I)$ is maximal with this property.

Let $\psi \colon X(T) \twoheadrightarrow X'$ be a homomorphism of $\mathrm{Gal}(L/K)$-modules such that X' is torsion-free with trivial $I_{L/K}$-action. Then $\psi(\mathrm{Tr}_I(X)) = n\psi(X)$, where n is the cardinality of $I_{L/K}$. Thus $\ker(\mathrm{Tr}_I) \subset \ker(\psi)$ since X' is torsion-free. Thus we have a homomorphism $X(T_I) \twoheadrightarrow X'$. Now, by [SGA3, VIII, Proposition 3.2] and [SGA1, VIII,Corollary 5.5], the map $X(T) \twoheadrightarrow X(T_I)$ corresponds to a homomorphism $T_I \longrightarrow T$ of algebraic tori which is a closed immersion.

Finally, since an unramified extension does not change the action of the inertia group, it is clear that the formation of T_I is compatible with such extensions. □

Proposition 2.1.4. *Let K be a local field and let G_K be a smooth and commutative algebraic K-group scheme. Then G_K has a maximal torus T_I with multiplicative reduction and the quotient $G' := G_K/T_I$ is represented by a smooth and commutative algebraic K-group.*

Further, $G' \otimes_K K^{\mathrm{nr}}$ contains no subgroup of the form $\mathbb{G}_{m,K^{\mathrm{nr}}}$. Moreover, $G' \otimes_K K^{\mathrm{nr}}$ has a subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$ if, and only if, $G_K \otimes_K K^{\mathrm{nr}}$ has such a subgroup.

Proof. The K-group G_K has a unique maximal torus T. This is a subscheme, so the homomorphism $T \longrightarrow G_K$ must be a closed immersion. Using the lemma above, we have a maximal subtorus T_I of T with multiplicative reduction and this is a subtorus of G_K. By [SGA3, VIa, §5.4, Theorem], the commutative algebraic K-group schemes form an abelian category, i.e., we have an (fpqc)-exact sequence

$$0 \longrightarrow T_I \longrightarrow G_K \longrightarrow G' \longrightarrow 0.$$

The K-torus T_I is smooth and flat over K and after changing bases with T_I the K-groups G_K and G' are isomorphic. Thus, by descent, G' is also smooth over K.

We now consider this sequence after a base change with K^{nr}. The preimage of a closed subgroup U of $G' \otimes_K K^{\mathrm{nr}}$ is a closed subgroup of $G_K \otimes_K K^{\mathrm{nr}}$ as well as an extension of U by $T_I \otimes_K K^{\mathrm{nr}} \cong \mathbb{G}^r_{m,K^{\mathrm{nr}}}$. Suppose $G' \otimes_K K^{\mathrm{nr}}$ has a subgroup of the form $\mathbb{G}_{m,K^{\mathrm{nr}}}$. Then $G_K \otimes_K K^{\mathrm{nr}}$ has a subgroup that is an extension of $\mathbb{G}_{m,K^{\mathrm{nr}}}$ by $\mathbb{G}_{m,K^{\mathrm{nr}}}$. By [SGA3, XVII, Proposition 7.1.1], this extension is a group scheme of multiplicative type. Now, by Cartier duality, the extension corresponds to an extension of $\mathbb{Z}$ by $\mathbb{Z}^r$ as $I = \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})$-modules. Since $\mathrm{Ext}^1_I(\mathbb{Z}^r, \mathbb{Z}) = \mathrm{H}^1(I, \mathbb{Z})^r = 0$, this extension is trivial. So $G_K \otimes_K K^{\mathrm{nr}}$ has a subgroup of the form $\mathbb{G}^{r+1}_{m,K^{\mathrm{nr}}}$. However, this is a contradiction to the fact that T_I is the maximal subtorus with multiplicative reduction of T.

If $G_K \otimes_K K^{\mathrm{nr}}$ has a subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$, then this also holds for $G' \otimes_K K^{\mathrm{nr}}$ because all homomorphisms from $\mathbb{G}_{m,K^{\mathrm{nr}}}$ to $\mathbb{G}_{a,K^{\mathrm{nr}}}$ are trivial, so that the quotient map on such a subgroup is an isomorphism.

Conversely, assume that $G' \otimes_K K^{\mathrm{nr}}$ has a subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$. Then $G_K \otimes_K K^{\mathrm{nr}}$ has a subgroup that is an extension of $\mathbb{G}_{a,K^{\mathrm{nr}}}$ by $\mathbb{G}^r_{m,K^{\mathrm{nr}}}$. However, by [SGA3, XVII, Theorem 6.1.1 A(ii)], such an extension is trivial and we can find a subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$ in $G_K \otimes_K K^{\mathrm{nr}}$. □

2.2 The identity component of a smooth sheaf

In analogy to [BX, 4.7], we will define a subsheaf of an abelian sheaf on the smooth site over the spectrum of a Henselian discrete valuation ring R that we will interpret in a certain way as an identity component. Incidentally, this definition only makes sense in the smooth topology, since a similarly defined étale sheaf would be trivial on the special fiber.

In this section, we let K be the quotient field of R and write k for the residue field of R.

Definition 2.2.1. Let $S = \operatorname{Spec} R$ be the spectrum of a Henselian discrete valuation ring and let $\mathcal{F}$ be an abelian sheaf on the smooth site over S. We define $\mathcal{F}^0$ to be the subsheaf that assigns to every smooth S-scheme T the sections $f \in \mathcal{F}(T)$ for which the following holds: for every étale point $u\colon \operatorname{Spec} A \longrightarrow T$ there exists

1. a valuation ring R' that is étale over A (and thus étale over R),
2. a fiberwise geometrically connected smooth S-scheme T' with a section $g \in \mathcal{F}(T')$ and
3. R'-valued points $u'_0, u'_1\colon \operatorname{Spec} R' \longrightarrow T'$ such that $g|_{u'_0} = 0$ and $g|_{u'_1}$ factors through $f|_u$.

Proposition 2.2.2. *For a smooth sheaf $\mathcal{F}$ over a Henselian discrete valuation ring R, $\mathcal{F}^0$ is a subsheaf of $\mathcal{F}$. Further, $j^*\mathcal{F} = j^*\mathcal{F}^0$, where $j\colon \operatorname{Spec} K \longrightarrow \operatorname{Spec} R$ is the canonical open immersion. If $\mathcal{F}$ is represented by a smooth R-group scheme F and $i\colon \operatorname{Spec} k \longrightarrow \operatorname{Spec} R$ is the (closed) immersion of the special fiber, then*

$$i^*\mathcal{F}^0 = i^*\big(F^0\big).$$

Further, if F has a connected generic fiber, then

$$\mathcal{F}^0 = F^0.$$

Proof. Let T be a smooth R-scheme. If $T = T_K$, then T has no étale points, so $\mathcal{F}^0(T) = \mathcal{F}(T)$. This means that $j^*\mathcal{F} = j^*\mathcal{F}^0$. Now let T be arbitrary. For a morphism $\psi\colon U \longrightarrow T$ we denote the restriction morphism of the sheaf $\mathcal{F}$ by ρ_ψ.

We have $0 \in \mathcal{F}^0(T)$, because for an étale point $u\colon \operatorname{Spec} A \longrightarrow T$ we have $\rho_u(0) = 0$ and this is equal to $\rho_{u'}(0)$ for $u'\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec} R$. Since R is geometrically connected, the conditions of the definition hold.

Now let $\tau\colon T_1 \longrightarrow T_2$ be a morphism in the category of smooth R-schemes. Then every étale point of T_1 induces an étale point of T_2 and thus $\rho_\tau \mathcal{F}^0(T_2) \subset \mathcal{F}^0(T_1)$.

Consequently, $\mathcal{F}^0$ is a subsheaf of sets. Now let $f, g \in \mathcal{F}^0(T)$. Then, as explained in [BX, after Definition 4.7], we have $f - g \in \mathcal{F}^0(T)$. This means that $\mathcal{F}^0$ is a subsheaf of groups.

Now let $\mathcal{F}$ be a sheaf represented by a smooth group scheme. We will show that a section $f \in \mathcal{F}^0(T)$ corresponds exactly to a morphism $T \longrightarrow F$, so that in the special fiber we have a factorization $T_k \longrightarrow F_k^0 \longrightarrow F_k$.

A section $f \in \mathcal{F}^0(T)$ corresponds to a morphism $f\colon T \longrightarrow F$ and for every étale point $u\colon \operatorname{Spec} A \longrightarrow T$ there exists a geometrically connected smooth scheme T', a morphism $g\colon T' \longrightarrow G$ and étale points $u_0', u_1'\colon \operatorname{Spec} R' \longrightarrow T'$, such that $g \circ u_0'$ factors through the unit section of G and $f \circ u \circ (\operatorname{Spec} R' \longrightarrow \operatorname{Spec} A)$ is equal to $g \circ u_1'$. This means that $f \circ u$ factors through F^0. Since the étale points have a dense image in the special fiber of T, $f_k\colon T_k \longrightarrow F_k$ must factor through F_k^0.

For a section $f\colon T \longrightarrow F$ such that f_k factors through F_k^0 and an étale point $u\colon \operatorname{Spec} A \longrightarrow T$, consider the geometrically connected scheme $T' := F^0 \otimes_R A$ and, as étale points, the unit section $u_0'\colon \operatorname{Spec} A \longrightarrow T'$ and the point u_1' induced by $u\colon \operatorname{Spec} A \longrightarrow T'$, which by definition means that $f \in \mathcal{F}^0$.

If the generic fiber of F is connected, the existence of a factorization $T_k \to F_k^0 \to F_k$ is equivalent to a factorization $T \to F^0 \to F$. Thus we have $\mathcal{F}^0 = F^0$.

In general we have an inclusion $i^*F^0 = i^*\left(F^0\right)^0 \longrightarrow i^*\mathcal{F}^0$. Here $\left(F^0\right)^0$ means the identity component (as defined above) of the sheaf represented by F^0. Conversely, we can construct all sections of $i^*\mathcal{F}^0$ from sections $T \longrightarrow F$ for which T has no connected component without an étale point. Thus we may compute i^* on the subsheaf F^0 and $i^*F^0 \longrightarrow i^*\mathcal{F}^0$ is therefore surjective. ☐

Thus, with this subsheaf, we can examine the identity component of a Néron model on the special fiber. For algebraic tori, this sheaf also represents the identity component of the Néron model over $\mathcal{O}_k$. We now investigate how this subsheaf is compatible with morphisms of smooth sheaves.

Proposition 2.2.3. *Let R be a Henselian discrete valuation ring and let $\psi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of abelian sheaves on the smooth site over $S = \operatorname{Spec} R$. Then this morphism induces morphisms $\psi^0 = \psi|_{\mathcal{F}^0}\colon \mathcal{F}^0 \longrightarrow \mathcal{G}^0$ and $\bar{\psi}\colon \Phi(\mathcal{F}) := \mathcal{F}/\mathcal{F}^0 \longrightarrow \Phi(\mathcal{G}) := \mathcal{G}/\mathcal{G}^0$.*

Proof. Since ψ is a morphism of functors into the category of abelian groups, given a section $f \in \mathcal{F}^0(T)$ with a fiberwise geometrically connected scheme T' and étale points u, u_0', u_1' (as required in Definition 2.2.1), their images under ψ also satisfy the conditions in Definition 2.2.1. Thus the first assertion is clear. This means that ψ induces a map of presheaves $\mathcal{F}/\mathcal{F}^0 \longrightarrow \mathcal{G}/\mathcal{G}^0$. Since sheafification is an exact functor from the category of presheaves to the category of sheaves, the second assertion follows. ☐

The functor $\mathcal{F} \rightsquigarrow \mathcal{F}^0$ satisfies the following

Proposition 2.2.4. [BX, 4.8] *Let $\phi\colon \mathcal{F}' \longrightarrow \mathcal{F}$ be an epimorphism of abelian sheaves on the smooth site over S. Then the induced morphism $(\mathcal{F}')^0 \longrightarrow \mathcal{F}^0$ is also an epimorphism.*

Proof. The proof of [BX, 4.8] carries over verbatim to our situation since there is only something to show on the special fiber. □

2.3 The sequence of groups of components

Let $h\colon G_1 \longrightarrow G_2$ be a homomorphism of smooth group schemes over an arbitrary base scheme S. This map induces a homomorphism $h^0\colon G_1^0 \longrightarrow G_2^0$ between the identity components. Therefore h also induces a homomorphism between the group of components. We can understand this globally as a morphism $\bar{h}\colon \Phi(G_1) \longrightarrow \Phi(G_2)$ of smooth or étale sheaves, or locally for $s \in S$ as a homomorphism $\bar{h}_s\colon \Phi((G_1)_s) \longrightarrow \Phi((G_2)_s)$ of étale $k(s)$-groups. We now want to investigate the effect of this map on short exact sequences.

Proposition 2.3.1. *Let K be a local field and let*

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

be a short exact sequence of smooth and commutative algebraic K-groups whose lft*-Néron models $\mathcal{G}_i$ exist over* $\operatorname{Spec} \mathcal{O}_K$. *Assume further that the induced sequence of Néron models*

$$0 \longrightarrow \mathcal{G}_1 \xrightarrow{\iota} \mathcal{G}_2 \longrightarrow \mathcal{G}_3 \longrightarrow 0$$

is exact in the smooth topology and set $\Phi(\mathcal{G}_i) := \mathcal{G}_i/\mathcal{G}_i^0$, where $\mathcal{G}_i^0$ is the subsheaf from Definition 2.2.1. Then there exists a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{G}_1/\iota^{-1}(\mathcal{G}_2^0) \longrightarrow \Phi(\mathcal{G}_2) \longrightarrow \Phi(\mathcal{G}_3) \longrightarrow 0.$$

Further, there exists an exact sequence of continuous $\operatorname{Gal}(k^{\mathrm{sep}}/k)$*-modules*

$$0 \longrightarrow \widetilde{\Phi} \longrightarrow \Phi((\mathcal{G}_2)_k) \longrightarrow \Phi((\mathcal{G}_3)_k) \longrightarrow 0,$$

where $\Phi((\mathcal{G}_i)_k)$ is the group of components of the k-groups $(\mathcal{G}_i)_k$ and $\widetilde{\Phi}$ is a quotient of $\Phi((\mathcal{G}_1)_k)$.

Proof. In the smooth topology, we have an exact and commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \iota^{-1}(\mathcal{G}_2^0) & \longrightarrow & \mathcal{G}_2^0 & \longrightarrow & \mathcal{G}_3^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}_1 & \xrightarrow{\iota} & \mathcal{G}_2 & \longrightarrow & \mathcal{G}_3 & \longrightarrow & 0. \end{array}$$

More precisely, commutativity follows from proposition 2.2.3 and exactness of the top row is verified as follows. Exactness at $\iota^{-1}(\mathcal{G}_2^0)$ holds because the preimage of a subsheaf is already a subsheaf (cf. [M, II,2.12 c)]). By proposition 2.2.4, the top row is also exact at $\mathcal{G}_3^0$. Exactness at $\mathcal{G}_2^0$ then follows from the exactness of the second row and the commutativity of the diagram. The snake lemma applied to the above diagram yields a short exact sequence of cokernels that corresponds to the claimed sequence

$$0 \longrightarrow \mathcal{G}_1/\iota^{-1}(\mathcal{G}_2^0) \longrightarrow \Phi(\mathcal{G}_2) \longrightarrow \Phi(\mathcal{G}_3) \longrightarrow 0.$$

The restriction i^* of this sequence to the smooth site over $s := \operatorname{Spec} k$ is exact. In particular, the quotients Φ are mapped to the (actual) group of components, since $i^*\mathcal{G}_i^0$ is represented by $\left(\mathcal{G}_i^0\right)_k$.

The restriction to the étale site over $\operatorname{Spec} k$ is exact by [M, III, Proposition 3.3]. By proposition 0.2.1, this restriction corresponds to an exact sequence of continuous $\operatorname{Gal}(k^{\text{sep}}/k)$-modules

$$0 \longrightarrow \widetilde{\Phi} \longrightarrow \Phi((\mathcal{G}_2)_k) \longrightarrow \Phi((\mathcal{G}_3)_k) \longrightarrow 0.$$

By the exactness of restriction, $\widetilde{\Phi}$ is a quotient of $\Phi((\mathcal{G}_1)_k)$. □

Theorem 2.3.2. *Let K be a local field and let $\mathcal{G}$ be a smooth, commutative and separated $\operatorname{Spec} \mathcal{O}_K$-group scheme which is an lft-Néron model of its generic fiber G_K. In particular, G_K is a smooth and commutative algebraic K-group. Then the group of components $\Phi(\mathcal{G}_k)$ of the special fiber of $\mathcal{G}$ is finitely generated as a Galois module.*

Proof. By Proposition 2.1.4 and [BLR, Theorem 10.2.2], we have an exact sequence of smooth and commutative algebraic K-groups

$$0 \longrightarrow T_I \longrightarrow G_K \longrightarrow G' \longrightarrow 0,$$

where T_I is a torus with multiplicative reduction and $G' \otimes_K K^{\text{nr}}$ does not contain subgroups of the form $\mathbb{G}_{m,K^{\text{nr}}}$ or $\mathbb{G}_{a,K^{\text{nr}}}$. By [X, Lemma 2.11], $\mathrm{R}^1 j_* T_I = 0$, whence proposition 2.3.1 yields an exact sequence

$$0 \longrightarrow \widetilde{\Phi} \longrightarrow \Phi(\mathcal{G}_k) \longrightarrow \Phi(\mathcal{G}'_k) \longrightarrow 0,$$

where $\widetilde{\Phi}$ is a quotient of the sheaf $\Phi(\mathcal{T}_I)$. Now, by [BLR, Theorem 10.2.1], the Néron model of G' is quasicompact, whence $\Phi(\mathcal{G}'_k)$ is represented by a finite $\operatorname{Gal}(k^{\text{sep}}/k)$-module. By Theorem 1.1.2, $\widetilde{\Phi}$ is represented by a finitely generated $\operatorname{Gal}(k^{\text{sep}}/k)$-module. This means that $\Phi(\mathcal{G}_k)$ is also represented by a finitely generated $\operatorname{Gal}(k^{\text{sep}}/k)$-module. □

Recall that a morphism $f\colon X \longrightarrow Y$ of schemes is called *quasi-compact* if there exists an open affine covering $(V_i)_{i\in I}$ of Y such that the inverse images $f^{-1}(V_i)$ in X are quasi-compact.

Proposition 2.3.3. *Let K be a local field and let G_2 be a smooth algebraic K-group with an lft-Néron model $\mathcal{G}_2$ over $\operatorname{Spec} \mathcal{O}_K$. Let G_1 be a smooth K-subgroup of G_2.*

Then there exists an lft*-Néron model $\mathcal{G}_1$ of G_1 and the map $\mathcal{G}_1 \longrightarrow \mathcal{G}_2$ induced by the inclusion on the generic fiber is quasi-compact.*

Proof. By [BLR, Theorem 10.1.4], the lft-Néron model of G_1 exists and can be obtained as a group smoothing of the schematic closure of G_1 in $\mathcal{G}_2$. So we have a diagram

$$\mathcal{G}_1 = \mathcal{G}^{(n)} \xrightarrow{\delta^{(n)}} \cdots \longrightarrow \mathcal{G}^{(1)} \xrightarrow{\delta^{(1)}} \overline{G_1} \subset \mathcal{G}_2,$$

where the $\delta^{(i)}$ are dilatations of appropriate closed subgroups of the special fiber. This follows from [BLR, Lemma 7.1.4] since it suffices (after making an étale base change if necessary) to construct the group smoothing on $\overline{G_1} \cap \mathcal{G}_2^0$.

According to [BLR, 3.2], a dilatation of a scheme can be constructed locally and the dilatation of an affine scheme is affine. This means that a dilatation is quasi-compact. A finite combination of quasi-compact morphisms is also quasi-compact.

By construction, we have a quasi-compact map $\mathcal{G}_1 \longrightarrow \mathcal{G}_2$ which corresponds to the inclusion $G_1 \subset G_2$ on the generic fiber. Thus the assertion follows from the Néron mapping property. □

We can now show that in the situation of Proposition 2.3.1 the quotient $\Phi((\mathcal{G}_1)_k) \twoheadrightarrow \widetilde{\Phi}$ has a finite kernel.

Proposition 2.3.4. *Let K be a local field and let G_2 be a smooth algebraic K-group with* lft-*Néron model $\mathcal{G}_2$ over* $\operatorname{Spec} \mathcal{O}_K$. *Let G_1 be a smooth (closed) K-subgroup of G_2. Then the corresponding map of Néron models induces a homomorphism of* $\operatorname{Gal}(k^{\text{sep}}/k)$*-modules*

$$\Phi\big((\mathcal{G}_1)_k\big) \longrightarrow \Phi\big((\mathcal{G}_2)_k\big)$$

with a finite kernel.

Proof. The morphism $\mathcal{G}_1 \longrightarrow \mathcal{G}_2$ induces a quasi-compact morphism $\iota\colon (\mathcal{G}_1)_k \longrightarrow (\mathcal{G}_2)_k$. Since the identity component $\big(\mathcal{G}_2^0\big)_k$ is of finite type over k, it is quasi-compact and therefore its preimage is quasi-compact. Thus we can cover $\iota^{-1}\big(\big(\mathcal{G}_2^0\big)_k\big)$ with finitely many connected components of $(\mathcal{G}_1)_k$. Since every connected component over k^{sep} decomposes into a finite number of translates of $\big(\mathcal{G}_1^0\big)_k$, only finitely many elements of $\Phi\big((\mathcal{G}_1)_k\big)$ lie in the kernel. $\square$

Chapter 3

Integral Models

In this chapter we consider the problem of extending a smooth and commutative K-group scheme of finite type to an integral model, i.e., a separated and flat $\mathcal{O}_K$-group scheme. This is motivated by the fact that the literature considers integral models of algebraic tori that are not Néron models. In [ChYu, §4], a so-called ft-Néron model is defined for algebraic tori T which is a smooth integral model $\mathcal{T}^{\mathrm{ft}}$ for which $\mathcal{T}^{\mathrm{ft}}(\mathcal{O}_K^{\mathrm{sh}})$ corresponds to the maximal bounded subgroup of $T(K^{\mathrm{nr}})$. On the other hand, in [VKM], [P] and [PV], Voskresenskii et. al. define a standard model which is an integral model with similar properties to the ft-Néron model. We will show that in the lft-Néron model there exists a unique open subgroup that induces the torsion part of the group of components, which we will define as the ft-Néron model.

We will show that for a smooth and commutative K-group scheme of finite type G_K a maximal bounded subgroup of $G_K(K^{\mathrm{nr}})$ exists if, and only if, G_K does not contain a subgroup of the form $\mathbb{G}_{a,K}$, i.e., if, and only if, it admits an lft-Néron model $\mathcal{G}$. We will identify the maximal bounded subgroup with the image of the points from $\mathcal{G}(\mathcal{O}_K^{\mathrm{sh}})$ that are mapped to torsion elements in the group of components.

Thus, our definition for algebraic tori is equivalent to that of Chai and Yu and an ft-Néron model exists if, and only if, an lft-Néron model exists. We will show that the ft-Néron model is caracterized by the lifting property for the étale points from the maximal restricted subgroup of K^{nr}-valued points and also has an extension property analogous to the Néron mapping property.

We will show that ft-Néron models are compatible with étale base change and Weil restriction. The advantage of the ft-Néron model lies in the fact that it is easier to compute than the lft-Néron model. For an algebraic torus T, this model is affine and can be obtained as a group smoothing of the schematic closure of T in $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$, where $n = \dim(T)$ and L is a splitting extension of T.

This schematic closure, which is itself an integral model of T, corresponds to the standard model of Voskresenskii et. al. Using an idea from [Edi], we establish a criterion for when a monomorphism of algebraic K-tori induces a closed immersion of the corresponding Néron models.

We will show that on the étale and the smooth sites the ft-Néron model is a left-exact functor. We will define a measure on the group of K^{nr}-valued points of a torus and identify these points with maps of the character group into the units of a splitting extension L/K^{nr} of $T_{K^{\mathrm{nr}}}$. A point belongs to the maximal bounded subgroup of $T(K^{\mathrm{nr}})$ if, and only if, the corresponding map φ takes its values in $\mathcal{O}_L^*$. This will be the case exactly when φ restricted to $X(T)^I$ has this property. Therefore, regarded as an étale sheaf, we have $\mathcal{T}^{\mathrm{ft}} = \underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})$.

The above means that the sequence constructed in Xarles' proof of [X, Theorem 1.1] is a particular case of the canonical sequence

$$0 \longrightarrow \mathcal{T}^{\mathrm{ft}} \longrightarrow \mathcal{T} \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), i_*\mathbb{Z})\,.$$

In Appendix B we will discuss the right-exactness of the ft-Néron model.

3.1 Integral models and Néron models

Let K be a local field and let G_K be a smooth K-group scheme of finite type. By an integral model of G_K we mean a flat and separated $\mathcal{O}_K$-group scheme G whose generic fiber as a group scheme is isomorphic to G_K.

The most useful type of integral models are probably the lft-Néron models: for algebraic tori, the Néron model always exists in our situation; one can even specify an explicit construction (cf. [BLR, Proposition 10.1.4]). Namely, if T is an algebraic K-torus, then there exists a finite Galois extension L/K such that $T \otimes_K L \cong \mathbb{G}_{m,L}^n$. This means that there is a closed immersion $T \longrightarrow \mathfrak{R}_{L/K}(\mathbb{G}_{m,L}^n)$ through which T can be identified with a subgroup of $\mathfrak{R}_{L/K}(\mathbb{G}_{m,L}^n)$. Now there is an lft-Néron model of $\mathbb{G}_{m,L}$ and thus also an lft-Néron model $\mathcal{R}$ of $\mathfrak{R}_{L/K}(\mathbb{G}_{m,L}^n)$. Then the group smoothing of the schematic closure of T in $\mathcal{R}$ is an lft-Néron model of T.

What is important in this construction is that the schematic closure is an integral model of T. Regarding this model, we cite the following result:

Lemma 3.1.1. [SGA3, VIII, 7.1] *Let R be a discrete valuation ring with quotient field K and set $S := \operatorname{Spec} R$. If G is an S-scheme and H is a closed subscheme of G_K (so that H is a subscheme of G), then the schematic closure $\bar{H}$ of H exists in G. This is a flat S-scheme with generic fiber $\bar{H}_K = H$ and is the only closed subscheme of G with these two properties.*

This construction is functorial with respect to such pairs (H, G) and commutes with fiber products. In particular, if G is an S-group scheme and H is a K-subgroup of G_K, then $\bar{H}$ is an S-subgroup of G.

Since the scheme $\mathcal{R}$ is no longer quasi-compact and the lft-Néron model of a torus T is generally neither affine nor quasi-compact, this construction is rather unwieldy for explicit calculations. That is why Ching-Li Chai and Jui-Kang Yu consider in [ChYu, §3] a so-called ft-Néron model. They write $T(K^{\mathrm{nr}})^{\mathrm{bd}}$ to denote the maximal bounded subgroup of $T(K^{\mathrm{nr}})$ and define their ft-Néron model as a smooth integral model $\mathcal{T}^{\mathrm{ft}}$ of T which satisfies $\mathcal{T}^{\mathrm{ft}}(\mathcal{O}_K^{\mathrm{sh}}) = T(K^{\mathrm{nr}})^{\mathrm{bd}}$. They state that this model can be constructed as a group smoothing of the schematic closure of T under the embedding

$$T \hookrightarrow \mathfrak{R}_{L/K}(T_L) \cong \mathfrak{R}_{L/K}(\mathbb{G}_{m,L}^n) \hookrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_{m,\mathcal{O}_L}^n)\,.$$

This construction corresponds to the construction of the lft-Néron model, except for the fact that the Néron model $\mathcal{R}$ is replaced by its identity component $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_{m,\mathcal{O}_L}^n)$ (cf. [NX, 2.4]).

We want to define the ft-Néron model in general, determine its properties and compare it with the lft-Néron model. Naturally, in the case that a Néron model of finite type exists, we also want to understand it as an ft-Néron model. Thus, in analogy to the considerations from [BLR, 10.2], we investigate the case of smooth and commutative K-group schemes of finite type.

Proposition 3.1.2. *Let G_K be a smooth and commutative K-group scheme of finite type. Then there exists a maximal bounded subgroup of $G_K(K^{\mathrm{nr}})$ if, and only if, G_K contains no subgroup of the form $\mathbb{G}_{a,K}$.*

A maximal bounded subgroup $G_K(K^{\mathrm{nr}})^{\mathrm{bd}}$ exists if, and only if, an lft-Néron model $\mathcal{G}$ of G_K exists. This subgroup corresponds to the preimage of the torsion part of $\Phi(\mathcal{G}_k)$ under the canonical surjection

$$G_K(K^{\mathrm{nr}}) = \mathcal{G}\big(\mathcal{O}_K^{\mathrm{sh}}\big) \longrightarrow \mathcal{G}_k(k^{\mathrm{sep}}) \longrightarrow \Phi(\mathcal{G}_k)(k^{\mathrm{sep}})\,.$$

For every bounded subgroup C of $G_K(K^{\mathrm{nr}})$, we have $C \subset G_K(K^{\mathrm{nr}})^{\mathrm{bd}}$.

Proof. Since G_K has no subgroup of the form $\mathbb{G}_{a,K}$ if, and only if, $G_K \otimes_K K^{\mathrm{nr}}$ has no subgroup of the form $\mathbb{G}_{a,K^{\mathrm{nr}}}$, we may assume, without loss of generality, that $K = K^{\mathrm{nr}}$. We now assume that a maximal bounded subgroup $B \subset G_K(K)$ exists. Suppose that there exists a K-subgroup $U_K = \mathbb{G}_{a,K} \hookrightarrow G_K$. Since we are considering K-group schemes, U_K is a closed subgroup. This means that $B \cap U_K(K)$ is a bounded subgroup of $U_K(K) = K$.

By the boundedness of B, there exist a finite covering by open affine subschemes $(V_i)_{i\in I}$ of G_K, closed immersions $V_i \longrightarrow \mathbb{A}_K^{n_i}$ and a decomposition $B = \bigcup_{i\in I} B_i$ such that the $B_i \subset V_i(K)$ correspond to bounded subsets of $\mathbb{A}_K^{n_i}(K)$. This means that $(U_K \cap V_i)_{i\in I}$ is an open affine cover of U_K, which shows that $B \cap U_K(K)$ is bounded in U_K.

The multiplication μ on G_K induces by restriction a group homomorphism $U_K \times_K G_K \longrightarrow G_K$. All subgroups $\pi^l \mathcal{O}_K \subset K = U_K(K)$ with $l \in \mathbb{Z}$ are bounded, but $U_K(K)$ itself is unbounded. So there exists an $l \in \mathbb{Z}$ such that $\pi^l \mathcal{O}_K$ is strictly larger than $B \cap U_K(K)$. Now $\pi^l \mathcal{O}_K \times B$ is a bounded subgroup of $U_K \times_K G_K$, so by [BLR, 1.1.4] the image of this subgroup under μ must be a bounded subgroup of $G_K(K^{\mathrm{nr}})$, which contradicts the maximality of B.

Now assume that G_K does not have a subgroup of the form $\mathbb{G}_{a,K}$. This means that there is an lft-Néron model $\mathcal{G}$ of G_K. This model has a group of components $\Phi(\mathcal{G}_k)$ which is finitely generated as an abelian group. Its torsion part is therefore a finite subgroup. Since $\mathcal{G}$ is smooth and $\mathcal{O}_K$ is Henselian, the map

$$G_K(K^{\mathrm{nr}}) = \mathcal{G}\big(\mathcal{O}_K^{\mathrm{sh}}\big) \longrightarrow \mathcal{G}_k(k^{\mathrm{sep}}) \longrightarrow \Phi(\mathcal{G}_k)(k^{\mathrm{sep}})$$

is an epimorphism of abelian groups. In $G_K(K)$, consider the preimage B of the torsion part of $\Phi(\mathcal{G}_k)$. This preimage is equal to the $\mathcal{O}_K$-valued points of the subscheme $\mathcal{G}^{\mathrm{ft}}$ of $\mathcal{G}$, which consists of the generic fiber and all connected components of $\mathcal{G}_k$ that are mapped onto a torsion element of $\Phi(\mathcal{G}_k)$. Since $\Phi(\mathcal{G}_k)$ becomes constant after a finite separable extension of k, there exists a finite Galois extension $\mathcal{O}_L \longrightarrow \mathcal{O}_K$ such that $\mathcal{G}^{\mathrm{ft}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is a union of the generic fiber and finitely many translates of the identity component of $\mathcal{G} \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Since the identity component is a quasi-compact open subscheme, $\mathcal{G}^{\mathrm{ft}}$ must therefore already be an open quasi-compact subscheme of $\mathcal{G}$. In particular, $\mathcal{G}^{\mathrm{ft}}$ is of finite type. Therefore B must be a bounded subgroup of $G_K(K)$ [BLR, 1.1.7].

Suppose C is a bounded subgroup of $G_K(K)$. By [BLR, 1.17 and 3.1.4], there exists a smooth $\mathcal{O}_K$-scheme H of finite type with generic fiber G_K such that $C \subset \mathrm{im}(H(\mathcal{O}_K) \longrightarrow H_K(K))$. Using the Néron mapping property, we obtain a morphism $H \longrightarrow \mathcal{G}$ and an identification of C with a subset of the $\mathcal{O}_K$-valued points of the image of H. However, the image of H in $\mathcal{G}_k$ is quasi-compact and can therefore be covered by a finite number of connected components of $\mathcal{G}_k$. Since every connected component decomposes into a finite number of translates of the identity component over $\mathcal{O}_K^{\mathrm{sh}}$, the image of C in $\Phi(\mathcal{G}_s)(k^{\mathrm{sep}})$ is finite. But, according to the requirement, it is also a subgroup. Thus, we must have $C \subset B$, thereby proving that B is the maximal bounded subgroup of $G_K(K)$. □

Next, we identify the ft-Néron model with an open subgroup of the lft-Néron model and show that an ft-Néron model can be characterized by a lifting property for certain étale points and a mapping property that is similar to the Néron mapping property.

Theorem 3.1.3. *Let G_K be a smooth and commutative K-group scheme of finite type and assume that G_K contains no subgroup of type $\mathbb{G}_{a,K}$. Let $\mathcal{G}$ be the lft-Néron model of G_K and let $\Phi(\mathcal{G}_k)$ be the group of components of $\mathcal{G}_k$. For a smooth integral model of finite type G^{ft} of G_K, the following are equivalent.*

(1) *We have $G^{\mathrm{ft}}(\mathcal{O}_K^{\mathrm{sh}}) = G_K(K^{\mathrm{nr}})^{\mathrm{bd}}$, where $G_K(K^{\mathrm{nr}})^{\mathrm{bd}}$ is the maximal bounded subgroup of $G_K(K^{\mathrm{nr}})$.*

(2) *Let Z be a smooth $\mathcal{O}_K$-scheme and let $u_K\colon Z_K \longrightarrow G_K$ be a K-morphism inducing a map $Z_K(K^{\mathrm{nr}}) \longrightarrow G_K\,(K^{\mathrm{nr}})^{\mathrm{bd}}$. Then there exists a unique extension of u_K to a morphism of $\mathcal{O}_K$-schemes $u\colon Z \longrightarrow G^{\mathrm{ft}}$.*

(3) *The model G^{ft} is isomorphic to the open subgroup $\mathcal{G}^{\mathrm{ft}}$ of $\mathcal{G}$ with generic fiber G_K whose special fiber consists of the connected components of $\mathcal{G}_k$ whose image under $\mathcal{G}_k \longrightarrow \Phi(\mathcal{G}_k)$ lies in the torsion part of $\Phi(\mathcal{G}_k)$.*

Such an integral model is called an ft-Néron model of G_K. *It exists under the conditions of the theorem, and without those conditions none of the given descriptions would make sense.*

Proof. Based on the description (3), we first establish the group structure. By [BLR, Theorem 10.2.2], under the stated assumptions there exists an lft-Néron model of G_K and its group of components $\Phi(\mathcal{G}_k)$ is finitely generated as an abelian group, whence the torsion part of $\Phi(\mathcal{G}_k)$ is finite. As we saw in the proof of proposition 3.1.2, the subset $\mathcal{G}^{\mathrm{ft}}$ of $\mathcal{G}$ defined above is in fact an open subscheme of finite type. The unit section trivially factors through $\mathcal{G}$. The inverse on $\mathcal{G}$ must be an isomorphism of $\mathcal{G}^{\mathrm{ft}}$, since the torsion part of $\Phi(\mathcal{G}_k)$ is a subgroup scheme of $\Phi(\mathcal{G}_k)$. In an analogous manner, multiplication must also factor through $\mathcal{G}^{\mathrm{ft}}$. Conversely, if G_K has a subgroup of the form $\mathbb{G}_{a,K}$, then neither an lft-Néron model of G_K nor a maximal bounded subgroup of $G_K(K^{\mathrm{nr}})$ exist. We now show the equivalence of the three descriptions:

(1) ⇒ (2): This follows using the idea in the proof of [BLR, 3.5.3]. Let Z be a smooth $\mathcal{O}_K$-scheme and let $u_k\colon Z_K \longrightarrow G_K^{\mathrm{ft}}$ be a K-morphism such that $Z_K(K^{\mathrm{nr}}) \subset G_K^{\mathrm{ft}}\,(K^{\mathrm{nr}})^{\mathrm{bd}}$. Without loss of generality, we may assume that Z is of finite type. Now let Γ be the schematic closure of the graph of u_K in $Z \times G^{\mathrm{ft}}$. Further, let $p\colon \Gamma \subset Z \times G^{\mathrm{ft}} \longrightarrow Z$ be the projection onto the first factor.

Since Z and G^{ft} are of finite type and $\mathcal{O}_K$ is Noetherian, Γ is of finite presentation. By Chevalley's theorem, we conclude that the image of $p_k\colon \Gamma_k \longrightarrow Z_k$ is constructible. By (1), the image contains the dense subset of all points $z_k \in Z_k(k^{\mathrm{sep}})$ that lift to a point $z \in Z(\mathcal{O}_K^{\mathrm{sh}})$. Thus, for every irreducible component of Z_k, its generic point must also lie in the image of p. This means that such a generic point η has a preimage $\xi \in \Gamma$ and the local ring $\mathcal{O}_{\Gamma,\xi}$ dominates the discrete valuation ring $\mathcal{O}_{Z,\eta}$. Since p is an isomorphism on the generic fiber and Γ is flat, the associated localizations must be isomorphic via π_K, and $\mathcal{O}_{\Gamma,\xi} \cong \mathcal{O}_{Z,\eta}$. This means that p is an isomorphism in an open neighborhood of η, and we therefore obtain an $\mathcal{O}_K$-rational map $Z \to G^{\mathrm{ft}}$.

Since G^{ft} is a smooth and separated group scheme, the existence of an $\mathcal{O}_K$-morphism $u\colon Z \longrightarrow G^{\mathrm{ft}}$ which extends u_K follows from Weil's extension theorem [BLR, Theorem 4.4.1]. This is unique because it is based on the dense subset of points $z_k \in Z_k(k)$ that lift to points $z \in Z(\mathcal{O}_K)$, which already determines it [EGA IV, 11.10].

(2) ⇒ (3): By the universal property of the lft-Néron model $\mathcal{G}$, there exists a map $G^{\mathrm{ft}} \longrightarrow \mathcal{G}$ which extends the identity on G_K. Since G^{ft} is quasi-compact, the image of G^{ft} is also quasi-compact and can therefore be covered by a finite number of translates of the identity component. Since this map has to be a group-scheme morphism by Néron's mapping property, it must factor through $\mathcal{G}^{\mathrm{ft}} \longrightarrow \mathcal{G}$.

By proposition 3.1.2, the group of $\mathcal{O}_K^{\mathrm{sh}}$-valued points of $\mathcal{G}^{\mathrm{ft}}$ is equal to the maximal bounded subgroup of G_K. Therefore, by (2), the identity on G_K lifts to a map $\mathcal{G}^{\mathrm{ft}} \longrightarrow G^{\mathrm{ft}}$. By the uniqueness of these maps, we must have $G^{\mathrm{ft}} \cong \mathcal{G}^{\mathrm{ft}}$.

(3) ⇒ (1) : This implication is clear by proposition 3.1.2. □

In order to investigate the ft-Néron model for algebraic tori, it is advisable to first describe the maximal bounded subgroup of K^{nr}-valued points. To do this, we want to define a modulus $\|.\|$ in a suitable manner.

So let T be an algebraic K-torus with character group $X(T)$ and let L be a finite and separable extension of K^{nr} over which T splits. Then, by Cartier duality,

$$\begin{aligned} T(L) &= \mathrm{Hom}_{\mathbb{Z}}\left(X(T), (K^{\mathrm{sep}})^*\right)^{\mathrm{Gal}(K^{\mathrm{sep}}/L)} = \mathrm{Hom}_{\mathbb{Z}}\left(X(T), L^*\right) \\ T\left(K^{\mathrm{nr}}\right) &= \mathrm{Hom}_{\mathbb{Z}}\left(X(T), (K^{\mathrm{sep}})^*\right)^{\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})} = \mathrm{Hom}_{\mathrm{Gal}(L/K^{\mathrm{nr}})}\left(X(T), L^*\right) \end{aligned}$$

Since T splits over L, there exists a trivialization $T_L \cong \mathbb{G}_{m,L}^n \cong \operatorname{Spec} L[X(T)]$. Using a $\mathbb{Z}$-basis $(\chi_1, \ldots, \chi_n)$ of $X(T)$, we can write

$$L[X(T)] = L\left[X_1, \ldots, X_n, Z\right] / \left(X_1 \cdot \ldots \cdot X_n \cdot Z - 1\right)$$

with variables X_i for each χ_i. Via the above a point $x \in T(L) = \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)$ becomes identified with a morphism

$$x\colon L[X(T)] \longrightarrow L,\ X_i \mapsto x(\chi_i) \in L^*.$$

To determine the size (i.e., modulus) of a point $x \in T(K^{\mathrm{nr}})$, it suffices to determine (using [BLR, Proposition 1.1.5]) the size of x as a point in $T_L(L)$.

Via the above identification, a point $x \in \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)^{\mathrm{Gal}(L/K^{\mathrm{nr}})}$ is identified with a $\mathrm{Gal}\left(L/K^{\mathrm{nr}}\right)$-equivariant map underlying a map of abelian groups $x\colon X(T) \longrightarrow L^*$. According to definition [BLR, 1.1.2], we consider the closed immersion

$$T_L \hookrightarrow \mathbb{A}_L^{n+1} \quad L\left[X_1, .., X_n, Z\right] \longrightarrow L\left[X_1, .., X_n, Z\right] / \left(X_1 \cdot \ldots \cdot X_n \cdot Z - 1\right)$$

and define the modulus of x as

$$\|x\| := \max\left\{\left|x\left(\chi_1\right)\right|, \ldots \left|x\left(\chi_n\right)\right|, \left|\frac{1}{\prod_{i=1,\ldots,n} x\left(\chi_i\right)}\right|\right\}.$$

We will write the group law on a torus multiplicatively. After trivialization, the multiplication on T becomes the component-wise multiplication on $\mathbb{G}_{m,L}^n$. This shows that, for $l \in \mathbb{N}$ and $x, y \in T(L)$, we have

$$\begin{aligned} \left\|x^l\right\| &= \|x\|^l \\ \left\|x^{-1}\right\|^{-1} &= \min\left\{\left|x\left(\chi_1\right)\right|, .., \left|x\left(\chi_n\right)\right|, \left|\frac{1}{\prod_{i=1,..n} x\left(\chi_i\right)}\right|\right\} \\ \|xy\| &\leq \|x\|\|y\|. \end{aligned}$$

Thus the subgroup generated by an element x is bounded if, and only if, all $x(\chi_i)$ have modulus one. By component-wise multiplication, two such elements again produce a bounded subgroup. Therefore, we have

$$T(K^{\mathrm{nr}})^{\mathrm{bd}} = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathcal{O}_L^*)^{\mathrm{Gal}(L/K^{\mathrm{nr}})}. \tag{3.1.3.1}$$

Based on the first description from Theorem 3.1.3, one can see that the ft-Néron model can be constructed similarly to the usual Néron model as a group smoothing of an integral model which $T(K^{\mathrm{nr}})^{\mathrm{bd}}$ lifts. In the case of algebraic tori, one can explicitly describe such an integral model.

Proposition 3.1.4. *The* ft*-Néron model of* $\mathbb{G}_{m,K}$ *is* $\mathbb{G}_{m,\mathcal{O}_K}$*. If* L/K *is a finite separable extension of local fields and* G' *is a smooth and commutative algebraic* L*-group with* ft*-Néron model* $(\mathcal{G}')^{\mathrm{ft}}$*, then* $\mathfrak{R}_{L/K}(G')$ *has an* ft*-Néron model and this is isomorphic to* $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{G}')^{\mathrm{ft}}\big)$*. The* ft*-Néron model is compatible with étale base changes.*

Proof. After constructing the lft-Néron model of $\mathbb{G}_{m,K}$, it is clear that the ft-Néron model of $\mathbb{G}_{m,K}$ is equal to $\mathbb{G}_{m,\mathcal{O}_K}$.

Now let G' be as stated. The lft-Néron model $\mathcal{G}'$ exists and contains $(\mathcal{G}')^{\mathrm{ft}}$ as an open and quasi-compact subgroup. The Weil restriction is compatible with open immersions and respects group scheme structures. Thus $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{G}')^{\mathrm{ft}}\big)$ is an open subgroup of the lft-Néron model $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{G}')$ of $\mathfrak{R}_{L/K}(G')$. Since $\mathcal{O}_K$ is Noetherian, the Weil restriction is also compatible with quasi-compactness, so this subgroup is also of finite type. Finally, $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{G}')^{\mathrm{ft}}\big)$ is trivially smooth.

Now we have an exact sequence

$$0 \longrightarrow (\mathcal{G}')^{\mathrm{ft}} \longrightarrow \mathcal{G}' \longrightarrow i_*\Phi(\mathcal{G}'_s)^{\vee\vee} \longrightarrow 0$$

on the étale site over $\mathcal{O}_L$. By the exactness of the Weil restriction functor, the above sequence induces an exact sequence

$$0 \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{G}')^{\mathrm{ft}}\big) \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{G}') \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\Big(i_*\Phi(\mathcal{G}'_s)^{\vee\vee}\Big) \longrightarrow 0$$

over $\mathcal{O}_K$. Now $\Phi(\mathcal{G}'_s)^{\vee}$ is torsion-free and the Weil restriction of this group is an induction of $\Phi(\mathcal{G}'_s)^{\vee\vee}$ as we saw in the proof of Theorem 1.2.1. This means that it is torsion-free again, so that the image of $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{G}')^{\mathrm{ft}}\big)\big(\mathcal{O}_K^{\mathrm{sh}}\big)$ in the group of components of $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{G}') \otimes_{\mathcal{O}_K} k$ contains the torsion part. By its quasi-compactness, the image cannot be larger than the torsion part, which means that we have shown that

$$\mathfrak{R}_{L/K}(G')\,(K^{\mathrm{nr}})^{\mathrm{bd}} = \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big((\mathcal{G}')^{\mathrm{ft}}\big)\big(\mathcal{O}_K^{\mathrm{sh}}\big)\,.$$

Now the claim follows from Theorem 3.1.3.

Since boundedness is compatible with finite separable base changes, the argument in [BLR, 1.2.2(c)] can also be applied to ft-Néron models. These are therefore compatible with étale base changes. □

Proposition 3.1.5. *Let* T *be an algebraic* K*-torus with a finite Galois splitting extension* L*. Consider the sequence of inclusions*

$$T \hookrightarrow \mathfrak{R}_{L/K}(T_L) \cong \mathfrak{R}_{L/K}\big(\mathbb{G}_{m,L}^n\big) \hookrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big(\mathbb{G}_{m,\mathcal{O}_L}^n\big)\,.$$

Let $\overline{T}$ *be the schematic closure of* T *in* $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big(\mathbb{G}_{m,\mathcal{O}_L}^n\big)$*. Then* $\overline{T}$ *is an affine integral model of* T *that is independent of the choice of* L*. Regarding the choice of* T*, it is compatible with étale base changes. Further,* $\overline{T}\big(\mathcal{O}_K^{\mathrm{sh}}\big) = T(K^{\mathrm{nr}})^{\mathrm{bd}}$*.*

Proof. The closed immersion $T \longrightarrow \mathfrak{R}_{L/K}(T_L)$ is a homomorphism of group schemes which maps $T(K^{\mathrm{nr}})^{\mathrm{bd}}$ into $\mathfrak{R}_{L/K}\big(\mathbb{G}_{m,L}^n\big)(K^{\mathrm{nr}})^{\mathrm{bd}}$. These points lift to $\mathcal{O}_K^{\mathrm{sh}}$-valued points of $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big(\mathbb{G}_{m,\mathcal{O}_L}^n\big)$.

This means that all points of $T\,(K^{\mathrm{nr}})^{\mathrm{bd}}$ lift to $\mathcal{O}_K^{\mathrm{sh}}$-valued points of $\overline{T}$. By construction, $\overline{T}$ is an affine scheme of finite type since $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$ and T are affine schemes. By [SGA3, VIII, Lemma 7.1], $\overline{T}$ is an integral model of T. This means that no other points of $T(K^{\mathrm{nr}})$ can lift to $\mathcal{O}_K^{\mathrm{sh}}$-valued points of $\overline{T}$ because these points would induce unbounded subgroups.

For independence from the choice of L, consider a finite extension M/L. Then the embedding $T \longrightarrow \mathfrak{R}_{\mathcal{O}_M/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_M})$ factors through the closed immersion $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L}) \longrightarrow \mathfrak{R}_{\mathcal{O}_M/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_M})$ induced by the closed immersion $\mathbb{G}^n_{m,\mathcal{O}_L} \hookrightarrow \mathfrak{R}_{\mathcal{O}_M/\mathcal{O}_L}(\mathbb{G}^n_{m,\mathcal{O}_M})$.

Finally, let K'/K be an unramified extension. Since the schematic closure is compatible with flat base change, $\overline{T} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ is the schematic closure of $T_{K'}$ in

$$\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} = \mathfrak{R}_{\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}/\mathcal{O}_{K'}}(\mathbb{G}_{m,\mathcal{O}_L} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'})\,.$$

Since K'/K is unramified, $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \cong \prod_{[L\cap K'\colon K]} \mathcal{O}_{L'}$, where L' is the composite of K' with L. We obtain similarly (see [NX, Proposition 2.2])

$$\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} = \prod_{[L\cap K'\colon K]} \mathfrak{R}_{\mathcal{O}_{L'}/\mathcal{O}_{K'}}\left(\mathbb{G}^n_{m,\mathcal{O}_{L'}}\right)$$

and the embedding of $T_{K'}$ here factors through the diagonal embedding

$$\Delta\colon \mathfrak{R}_{\mathcal{O}_{L'}/\mathcal{O}_{K'}}\left(\mathbb{G}^n_{m,\mathcal{O}_{L'}}\right) \hookrightarrow \prod_{[L\cap K'\colon K]} \mathfrak{R}_{\mathcal{O}_{L'}/\mathcal{O}_{K'}}\left(\mathbb{G}^n_{m,\mathcal{O}_{L'}}\right).$$

Since the schemes considered above are separated, the diagonal embedding is a closed immersion, whence the schematic closure also factors through Δ. □

3.2 Néron models and closed immersions

Proposition 3.2.1. *The* ft*-Néron model of an algebraic K-torus T is equal to the group smoothing of the schematic closure $\overline{T}$ of T under the immersion $T \longrightarrow \mathfrak{R}_{L/K}(T_L) \cong \mathfrak{R}_{L/K}(\mathbb{G}^n_{m,L}) \hookrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$. Further, the following are equivalent:*

1. *$\overline{T}$ is smooth.*
2. *The canonical map $\mathcal{T}^{\mathrm{ft}} \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$ is a closed immersion.*
3. *The canonical map $\mathcal{T} \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{G}^n_{\mathcal{O}_L})$ is a closed immersion.*
   Further, $\mathcal{G}^n_{\mathcal{O}_L}$ is the lft*-Néron model of $\mathbb{G}^n_{m,L}$ over $\mathcal{O}_L$.*

Proof. (1) ⇒ (2): Since $\overline{T}$ is an integral model of finite type, the group smoothing $\mathcal{T}^{\mathrm{ft}}$ of $\overline{T}$ exists. By Proposition 3.1.5, $\mathcal{T}^{\mathrm{ft}}$ is an ft-Néron model of T. If $\overline{T}$ is already smooth, then the smoothing is not necessary and the canonical map $\mathcal{T}^{\mathrm{ft}} \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$ corresponds to the closed immersion $\overline{T} \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$.

(2) ⇒ (3): Similarly, $\mathcal{T}$ is the group smoothing of the schematic closure of T in $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{G}^n_{\mathcal{O}_L})$. Since this schematic closure is an $\mathcal{O}_K$-group scheme, it is smooth if it is smooth in a neighborhood of the identity. The latter can be checked on the open subgroup $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$. By (2) and the uniqueness statement in [SGA3, VIII, 7.1], the schematic closure in a neighborhood of the identity equals the canonical embedding $\mathcal{T}^{\mathrm{ft}} \longrightarrow \mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}^n_{m,\mathcal{O}_L})$ and is, therefore, smooth. This means that no smoothing is necessary and the canonical morphism is a closed immersion.

(3) ⇒ (1) Again by [SGA3, VIII, Lemma 7.1], the schematic closure of T in $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}(\mathcal{G}^n_{\mathcal{O}_L})$ is smooth. Since the schematic closure is locally determined, $\overline{T}$ is an open subscheme of this schematic closure, hence, in particular, smooth. □

In [VKM, §5, Proposition 6], the schematic closure $\overline{T}$ just considered is identified with the so-called standard model (see loc. cit. and [PV, P]). By the proof of the last two propositions, it is clear that the group smoothing of this closure corresponds to the ft-Néron model (cf. [P, §10, Proposition 8] and [VKM, §5, Proposition 7]). In [Edi, Constructions 2.3 and 2.4], Edixhoven defined for a finite Galois extension $S' \longrightarrow S$ and an S-scheme X an action of $G := \mathrm{Gal}(S'/S)$ on $X' := \mathfrak{R}_{S'/S}(X \times_S S')$. He then considered the functor of G-invariant points of X' [Edi, §3] and showed that, for a separated X, this functor is represented by a closed subscheme. From the explicit construction one sees that the canonical closed immersion $X \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S')$ corresponds to the immersion $(X')^G \longrightarrow X'$ of the representing scheme of G-invariant points.

Now if T is an algebraic K-torus with a finite Galois splitting extension L, then we have on $\mathfrak{R}_{L/K}(T_L) \cong \mathfrak{R}_{L/K}\big(\mathbb{G}_{m,L}^d\big)$ an equivariant $G := \mathrm{Gal}(L/K)$-action so that T corresponds to the subscheme of G-invariant points. By the equivariance, this action extends canonically to an action on $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big(\mathbb{G}_{m,\mathcal{O}_L}^d\big)$.

The closed subscheme of G-invariant points of $\mathfrak{R}_{\mathcal{O}_L/\mathcal{O}_K}\big(\mathbb{G}_{m,\mathcal{O}_L}^d\big)$ is an $\mathcal{O}_K$-model of T and by [Edi, Proposition 3.4] one can see that this model is smooth if L/K is tamely ramified. By [SGA3, VIII, Lemma 7.1], this model must already be the schematic closure of T. In particular, the standard model is then equal to the ft-Néron model.

As an application we obtain the following statement:

Proposition 3.2.2 (cf. [BLR, Theorem 7.5.4] and [Edi, Theorem 6.1])**.** *Let $\iota\colon T_1 \longrightarrow T_2$ be a monomorphism of algebraic K-tori and assume that T_1 splits over a tamely ramified extension of K. Then the induced map $\mathcal{T}_1 \longrightarrow \mathcal{T}_2$ of the Néron models is a closed immersion.*

Proof. Let L/K be a common splitting extension of T_1 and T_2. Then we have a commutative diagram

$$\begin{array}{ccc} T_1 & \hookrightarrow & T_2 \\ \downarrow & & \downarrow \\ \mathfrak{R}_{L/K}((T_1)_L) & \hookrightarrow & \mathfrak{R}_{L/K}((T_2)_L) \end{array}$$

in which all maps are closed immersions. For the top horizontal map one uses that monomorphisms of diagonalizable group schemes are closed immersions and closed immersions are compatible with descent. The vertical maps are the canonical embeddings, which are closed immersions by the separation of the tori T_i. On the bottom row we use the compatibility of the Weil restriction with closed immersions.

Let $\mathcal{R}$ be the Néron model of $\mathfrak{R}_{L/K}((T_2)_L)$. By assumption, the schematic closure of T_1 in $\mathcal{R}$ is smooth and equal to the Néron model of T_1. This embedding now factors through the map $j_*\iota$ between Néron models induced by ι

$$\begin{array}{ccc} \mathcal{T}_1 & \xrightarrow{j_*\iota} & \mathcal{T}_2 \\ \downarrow & \swarrow & \\ \mathcal{R} & & \end{array}$$

This means that $j_*\iota$ must be a closed immersion, since $\mathcal{T}_2 \longrightarrow \mathcal{R}$ is separated [H, II, Example 4.8 and Corollary 4.6(e)]. □

3.3 The ft-Néron model of a torus as a sheaf

The formation of the Néron model corresponds to the left-exact functor j_* on the étale and smooth sites over $\mathrm{Spec}\, K$. In this sense, a short exact sequence of commutative K-group schemes induces a left exact sequence of Néron models on the corresponding site over $\operatorname{Spec}\mathcal{O}_K$. We now show that this also applies to the ft-Néron models.

Proposition 3.3.1. *Let K be a local field and let*

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

be a short exact sequence of smooth and commutative algebraic K-groups for which lft-*Néron models exist. Then the above sequence induces a left-exact sequence on the étale and smooth sites over $\mathcal{O}_K$ of the corresponding ft-Néron models*

$$0 \longrightarrow \mathcal{G}_1^{\mathrm{ft}} \longrightarrow \mathcal{G}_2^{\mathrm{ft}} \longrightarrow \mathcal{G}_3^{\mathrm{ft}},$$

where the morphisms between the sheaves are extensions of the corresponding morphisms between the K-group schemes via the universal property of the ft-*Néron model.*

Proof. If Z is a smooth $\mathcal{O}_K$-scheme, $Z \longrightarrow \mathcal{G}_i^{\mathrm{ft}}$ is an $\mathcal{O}_K$-morphism and $u_K\colon Z_K \longrightarrow G_i$ is a K-morphism, then the $\mathcal{O}_K$-morphism $u\colon Z \longrightarrow \mathcal{G}_i$ that extends u_K is the composition $Z \longrightarrow \mathcal{G}_i^{\mathrm{ft}} \hookrightarrow \mathcal{G}$. Thus the exact sequence $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \longrightarrow \mathcal{G}_3$ induces a sequence of étale sheaves

$$0 \longrightarrow \mathcal{G}_1^{\mathrm{ft}} \longrightarrow \mathcal{G}_2^{\mathrm{ft}} \longrightarrow \mathcal{G}_3^{\mathrm{ft}},$$

where the intervening maps are as in the statement. Further, a morphism $u_K\colon Z_K \longrightarrow G_i$ with an extension $Z \longrightarrow \mathcal{G}_i^{\mathrm{ft}}$ induces a K-morphism $Z_K \longrightarrow G_{i+1}$ with an extension $Z \longrightarrow \mathcal{G}_{i+1}^{\mathrm{ft}}$, because the image of a bounded subset is bounded again and the morphism $G_i \longrightarrow G_{i+1}$ is a group homomorphism.

To examine the exactness of this sequence in the étale topology, it suffices to look at the stalks. The sequence of stalks at the generic fiber is exact by assumption and the sequence of stalks at the special fiber corresponds to the sequence

$$0 \longrightarrow G_1(K^{\mathrm{nr}})^{\mathrm{bd}} \xrightarrow{\alpha} G_2(K^{\mathrm{nr}})^{\mathrm{bd}} \xrightarrow{\beta} G_3(K^{\mathrm{nr}})^{\mathrm{bd}}$$

with canonical arrows. It is clear that the sequence at $G_1\left(K^{\mathrm{nr}}\right)^{\mathrm{bd}}$ is exact. For the exactness at $G_2\left(K^{\mathrm{nr}}\right)^{\mathrm{bd}}$ we need only check that $\operatorname{Im}\alpha \supset \ker\beta$. By the left exactness of the sequence of the lft-Néron models, an $x \in \ker\beta$ has a preimage $z \in G_1(K^{\mathrm{nr}})$. But since $\left\{x^l \mid l \in \mathbb{Z}\right\} \subset G_2(K^{\mathrm{nr}})$ is bounded by assumption and $G_1 \longrightarrow G_2$ is a closed immersion, its preimage $\left\{z^l \mid l \in \mathbb{Z}\right\}$ must also be bounded in G_1, whence $z \in G_1(K^{\mathrm{nr}})^{\mathrm{bd}}$. In the smooth topology, we also need only check that $\operatorname{Im}\alpha \supset \ker\beta$. So let Z be a smooth $\mathcal{O}_K$-scheme and let $f_2\colon Z \longrightarrow \mathcal{G}_2^{\mathrm{ft}}$ be a morphism whose composition with $\mathcal{G}_2^{\mathrm{ft}} \longrightarrow \mathcal{G}_3^{\mathrm{ft}}$ is trivial. By the left-exactness of the sequence of the lft-Néron models, f_2 factors smoothly-locally through a section of $\mathcal{G}_2$ over $\mathcal{G}_1$. However, the image of these factorizations can only affect components of $(\mathcal{G}_1)_k$ which are in the torsion part of $\Phi((\mathcal{G}_1)_k)$ because, under the quasi-compact group homomorphism $(\mathcal{G}_1)_k \longrightarrow (\mathcal{G}_2)_k$, these components map exactly into the torsion part of $\Phi((G_2)_k)$. $\square$

In Appendix B we will address the question of right-exactness (in the case of algebraic K-tori).

We now want to examine the étale sheaf represented by the ft-Néron model in more detail in the case of an algebraic K-torus T. Up to this point we have tested the boundedness of a set of points from $T(K^{\mathrm{nr}})$ on the whole character group. However, according to Xarles' description of the free part, the boundedness (at least for perfect residue fields) should be tested on $X(T)^I = \operatorname{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$. In fact, we have

Proposition 3.3.2. *Let T be an algebraic K-torus with character group $X(T)$. Let L/K^{nr} be a finite Galois splitting extension of $T \otimes_K K^{\mathrm{nr}}$ with Galois group $I = \mathrm{Gal}(L/K^{\mathrm{nr}})$. Then*

$$T\left(K^{\mathrm{nr}}\right)^{\mathrm{bd}} = \left\{ f \in \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)^I \mid f(X) \in (\mathcal{O}_K^{\mathrm{sh}})^* \text{ for } x \in X(T)^I \right\},$$

i.e., the maximal bounded subgroup consists exactly of those points which, under the canonical map $T \longrightarrow T^I$, are mapped to a point that generates a bounded subgroup of $T^I(K^{\mathrm{nr}})$.

Proof. Recall the sequence (6)

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0$$

and its Cartier dual (7)

$$0 \longrightarrow X(T)^I \longrightarrow X(T) \longrightarrow X(\widetilde{T}) \longrightarrow 0.$$

Taking K^{nr}-valued points in the above sequences, we obtain an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_I\left(X(\widetilde{T}), L^*\right) \longrightarrow \mathrm{Hom}_I(X(T), L^*) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(X(T)^I, (K^{\mathrm{nr}})^*\right).$$

Since $X(T)^I$ is a saturated submodule of $X(T)$, there exists a $\mathbb{Z}$-basis $(\chi_1, \ldots, \chi_n)$ of $X(T)$ such that $(\chi_1, \ldots, \chi_d)$ is a $\mathbb{Z}$-basis of $X(T)^I$. Now a point $x \in T(K^{\mathrm{nr}}) = \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)^I$ belongs to the maximal bounded subgroup of $T(K^{\mathrm{nr}})$ if, and only if, every $x(\chi_i)$ lies in $\mathcal{O}_L^*$. Thus it can be shown that for a point x with $x(\chi_1), \ldots, x(\chi_d) \in \mathcal{O}_L^*$ it necessarily follows that $x(\chi_{d+1}), \ldots, x(\chi_n) \in \mathcal{O}_L^*$.

Since the modulus is compatible with exponentiation, it suffices to show the above for $x' := x^r$, where $r = [L \colon K^{\mathrm{nr}}]$ is the cardinality of I. Now x' corresponds to the map induced by the map $\chi_i \mapsto x_i^r$. Further, $x_1, \ldots, x_d \in (\mathcal{O}_K^{\mathrm{sh}})^*$ since x is an I-morphism and the $\chi_1, \ldots, \chi_d$ are I-invariant.

Consider the map $y \in \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*) = T(L)$ which is induced by the assignment

$$\chi_1 \mapsto x_1, \ldots, \chi_d \mapsto x_d, \chi_{d+1} \mapsto 1, \ldots, \chi_n \mapsto 1.$$

The map $y' := \prod_{\sigma \in I} \sigma \circ y$ is invariant under the action of I, i.e., $y' \in T(K^{\mathrm{nr}})$. For the I-invariant characters $(\chi_i)_{i=1,\ldots,d}$, we have $\sigma \circ y(\chi_i) = \sigma\big(y(\sigma^{-1}(\chi_i))\big) = \sigma(y(\chi_i))$. Consequently, $y'(\chi_i) = N_{L/K^{\mathrm{nr}}}(x_i)$ for $i = 1, \ldots, d$. But since $x_i \in (\mathcal{O}_K^{\mathrm{sh}})^*$, we have $y'(\chi_i) = x_i^r$ for $i = 1, \ldots, d$. The remaining characters $(\chi_j)_{j=d+1,\ldots,n}$ are obviously mapped to products of powers of elements of the form $\sigma(x_i)$ with $\sigma \in I$ and $i = 1, \ldots, d$. Since the x_i are in $(\mathcal{O}_K^{\mathrm{sh}})^*$, these characters must also have their images contained in $(\mathcal{O}_K^{\mathrm{sh}})^*$.

Now the map $T(K^{\mathrm{nr}}) \longrightarrow T^I(K^{\mathrm{nr}})$ is the restriction to $X(T)^I$, so x'/y' lies in the kernel of this map. By left-exactness of sections over K^{nr}, x'/y' must therefore lie in the image of $\widetilde{T}(K^{\mathrm{nr}})$.

Now $\widetilde{T} \otimes_K K^{\mathrm{nr}}$ is an anisotropic torus, whence

$$\widetilde{T}(K^{\mathrm{nr}}) = \mathrm{Hom}_I\left(X(\widetilde{T}), L^*\right) = \mathrm{Hom}_{\mathbb{Z}}\left(X(T)/X(T)^I, L^*\right)^I$$

is bounded [BLR, Theorem 10.2.1]. This means that all maps in $\mathrm{Hom}_{\mathbb{Z}}\left(X(T)/X(T)^I, L^*\right)^I$ are induced by assignments

$$\chi_{d+1} \mapsto x_{d+1}, \ldots, \chi_n \mapsto x_n$$

with values $x_{d+1}, \ldots, x_n \in \mathcal{O}_L^*$.

The map $\widetilde{T}(K^{\mathrm{nr}}) \longrightarrow T(K^{\mathrm{nr}})$ corresponds to the assignment $\chi_1, .., \chi_d \mapsto 1$. For $i = d + 1, \ldots, n$, we conclude that

$$x_i^r = x'(\chi_i) = y'(\chi_i) \cdot (x'(\chi_i)/y'(\chi_i)) \in \mathcal{O}_L^*$$

whereby the x_i are actually in $\mathcal{O}_L^*$. □

We can now describe the ft-Néron model as an étale sheaf:

Proposition 3.3.3. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Then the sheaf $\underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})$ on $(\mathcal{O}_K)_{\text{ét}}$ is represented by the* ft*-Néron model of T:*

$$\underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) = \underline{\mathrm{Hom}}_{(\mathcal{O}_K)_{\text{ét}}}(-, \mathcal{T}^{\mathrm{ft}}).$$

Proof. It suffices to check the statement for connected étale $\mathcal{O}_K$-schemes U. If $U = \operatorname{Spec} K'$ for a finite separable field extension K'/K, then Cartier duality yields a natural isomorphism

$$\underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})(U) = \mathrm{Hom}_{K'}(\underline{X}(T), \mathbb{G}_{m,K'}) \cong T(K').$$

The ft-Néron model $\mathcal{T}^{\mathrm{ft}}$ has as its generic fiber the torus T itself, so the claim holds on the generic fiber.

If $U = \operatorname{Spec} \mathcal{O}_{K'}$ for a finite unramified extension K'/K, then we have

$$\underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})(U) = \mathrm{Hom}_{\mathcal{O}_{K'}}\left(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_{K'}}\right).$$

We will investigate the right-hand side of the above equation using the decomposition theorem [M, II, Theorem 3.10].

A morphism ψ from $j_*\underline{X}(T)$ to $\mathbb{G}_{m,\mathcal{O}_K}$ corresponds to a pair of morphisms

$$\begin{pmatrix} \psi_{\bar{\eta}}\colon X(T) & \longrightarrow & (K^{\mathrm{sep}})^* \\ \psi_{\bar{s}}\colon X(T)^I & \longrightarrow & (\mathcal{O}_{K'}^{\mathrm{sh}})^* \end{pmatrix},$$

where $\psi_{\bar{\eta}}$ is a $G_{K'} = \mathrm{Gal}(K^{\mathrm{sep}}/K')$-morphism and $\psi_{\bar{s}}$ is a $G_{k'} = \mathrm{Gal}(k^{\mathrm{sep}}/k')$ morphism. In addition, the diagram

$$\begin{array}{ccccc} X(T)^I & \xrightarrow{\psi_{\bar{s}}} & (\mathcal{O}_{K'}^{\mathrm{sh}})^* & & \\ \| & & \downarrow & & \\ X(T)^I & \xrightarrow[\psi_{\bar{\eta}}]{} & ((K^{\mathrm{sep}})^*)^I & = & K^{\mathrm{nr}*} \end{array}$$

commutes. But this means that $\underline{\mathrm{Hom}}\left(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_{K'}}\right)(U)$ consists exactly of pairs $\left(\psi_{\bar{\eta}}, \psi_{\bar{\eta}}|_{X(T)^I}\right)$ for which $\psi_{\bar{\eta}}|_{X(T)^I}$ assumes values in $\mathcal{O}_{K'}^{\mathrm{sh}}$. The restriction of a morphism $\psi_{\bar{\eta}}$ must be understood as the image under the morphism

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(X(T), (K^{\mathrm{sep}})^*)^{G_{K'}} \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\left(X(T)^I, (K^{\mathrm{sep}})^*\right)^{G_{K'}} \\ = & \mathrm{Hom}_{\mathbb{Z}}\left(X(T)^I, (K^{\mathrm{nr}})^*\right)^{G_{K'}} \end{aligned}$$

induced by the inclusion $X(T)^I \longrightarrow X(T)$.

Thus the elements of $\underline{\mathrm{Hom}}\left(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_{K'}}\right)(U)$ correspond exactly to the points of $T(K') = \mathrm{Hom}_{\mathbb{Z}}(X(T), (K^{\mathrm{sep}})^*)^{G_{K'}}$ which yield characters on $X(T)^I$ that only assume values in $\mathcal{O}_{K'}^{\mathrm{sh}}$. But these are exactly the points of the maximal bounded subgroup of $T(K')$, whence

$$\underline{\mathrm{Hom}}\left(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_{K'}}\right)(U) = \mathcal{T}^{\mathrm{ft}}(U).$$

Via Cartier duality on the generic fiber, one sees that this identification is natural, i.e., it induces a sheaf isomorphism. □

Chapter 4

Exactness properties of the Néron model

If one examines the Néron model of an algebraic torus T using cohomological methods, as Xavier Xarles does in his work [X], it is important to know under what conditions the formation of the Néron model is an exact functor, i.e., when $\mathrm{R}^1 j_* T = 0$.

We first consider this problem in the étale topology. Since j_* is the identity on the generic fiber, $\mathrm{R}^1 j_* T$ is a skyscraper sheaf, so one can test the triviality of $\mathrm{R}^1 j_* T$ in the stalk at $\bar{s}$. Using a base change theorem and Hilbert's Theorem 90, we see that relative to a splitting extension L/K^{nr} of $T_{K^{\mathrm{nr}}}$ this stalk can be determined as $\mathrm{H}^1(\mathrm{Gal}(L/K^{\mathrm{nr}}), \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*))$. Note that, as a cohomology group with respect to a finite group, $\left(\mathrm{R}^1 j_* T\right)_{\bar{s}}$ is a torsion group.

In the case of a perfect residue field or in the case that L/K^{nr} is tamely ramified, the norm residue group of L/K^{nr} is trivial, so that L^* is a cohomologically trivial $\mathrm{Gal}(L/K^{\mathrm{nr}})$-module. Since $X(T)$ is torsion-free, the same applies to $\mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)$, whence $\mathrm{R}^1 j_* T = 0$. Via a spectral sequence argument, we conclude that the property $\mathrm{R}^1 j_* T = 0$ also holds for Weil restrictions of T.

For arbitrary tori we can decompose the splitting extension L/K^{nr} into a part of order a power of p and a tamely ramified part. We thus see that both the norm residue group of a finite separable extension of K^{nr} and $\left(\mathrm{R}^1 j_* T\right)_{\bar{s}}$ are p-primary torsion groups.

Finally, we show that for a norm-one torus T_N with respect to L/K^{nr} the group $\left(\mathrm{R}^1 j_* T_N\right)_{\bar{s}}$ is equal to the norm residue group of this extension. We use examples to show that norm residue groups are generally infinite and can be non-trivial even when there is no residual ramification.

Next we consider $\mathrm{R}^1 j_* T$ in the smooth topology. This is necessary, since the torsion part of the group of components can only be determined in the smooth topology via Xarles' methods. The existence of a short exact sequence of groups of components in the étale topology also requires the vanishing of a certain smooth sheaf $\mathrm{R}^1 j_* T_1$.

We show that $\mathrm{R}^1 j_*$ vanishes for the multiplicative group and for Weil restrictions of the multiplicative group. In the first case we use the definition of $\mathrm{R}^1 j_* \mathbb{G}_{m,K}$ as a sheafification of certain cohomology groups, which we can consider as Picard groups. We trace the second case back to the first one using a spectral sequence argument. Using the Hochschild-Serre spectral sequence, we can relate $\mathrm{R}^1 j_* T$ for any tori T to $\mathrm{R}^1 j_* \mathbb{G}_{m,K}$ and show that $\mathrm{R}^1 j_* T$ is a $[L\colon K^{\mathrm{nr}}]$-torsion sheaf if L is a splitting extension of $T_{K^{\mathrm{nr}}}$.

To see that $\mathrm{R}^1 j_* T$ is, in fact, a p-primary torsion sheaf, we show that the smooth sheaf $\mathrm{R}^1 j_* T$ restricted to the étale site is equal to the étale sheaf $\mathrm{R}^1 j_* T$, since this restriction is exact and $\mathrm{R}^1 j_*$ can always be written as the cokernel of a homomorphism of Néron models. Since the identity component of a smooth group scheme is an l-divisible sheaf for every $l \in \mathbb{N}$ such that $p \nmid l$, there would have to be a non-p-primary torsion part of $\mathrm{R}^1 j_* T$ that factors through the group of components and can therefore be computed étale-wise. Thus $\mathrm{R}^1 j_* T$ is, as in the case of the étale topology, a p-primary torsion sheaf.

This also implies that $\mathrm{R}^1 j_* T = 0$ if T trivializes over a tamely ramified extension. Further, we see that in the smooth topology we have $\mathrm{Hom}\big(\mathrm{R}^1 j_* T, i_*\mathbb{Z}\big) = \mathrm{Ext}^1\big(\mathrm{R}^1 j_* T, i_*\mathbb{Z}\big[p^{-1}\big]\big) = 0$.

We give an explicit example that $\mathrm{R}^1 j_* T \neq 0$ can happen, even if the residue field is perfect but T only trivializes over a wildly ramified extension. As a supplement, in the last section we give a brief description of the functors j_* and $\mathrm{R}^1 j_*$ applied to étale group schemes.

4.1 $\mathrm{R}^1 j_* T$ as an étale sheaf

Proposition 4.1.1. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Further, let L/K^{nr} be a finite Galois extension such that T_L is split and let $I := \mathrm{Gal}(L/K^{\mathrm{nr}})$. Then, in the étale topology, $R^1 j_* T$ is a skyscraper sheaf and $\big(R^1 j_* T\big)_{\bar{s}} = H^1(I, \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*))$.*

Proof. It is clear that $\mathrm{R}^1 j_* T$ is a skyscraper sheaf. The stalk in the special fiber can be determined using [M, III, Theorem 1.15]. We obtain

$$\big(\mathrm{R}^1 j_* T\big)_{\bar{s}} = \mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}}), T(K^{\mathrm{sep}}))\,.$$

Let L/K^{nr} be a finite Galois extension with $I := \mathrm{Gal}(L/K^{\mathrm{nr}})$ such that T splits over L. Then, by the exactness of the direct limit functor, the canonical inflation-restriction sequence yields an exact sequence

$$0 \longrightarrow \mathrm{H}^1\Big(I, T(K^{\mathrm{sep}})^{G_L}\Big) \longrightarrow \mathrm{H}^1(G_{K^{\mathrm{nr}}}, T(K^{\mathrm{sep}})) \longrightarrow \mathrm{H}^1(G_L, T(K^{\mathrm{sep}}))\,,$$

where $G_L := \mathrm{Gal}(K^{\mathrm{sep}}/L)$ and $G_{K^{\mathrm{nr}}} := \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})$. By Cartier duality, it follows that $T(K^{\mathrm{sep}}) = \mathrm{Hom}_{\mathbb{Z}}(X(T), (K^{\mathrm{sep}})^*)$. Since G_L acts trivially on $X(T)$, we have

$$\mathrm{H}^1(G_L, T(K^{\mathrm{sep}})) = \mathrm{H}^1\left(G_L, (K^{\mathrm{sep}})^*\right)^d = 0$$

by Hilbert's Theorem 90, where d is the rank of $X(T)$. □

We immediately get some interesting corollaries.

Corollary 4.1.2. *In the étale topology, $\big(R^1 j_* T\big)_{\bar{s}}$ is compatible with étale base changes. For products of algebraic tori, we have*

$$\big(R^1 j_* T_1 \times_K T_2\big)_{\bar{s}} = \big(R^1 j_* T_1\big)_{\bar{s}} \oplus \big(R^1 j_* T_2\big)_{\bar{s}}\,.$$

Proof. Since an étale base change does not change the $\mathrm{Gal}\,(K^{\mathrm{sep}}/K^{\mathrm{nr}})$-module structure of the character group, the first assertion is clear. The second assertion follows from the fact that $X\left(T_1 \times_K T_2\right) = X\left(T_1\right) \oplus X\left(T_2\right)$ (in the category of Galois modules) and both $\mathrm{Hom}_{\mathbb{Z}}(\cdot, L^*)$ and $\mathrm{H}^1(I, \cdot)$ are compatible with finite sums. □

Corollary 4.1.3. *In the étale topology, $R^1 j_* T$ is a torsion sheaf. In particular,*

$$\underline{\mathrm{Hom}}\big(R^1 j_* T, i_*\mathbb{Z}\big) = 0.$$

Proof. Since $\left(\mathrm{R}^1 j_* T\right)_{\bar{s}}$ is a cohomology group with coefficients in a finite group I, multiplication by the cardinality of I annihilates every element of $\left(\mathrm{R}^1 j_* T\right)_{\bar{s}}$. Clearly this also applies to every section of the skyscraper sheaf $\mathrm{R}^1 j_* T$. □

By [S, IX, §5, Theorem 9], for a finite group G and two G-modules A and B, $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ is a cohomologically trivial G-module if either A or B is a cohomologically trivial G-module and $\mathrm{Ext}^1_{\mathbb{Z}}(A, B) = 0$. Since the character group of an algebraic torus is torsion-free, $\mathrm{Ext}^1_{\mathbb{Z}}(X(T), L^*) = 0$. This shows that for any algebraic torus T over a local field with a perfect residue field we have $\mathrm{R}^1 j_* T = 0$, because L^* is cohomologically trivial as a $\mathrm{Gal}(L/K^{\mathrm{nr}})$-module for every finite separable extension L/K^{nr}. This follows from the fact that the absolute Brauer group of K^{nr} is trivial. Note that this no longer holds for an arbitrary residue field, but we still get:

Lemma 4.1.4. *Let L/K be a finite Galois and tamely ramified extension of a discretely valued and non-archimedean strictly Henselian field. Then the norm residue group $K^*/N_{L/K}L^*$ is trivial. In particular, L^* is a cohomologically trivial* $\mathrm{Gal}(L/K)$*-module.*

Proof. For the norm map we have an exact and commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_L^* & \longrightarrow & L^* & \xrightarrow{\nu_L} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle N_{L/K}} & & \\ 0 & \longrightarrow & \mathcal{O}_K^* & \longrightarrow & K^* & \xrightarrow{\nu_K} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Because of the (total) tame ramification, $N_{L/K}(\pi_L)$ is a uniformizing element in $\mathcal{O}_K$. This means that the norm map on the value groups is bijective and the snake lemma yields an isomorphism

$$K^*/N_{L/K}L^* \cong \mathcal{O}_K^*/N_{L/K}\mathcal{O}_L^*.$$

Thus it suffices to check that the norm is surjective on integral units. Restricted to elements from $\mathcal{O}_K$, the norm map is the exponentiation by $e := [L\colon K]$. By assumption $p \nmid e$, so that for any $x \in \mathcal{O}_K^*$ the polynomial $X^e - x$ is a primitive and separable polynomial. Modulo π_K, this polynomial splits into linear factors since, according to the assumption, k is separably closed and $\bar{x} \neq 0$. Using Hensel's lemma, $X^e - x$ must then split into linear factors in $\mathcal{O}_K$. This means that there is even a preimage of x in $\mathcal{O}_K$ under the standard map.

Thus for L^* the zero-th (Tate) cohomology group with respect to $\mathrm{Gal}(L/K)$ is trivial, whence L^* is cohomologically trivial by Hilbert's Theorem 90 and [S, IX, §5, Theorem 8]. □

Corollary 4.1.5. *Let L/K be a finite separable and tamely ramified extension of local fields and let T be an algebraic K-torus that splits over L. Then $R^1 j_* T = 0$ in the étale topology.*

Proof. Without loss of generality, we may assume that $K = K^{\mathrm{nr}}$ and L/K is Galois. By [S, IX, §5, Theorem 9] and the argument above, it suffices to check that L^* is a cohomologically trivial $I := \mathrm{Gal}(L/K)$-module. But this is the assertion of the previous lemma. □

The property that $\mathrm{R}^1 j_* T = 0$ is compatible with Weil restrictions relative to finite separable extensions of local fields. We will show this using an idea from the proof of [BX, 4.2]:

Proposition 4.1.6. *Let L/K be a finite separable and tamely ramified extension of local fields and let T' be an algebraic L-torus such that $R^1(j_L)_* T' = 0$ in the étale topology over $\mathcal{O}_L$. Then $R^1 j_* \mathfrak{R}_{L/K}(T') = 0$ in the étale topology over $\mathcal{O}_K$.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} L & \overset{j_L}{\hookrightarrow} & \operatorname{Spec} \mathcal{O}_L \\ {\scriptstyle \rho_K}\downarrow & & \downarrow{\scriptstyle \rho} \\ \operatorname{Spec} K & \overset{j}{\hookrightarrow} & \operatorname{Spec} \mathcal{O}_K \end{array}$$

and regard $\mathfrak{R}_{L/K}(T')$ as the étale sheaf $(\rho_K)_* T'$. By the above diagram, we have $\rho_* \circ (j_L)_* = j_* \circ (\rho_K)_*$ on the respective étale sites. Further, flabby sheaves are acyclic for direct images and the direct image of a flabby sheaf is flabby again [M, III, 1.14 and 1.19]. Thus we may consider Leray spectral sequences [M, III, Theorem 1.18]:

By [M, II, Proposition 3.6], the Weil restriction as a functor h_* for a finite morphism of schemes h is exact in the étale topology. In particular, ρ_* and $(\rho_K)_*$ are exact functors.

So we obtain the following exact sequence of terms of low degree from the Leray spectral sequence for $\rho_* \circ (j_L)_*$:

$$0 \longrightarrow \left(\mathrm{R}^1 \rho_*\right) (j_L)_* T' \longrightarrow \mathrm{R}^1 \left(\rho_* \circ (j_L)_*\right) T' \longrightarrow \rho_* \mathrm{R}^1 (j_L)_* T'.$$

Since ρ_* is exact, the first term of this sequence vanishes. By the hypothesis on T', the third term must also vanish. Consequently

$$\mathrm{R}^1(j_* \circ (\rho_K)_*)\, T' = \mathrm{R}^1(\rho_* \circ (j_L)_*)\, T' = 0.$$

Now, from the Leray spectral sequence for $j_* \circ (\rho_K)_*$ we obtain the exact sequence of low degree terms

$$0 \longrightarrow \left(\mathrm{R}^1 j_*\right)(\rho_K)_* T' \longrightarrow \mathrm{R}^1(j_* \circ (\rho_K)_*)\, T' = 0,$$

which yields the proposition. □

In order to obtain further results, we investigate how extensions of local fields can be decomposed.

Proposition 4.1.7. *Let L/K be a finite Galois extension of local fields. Then, possibly after making a finite separable extension of L, one can find a sequence of fields $K \subseteq K^{\mathrm{nr}} \subseteq K_{\mathrm{ins}} \subseteq K_{(p)} \subseteq L$ such that*

- *all extensions are finite and separable*
- *K^{nr}/K is Galois and unramified.*
- *a uniformizing element in K is also a uniformizing element in K_{ins} and the extension $K_{\mathrm{ins}}/K^{\mathrm{nr}}$ is totally ramified with purely inseparable residue field extension (we say that it is* residually ramified*)*
- *$K_{(p)}/K_{\mathrm{ins}}$ is totally ramified of p-primary degree with trivial residue field extension.*
- *$L/K_{(p)}$ is totally and tamely ramified.*

Moreover, it is necessary to make an extension of L only if the associated extension of residue fields has an inseparable part.

Proof. The existence of the extension $K \subseteq K^{\mathrm{nr}}$ is well-known. By [AS, Appendix, Corollary 2], it follows that one can find a finite extension K_{ins} of K^{nr} such that the corresponding extension of the residue field is purely inseparable and a uniformizing element of L_{nr} is also a uniformizing element in K_{ins}. However, K_{ins} is a priori only a subfield of a finite separable extension L' of L, but the extension $K_{\mathrm{ins}} \subseteq L'$ does not ramify residually.

Now we replace L with the normal closure of L'. The extension $K_{\mathrm{ins}} \subseteq L$ induces a (possibly trivial) separable extension of the residue fields. By correspondingly increasing K^{nr} and K_{ins}, $K_{\mathrm{ins}} \subseteq L$ is then totally ramified and Galois with a trivial extension of the residue field.

By [S, IV, §2, Corollary 4], there exists in $G := \mathrm{Gal}(L/K_{\mathrm{ins}})$ a cyclic subgroup Z of order prime to $p = \mathrm{char}(k)$ and a normal subgroup N of p-power order such that $G = N \rtimes Z$. But this means that there is an intermediate field $K_{\mathrm{ins}} \subset L^Z \subset L$ such that $\mathrm{Gal}\big(L/L^Z\big)$ is equal to Z, i.e., it is totally and tamely ramified, and $K_{\mathrm{ins}} \subset L^Z$ is totally and wildly ramified of p-primary order, but in general without inseparable part. If we set $K_{(p)} := L^Z$, the claim follows. □

Corollary 4.1.8. *Let K be a strictly Henselian local field and let L be a finite Galois extension. Then the norm residue group $K^*/N_{L/K}L^*$ is a p-group.*

Proof. Without loss of generality, we can enlarge L so that there is a decomposition of L/K as in proposition 4.1.7, because such an extension at worst increases the norm residue group.

The norm residue group of the tamely ramified extension $K_{(p)} \subset L$ is trivial and therefore $K^*/N_{L/K}(L^*) \subset K^*/N_{K_{(p)}/K}(K^*_{(p)})$. Since $K = K^{\mathrm{nr}}$, this is an extension of p-primary order and the norm residue group must be a p-group. □

Proposition 4.1.9. *Let K be a local field and let T be an algebraic K-torus. Let $p = \mathrm{char}(k)$. Then, in the étale topology over $\mathcal{O}_K$, $R^1 j_* T$ is a p-primary torsion sheaf. More precisely, there exists a power p^r, with $r \in \mathbb{N}$, such that the multiplication by p^r on $R^1 j_* T$ is the zero map.*

Proof. Without loss of generality, we may assume that $K = K^{\mathrm{nr}}$. Let L/K be a finite Galois splitting extension of T. Using proposition 4.1.7, let $K_{(p)}$ be an intermediate field such that $K_{(p)}/K$ is an extension of p-primary degree and $L/K_{(p)}$ is tamely ramified. We set $G := \mathrm{Gal}(L/K), H := \mathrm{Gal}\big(L/K_{(p)}\big)$ and $J := \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)$.

Now $\big(\mathrm{R}^1 j_* T\big)_{\bar{s}} = \mathrm{H}^1(G, J)$ and $\mathrm{H}^1(H, J) = 0$ because L^* is a cohomologically trivial H-module. Further, the composition

$$\mathrm{H}^1(G, J) \xrightarrow{\mathrm{res}} \mathrm{H}^1(H, J) \xrightarrow{\mathrm{cor}} \mathrm{H}^1(G, J)$$

is multiplication by $(G \colon H) = p^r := \big[K_{(p)} \colon K\big]$ (see [S, VII,§7 Proposition 6]). But since $\mathrm{H}^1(H, J) = 0$, this map must be the zero map. □

Corollary 4.1.10. *Let K be a local field and let T be an algebraic K-torus. Let L/K^{nr} be a finite Galois splitting extension of $T_{K^{\mathrm{nr}}}$. Let $I := \mathrm{Gal}(L/K^{\mathrm{nr}})$ and let I_p be the p-Sylow subgroup of I. Then $R^1 j_* T_{\bar{s}} = H^1\left(I_p, \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)\right)^{I/I_p}$.*

Proof. Note that I_p is a normal subgroup of I (cf. [S, IV, §2, Corollary 4]) and the quotient $H := I/I_p$ has order prime to p. Let $J := \mathrm{Hom}_{\mathbb{Z}}(X(T), L^*)$. There exists an exact sequence (see [S, VII, §6, note at the end])

$$0 \longrightarrow \mathrm{H}^1\big(H, J^{I_p}\big) \xrightarrow{\alpha} \mathrm{H}^1(I, J) \xrightarrow{\beta} \mathrm{H}^1\left(I_p, J\right)^H \xrightarrow{\gamma} \mathrm{H}^2\big(H, J^{I_p}\big)\,.$$

Now $\mathrm{H}^1\big(H, J^{I_p}\big)$ and $\mathrm{H}^2\big(H, J^{I_p}\big)$ are torsion groups which are annihilated by the order h of H. Conversely, $\big(\mathrm{R}^1 j_* T\big)_{\bar{s}} = \mathrm{H}^1(I, J)$ and $\mathrm{H}^1\left(I_p, J\right)^H$ are torsion groups that are annihilated by a power of p. Because p and h are coprime, the morphisms α and γ must be the zero map, i.e., we have an isomorphism $\mathrm{H}^1(I, J) \cong \mathrm{H}^1\left(I_p, J\right)^{I/I_p}$. □

We now consider $\mathrm{R}^1 j_* T$ using norm-one tori as an example.

Proposition 4.1.11. *Let L/K be a finite Galois extension of local fields and let T be a K-torus that splits over L. Then the canonical short exact sequence*

$$0 \longrightarrow T \longrightarrow \mathfrak{R}_{L/K}(T_L) \longrightarrow T' \longrightarrow 0$$

induces an isomorphism

$$\big(R^1 j_* T\big)_{\bar{s}} \cong \operatorname{coker}\big(\mathfrak{R}_{L/K}(T_L)(K^{\mathrm{nr}}) \longrightarrow T'(K^{\mathrm{nr}})\big)\,.$$

Proof. Since T_L is split, $\mathrm{R}^1 j_* \mathfrak{R}_{L/K}(T_L) = 0$. This means that the long exact sequence of the Néron models induced by the sequence of the proposition yields an isomorphism

$$\mathrm{R}^1 j_* T \cong \operatorname{coker}\big(j_* \mathfrak{R}_{L/K}(T_L) \longrightarrow j_* T'\big)\,.$$

Since the taking of stalks is an exact functor, by the Néron mapping property we have $(j_* T')_{\bar{s}} = j_* T'\big(\mathcal{O}_K^{\mathrm{sh}}\big) = T'(K^{\mathrm{nr}})$ and similarly $\big(j_* \mathfrak{R}_{L/K}(T_L)\big)_{\bar{s}} = \mathfrak{R}_{L/K}(T_L)(K^{\mathrm{nr}})$. The proposition is now clear. ☐

From the above proposition one immediately sees that, if T_N is the norm-one torus associated to a finite Galois extension L/K, then $\big(\mathrm{R}^1 j_* T_N\big)_{\bar{s}}$ is equal to the norm residue group of this extension. Thus any extension of local fields with a nontrivial inseparable residual extension yields an example of a torus with nontrivial $\mathrm{R}^1 j_* T$. If L/K is totally ramified, then we let $e_{L/K}$ be the ramification index, that is, the uniquely determined natural number $e_{L/K}$ such that $\pi_L^{e_{L/K}} \equiv \pi_K \bmod \pi_L$, and $\delta := [L\colon K]/e_{L/K}$.

Now, considered as abelian groups,

$$K^*/N_{L/K} L^* = \mathcal{O}_K^*/N_{L/K}\mathcal{O}_L^* \oplus \mathbb{Z}/\delta\mathbb{Z},$$

since the units of a local field are the direct sum of the units of the associated valuation ring and the section $\pi^{\mathbb{Z}}$ of the value group.

Using norm-one tori, we can also find examples where $\mathrm{R}^1 j_* T_N \neq 0$ in which no inseparable residual extension occurs.

Lemma 4.1.12. *Let K be a strictly Henselian local field and let L/K be a finite separable extension which induces a trivial extension of the residue fields. Let k be the residue field and let p^r be the highest power of p that divides the degree $[L\colon K]$. Then there exists a surjection*

$$K^*/N_{L/K} L^* \twoheadrightarrow k^*/(k^{p^r})^*.$$

Proof. Since, by hypothesis, the norm map on the value groups is surjective, the residual norm group must be equal to the residual norm group of the extension of the valuation rings. Let U^1 denote the group of 1-units. Then, by the snake lemma, the lemma follows from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_L^1 & \longrightarrow & \mathcal{O}_L^* & \longrightarrow & k^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle N_0} & & \\
0 & \longrightarrow & U_K^1 & \longrightarrow & \mathcal{O}_K^* & \longrightarrow & k^* & \longrightarrow & 0,
\end{array}$$

taking into account the fact that the norm map on the residue fields induces the exponentiation by $[L\colon K]$. Since k is separably closed, the cokernel of the exponentiation by $[L\colon K]$ is equal to the cokernel of the exponentiation by p^r. ☐

If k is not a perfect field, then $k \supset k^{p^r}$ is a non-trivial field extension, so the quotient $k^*/(k^{p^r})^*$ is also non-trivial. This means that the norm residue group is an infinite group since the preimage of $k^* - (k^{p^r})^*$ under the canonical projection $\mathcal{O}_K \longrightarrow k$ cannot lie in $N_{L/K}(L^*)$. Indeed, by [S, II, §4, Proposition 5], one can write all elements of $\mathcal{O}_K$ uniquely as power series $\sum_{i=0}^{\infty} s_i \pi_k^i$, where the s_i come from a representative system for k in $\mathcal{O}_K$. This means that all power series for which, after reduction, s_0 lies in $k^* - (k^{p^r})^*$ cannot lie in the image of the norm map.

Using such norm-one tori as examples, it is easy to see that $\mathrm{R}^1 j_* T$ is generally not a constant sheaf.

For let T_N be a norm-one torus with respect to a totally ramified extension L/K of degree $p = 2 = \mathrm{char}(k)$ with $e_{L/K} = [L\colon K]$. We assume that the residue field k is neither perfect nor separably closed. More precisely, there exists an element $\overline{z} \in k$ such that $\sqrt{\overline{z}} \notin k^{\mathrm{sep}}$ and its irreducible and separable polynomial $Y^2 + \bar{a}Y + \bar{X}$ is in $k[Y]$. Let z be a preimage of $\overline{z}$ in K^{nr}. Then $z \notin N_{L/K}(L^{\mathrm{nr}})^*$. Now let $\tilde{z}$ be a root of the polynomial $Y^2 + aY + z$, where $a \in \mathcal{O}_K^*$ is a preimage of $\bar{a}$. This polynomial defines an unramified extension, i.e., $\tilde{z} \in K^{\mathrm{nr}}$. If σ is an element of $\mathrm{Gal}(K^{\mathrm{nr}}/K)$ which does not leave $\tilde{z}$ fixed, then $\sigma(\tilde{z})/\tilde{z} = z/\tilde{z}^2$.

Since on K^{nr} the norm map to L/K is a square, it follows that $\sigma(\tilde{z})/\tilde{z} \notin N_{L/K}(L^{\mathrm{nr}})^*$, that is, the images of $\tilde{z}$ and $\sigma(\tilde{z})$ in the norm residue group are not equal.

However, we will see later that only the group of components of $\mathrm{R}^1 j_* T$ is really relevant for the structure of the Néron model. For a norm-one torus, this corresponds to the Galois group acting trivially on the quotient of the value groups.

4.2 $\mathrm{R}^1 j_* T$ as a smooth sheaf

First we prove a well-known result (see, e.g., [X, 2.14]). We give a proof because we will need some of the considerations from this proof in later proofs.

Proposition 4.2.1. *Let K be a local field. Then $R^1 j_* \mathbb{G}_{m,K} = 0$ in the smooth topology over* $\mathrm{Spec}\, \mathcal{O}_K$.

Proof. We show that $\mathrm{R}^1 j_* \mathbb{G}_{m,K}(Y) = 0$ for all smooth $\mathcal{O}_K$-schemes Y. Since $\mathrm{R}^1 j_* \mathbb{G}_{m,K}$ is the smooth sheaf associated to the presheaf $V \mapsto \mathrm{Pic}(V_K)$, it suffices to show that the étale sheaf associated to $V \mapsto \mathrm{Pic}(V_K)$ vanishes on the étale site over Y. This in turn holds if this sheaf vanishes Zariski-locally. So we will show that, if $Y \longrightarrow \mathrm{Spec}\, \mathcal{O}_K$ is a smooth morphism , $y \in Y$ is a point and $Y' := \mathrm{Spec}\, \mathcal{O}_{Y,y}$, then $\mathrm{Pic}(Y'_K) = 0$. To do this, we will show that the affine ring of Y'_K, i.e. $\mathcal{O}_{Y,y}\left[\pi^{-1}\right]$, is integral and factorial. The former property implies that $\mathrm{Pic}(Y'_K)$ is equal to the divisor class group of Y'_K. The latter group is then trivial because $\mathcal{O}_{Y,y}\left[\pi^{-1}\right]$ is factorial.

Since $\mathcal{O}_K$ is regular as a discrete valuation ring, Y' is regular again as a local ring scheme of a smooth $\mathcal{O}_K$-scheme. A regular local ring is integral and factorial, so $\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_K} K = \mathcal{O}_{Y,y}\left[\pi^{-1}\right]$ is an integral ring. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{Y,y}$. If π is not in $\mathfrak{m}$, then $\mathcal{O}_{Y,y}\left[\pi^{-1}\right] = \mathcal{O}_{Y,y}$ and the claim is clear. So let $\pi \in \mathfrak{m}$. Since the scheme Y is smooth, π does not lie in $\mathfrak{m}^2$. This means that $\mathcal{O}_{Y,y}\left[\pi^{-1}\right]$ is factorial (cf. [BIV, proof of proposition 14.33]), which means that $\mathrm{Pic}(Y'_K) = 0$. □

Proposition 4.2.2. *Let K be a local field and let L be a finite separable extension of K. Then $R^1 j_* \mathfrak{R}_{L/K}(\mathbb{G}_{m,L}) = 0$ in the smooth topology.*

Proof. The proof of proposition 4.1.6 can be copied verbatim with $T_L := \mathbb{G}_{m,L}$ once we prove that the Weil restriction with respect to a finite morphism of schemes is an exact functor in the smooth topology.

So let $h\colon X' \longrightarrow X$ be a finite morphism of schemes. Let us denote by f_*, respectively f'_*, the restriction from the smooth site $(\mathrm{sm})/X$ (or $(\mathrm{sm})/X'$) to the étale site $(\text{ét})/X$ (resp. $(\text{ét})/X'$). Then we have a commutative diagram of morphisms of sites

$$\begin{array}{ccc} (\mathrm{sm})/X' & \xrightarrow{h_*^{\mathrm{sm}}} & (\mathrm{sm})/X \\ {\scriptstyle f'_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ (\text{ét})/X' & \xrightarrow{h_*^{\text{ét}}} & (\text{ét})/X. \end{array}$$

Let $\mathcal{F}$ be a smooth sheaf over X'. By the Leray spectral sequence [M, III, Theorem 1.18] and the exactness of f_* [M, III, Proposition 3.3], the following sequence of low-degree terms is exact

$$0 \longrightarrow \mathrm{R}^1 h_*^{\text{ét}}(f'_*\mathcal{F}) \longrightarrow \mathrm{R}^1\left(h_*^{\text{ét}} \circ f'_*\right)\mathcal{F} \longrightarrow h_*^{\text{ét}}\mathrm{R}^1 f'_*\mathcal{F}.$$

Since $h_*^{\text{ét}}$ is an exact functor on the étale sites, we have

$$\mathrm{R}^1(f_* \circ h_*^{\mathrm{sm}})\,\mathcal{F} = \mathrm{R}^1\left(h_*^{\text{ét}} \circ f'_*\right)\mathcal{F} = 0.$$

This gives the other sequence of terms of low degree:

$$0 = \mathrm{R}^1(f_* \circ h_*^{\mathrm{sm}})\,\mathcal{F} \longrightarrow f_*\mathrm{R}^1 h_*^{\mathrm{sm}}\mathcal{F} \longrightarrow \mathrm{R}^2 f_*(h_*^{\mathrm{sm}}\mathcal{F}) = 0.$$

Thus the restriction of $\mathrm{R}^1 h_*^{\mathrm{sm}}$ to the étale site over X vanishes. Now let $\pi\colon U \longrightarrow X$ be a smooth morphism. Then the restriction $\pi^*\colon (\mathrm{sm})/X \longrightarrow (\mathrm{sm})/U$ is an exact functor that maps flabby sheaves to flabby sheaves. Similarly, we can also consider the restriction to the morphism $\pi'\colon U' := U \times_X X' \longrightarrow X'$. Then $(\pi')^*$ is also exact and maps flabby sheaves to flabby sheaves.

If we look at the diagram similar to the above

$$\begin{array}{ccc} (\mathrm{sm})/X' & \xrightarrow{(\pi')^*} & (\mathrm{sm})/U' \\ {\scriptstyle h_*}\downarrow & & {\scriptstyle (h_U)_*}\downarrow \\ (\mathrm{sm})/X & \xrightarrow{\pi^*} & (\mathrm{sm})/U, \end{array}$$

we get $\pi^*\mathrm{R}^1 h_*\mathcal{F} = \left(\mathrm{R}^1(h_U)_*\right)(\pi')^*\,\mathcal{F}$.

This means that $\mathrm{R}^1 h_*\mathcal{F}(U) = 0$ by replacing $h\colon X' \longrightarrow X$ with a morphism $h_U\colon U' \longrightarrow U$. Thus $\mathrm{R}^1 h_*\mathcal{F} = 0$. □

This result gives us the opportunity to compare the smooth sheaf $\mathrm{R}^1 j_*T$ and the étale sheaf $\mathrm{R}^1 j_*T$.

Proposition 4.2.3. *Let $f\colon (\mathrm{sm})/\mathcal{O}_K \longrightarrow (\text{ét})/\mathcal{O}_K$ be the restriction from the smooth to the étale site. Then for every algebraic K-torus T we have*

$$f_* R^1_{\mathrm{sm}} j_*T = R^1_{\text{ét}} j_*T.$$

Proof. By [M, III, 3.3], f_* is an exact functor. Let L/K be a splitting extension for T and consider the exact sequence of smooth sheaves

$$0 \longrightarrow T \longrightarrow \mathfrak{R}_{L/K}(T_L) \longrightarrow T' \longrightarrow 0.$$

This induces a long exact sequence

$$0 \longrightarrow j_*T \longrightarrow j_*\mathfrak{R}_{L/K}(T_L) \longrightarrow j_*T' \longrightarrow \mathrm{R}^1 j_*T \longrightarrow 0.$$

Applying f_* yields an exact sequence of étale sheaves

$$0 \longrightarrow f_* j_* T \longrightarrow f_* j_* \mathfrak{R}_{L/K}(T_L) \longrightarrow f_* j_* T' \longrightarrow f_* \mathrm{R}^1 j_* T \longrightarrow 0.$$

Since the Néron models are representable sheaves, this sequence is isomorphic to the sequence

$$0 \longrightarrow j_* T \longrightarrow j_* \mathfrak{R}_{L/K}(T_L) \longrightarrow j_* T' \longrightarrow f_* \mathrm{R}^1 j_* T \longrightarrow 0.$$

Now the (étale) cokernel of the map $j_* \mathfrak{R}_{L/K}(T_L) \longrightarrow j_* T'$ is equal to the étale $\mathrm{R}^1 j_* T$. □

Thus, the étale sheaf $\mathrm{R}^1 j_* T$ is trivial if the smooth $\mathrm{R}^1 j_* T$ is trivial. As the example at the end of this section shows, the étale sheaf $\mathrm{R}^1 j_* T$ can be trivial without the smooth $\mathrm{R}^1 j_* T$ being trivial.

Proposition 4.2.4. *Let K be a local field and let T be an algebraic K-torus. Let L/K be a finite Galois extension splitting T and let $e := [L\colon K^{\mathrm{nr}}]$ be the ramification index of L/K. Then $R^1 j_* T$ is an e-torsion sheaf.*

Proof. Since $\mathrm{R}^1 j_* T$ is a sheaf and $\operatorname{Spec} \mathcal{O}_{K^{\mathrm{nr}}} \longrightarrow \mathcal{O}_K$ is an étale cover, it suffices to check the e-torsion property on the smooth site over $\mathcal{O}_{K^{\mathrm{nr}}}$. Further, since the formation of the Néron model is compatible with étale base changes, we may assume that L/K is totally ramified. Set $G := \mathrm{Gal}(L/K)$ and let $U \longrightarrow \operatorname{Spec} \mathcal{O}_K$ be a smooth morphism. Then the morphism

$$\rho\colon U_L := U \otimes_{\mathcal{O}_K} \mathcal{O}_L \otimes_{\mathcal{O}_L} L = U \otimes_{\mathcal{O}_K} K \otimes_K L \longrightarrow U \otimes_{\mathcal{O}_K} K =: U_K$$

is Galois, since L/K is Galois.

Thus, the Hochschild-Serre spectral sequence [M, III, Theorem 2.20] yields an exact sequence

$$0 \longrightarrow \mathrm{H}^1\big(G, \mathrm{H}^0(U_L, T)\big) \longrightarrow \mathrm{H}^1(U_K, T) \longrightarrow \mathrm{H}^1\,(U_L, T)^G.$$

Now $\mathrm{R}^1 j_* T$ is the sheaf associated to the smooth presheaf $U \mapsto \mathrm{H}^1(U_K, T)$. For a smooth morphism $V \longrightarrow \operatorname{Spec} \mathcal{O}_K$ and a section $s \in \mathrm{R}^1 j_* T(V)$, we can find a smooth covering $(U_i)_{i \in I}$ of V such that s is the glueing of sections $a(s_i)$, where a is the sheafification functor and $s_i \in \mathrm{H}^1((U_i)_K, T)$. The section s is an e-torsion element if all s_i are e-torsion elements. So consider for U_i the exact sequence induced by the Hochschild-Serre spectral sequence. Since the group order e annihilates all elements from $\mathrm{H}^1(G, T((U_i)_L))$, it suffices to check that the s_i in $\mathrm{H}^1((U_i)_L, T)$ are zero.

Now T splits over L and $\mathrm{H}^1((U_i)_L, T) = \operatorname{Pic}((U_i)_L)^d$, where d is the dimension of T. Let $y \in U_i$ be a point and let $\mathcal{O}_{U_i,y}$ be its local ring. Then $\mathcal{O}_{U_i,y} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is a finite ring extension. Since finite algebras split into a product of local rings over a Henselian local ring, we see that, after étale extension of U_i, the ring $\mathcal{O}_{U_i,y} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is a product of local rings. Since Pic commutes with finite products, we may assume that $\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is a regular local ring . As we saw in the proof of the equality $\mathrm{R}^1 j_* \mathbb{G}_{m,K} = 0$, we conclude that $\operatorname{Pic}\big(\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_K} \mathcal{O}_L\big[\pi^{-1}\big]\big) = 0$. Summarizing, after a suitable refinement, we can assume that $s_i = 0$ in $\mathrm{H}^1((U_i)_L, T)$, whence the claim follows. □

On the other hand, the smooth sheaf $\mathrm{R}^1 j_* T$ is also a p-primary torsion sheaf:

Proposition 4.2.5. *Let K be a local field and let T be an algebraic K-torus. Then the smooth sheaf $R^1 j_* T$ is a p-primary torsion sheaf.*

Proof. Let L/K be a finite splitting extension. We consider the exact sequence of smooth sheaves for T

$$0 \longrightarrow j_*T \longrightarrow j_*\mathfrak{R}_{L/K}(T_L) \longrightarrow j_*T' \longrightarrow \mathrm{R}^1 j_*T \longrightarrow 0.$$

Since $\mathrm{R}^1 j_*T$ is an abelian sheaf, this sheaf can be clearly decomposed into a sum

$$\mathrm{R}^1 j_*T = R_p \oplus R'$$

of a p-primary torsion part R_p and a prime-to-p-primary torsion part R'. We embed the Néron model j_*T' into the exact sequence

$$0 \longrightarrow j_*(T')^0 \longrightarrow j_*T' \longrightarrow i_*\Phi' \longrightarrow 0$$

and show that the morphism

$$\delta\colon j_*T' \longrightarrow \mathrm{R}^1 j_*T \longrightarrow R'$$

factors through $i_*\phi'$ or, equivalently, that $j_*(T')^0 \longrightarrow R'$ is the zero map:

By proposition 4.2.4, there exists an $l \in \mathbb{N}$ relatively prime to p such that multiplication by l is the zero map on R'. Since j_*T^0 is a smooth connected abelian group scheme, multiplication by l is étale and surjective.

Thus let $U \longrightarrow \operatorname{Spec} \mathcal{O}_K$ be a smooth morphism and let $s \in \operatorname{Hom}_{\mathcal{O}_K}(U, j_*(T')^0)$ be a section. By assumption, multiplication by l is an (étale) cover $j_*(T')^0 \longrightarrow j_*(T')^0$, so we have a Cartesian diagram

$$\begin{array}{ccc} U' := U \times_{j_*(T')^0} j_*(T')^0 & \xrightarrow{s_l} & j_*(T')^0 \\ {\scriptstyle p_U}\big\downarrow & & \big\downarrow{\scriptstyle \cdot l} \\ U & \xrightarrow[s]{} & j_*(T')^0, \end{array}$$

where $p_U : U' \longrightarrow U$ is an étale cover. By definition, we have

$$\mathrm{res}_{U',U}(s) = s \circ p_U = l \cdot s_l,$$

that is, the restriction of s to U' is l times the sum of the section $s_l \in \operatorname{Hom}_{\mathcal{O}_K}\big(U', j_*(T')^0\big)$. Thus we obtain a commutative diagram

$$\begin{array}{ccccc} j_*(T')^0(U') & & \xrightarrow{\quad\delta\quad} & & R'(U') \\ \uparrow & l \cdot s_l & \longmapsto & l \cdot \delta(s_l) = 0 & \uparrow \\ \scriptstyle\mathrm{res} & \uparrow & & \uparrow & \scriptstyle\mathrm{res} \\ & s & \longmapsto & \delta(s) & \\ j_*(T')^0(U) & & \xrightarrow{\quad\delta\quad} & & R'(U). \end{array}$$

Since $U' \longrightarrow U$ is a cover, the associated restriction is injective, whence $\delta(s) = 0$.

To establish the assertion of the proposition, it is now sufficient to show that $R'(U) = 0$ for any smooth morphism $U \longrightarrow \operatorname{Spec} \mathcal{O}_K$.

So let a section $s \in R'(U)$ be given. By surjectivity, there exists a smooth cover $(U_i)_{i \in I}$ of U and sections $s_i \in j_*T'(U_i)$ such that $\delta(s_i) = s|_{U_i}$.

As shown above, $\overline{s_i} \in i_*\Phi'(U_i) = \mathrm{Hom}_{\mathcal{O}_K}(U_i, i_*\Phi')$ are the preimages of the $s|_{U_i}$ under the map $i_*\Phi' \longrightarrow R'$. Now $\overline{s_i}\colon U_i \longrightarrow i_*\Phi'$ is a morphism in the category of smooth $\mathcal{O}_K$-schemes and for the associated restriction map we obtain

$$\mathrm{res}_{\overline{s_i}}\colon \mathrm{Hom}_{\mathcal{O}_K}(i_*\Phi', i_*\Phi') \longrightarrow \mathrm{Hom}_{\mathcal{O}_K}(U_i, i_*\Phi') \quad f \mapsto f \circ \overline{s_i}.$$

This means that for the corresponding restriction map on $\mathrm{R}^1 j_*T$ we have $\mathrm{res}_{\overline{s_i}}(\delta(\mathrm{Id}_\Phi)) = \delta(s_i)$. But since $i_*\Phi'$ is an étale $\mathcal{O}_K$-scheme and the restriction of the smooth sheaf $\mathrm{R}^1 j_*T$ is equal to the étale sheaf $\mathrm{R}^1 j_*T$, $\delta(\mathrm{Id}_\Phi)$ is a p-primary torsion element. Thus $R'(U) = 0$. ☐

We immediately obtain the following corollaries:

Corollary 4.2.6. *Let K be a local field and let T be an algebraic K-torus which splits over a tamely ramified extension. Then $R^1 j_*T = 0$ in the smooth topology.*

Proof. This is clear from the previous two propositions since an abelian group that is annihilated by two coprime numbers is trivial. ☐

Corollary 4.2.7. *Let K be a local field, let T be a K-torus and let $\mathcal{K}$ be a subsheaf of the smooth sheaf $R^1 j_*T$. Then, in the smooth topology over $\mathcal{O}_K$, for an appropriate $r \in \mathbb{N}$ and for every $i \in \mathbb{N}$, we have*

$$\underline{\mathrm{Hom}}(\mathcal{K}, i_*\mathbb{Z}) = 0$$
$$p^r \cdot \underline{\mathrm{Ext}}^i(\mathcal{K}, i_*\mathbb{Z}) = 0$$
$$\underline{\mathrm{Ext}}^i\big(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]\big) = 0.$$

Proof. The first statement is clear since $\mathrm{R}^1 j_*T$ is a torsion sheaf. If r is equal to the p-valuation of the degree of L/K^{nr} for a splitting extension L of T, then multiplication by p^r is the zero map on $\mathrm{R}^1 j_*T$ and thus also on $\mathcal{K}$. Now $\underline{\mathrm{Ext}}^i(\mathcal{K}, i_*\mathbb{Z})$ is a quotient of a subsheaf of $\underline{\mathrm{Hom}}(\mathcal{K}, \mathcal{I})$ for a suitable injective sheaf $\mathcal{I}$ and multiplication by p^r on $\underline{\mathrm{Hom}}(\mathcal{K}, \mathcal{I})$ induces multiplication with p^r on $\underline{\mathrm{Ext}}^i(\mathcal{K}, i_*\mathbb{Z})$. Let U be a smooth $\mathcal{O}_K$-scheme. Then $\underline{\mathrm{Hom}}(\mathcal{K}, \mathcal{I})(U) = \underline{\mathrm{Hom}}(\mathcal{K}|_U, \mathcal{I}|_U)$ and an element f in the latter group is a family $f_V\colon \mathcal{K}(V) \longrightarrow \mathcal{I}(V)$ of homomorphisms of abelian groups for all smooth U-schemes V. For an $x \in \mathcal{K}(V)$ we have $(p^r \cdot f)(X) = p^r f(X) = f(p^r x) = f(0) = 0$.

For the third equation, note that multiplication by p^r induces an isomorphism $\mathbb{Z}[p^{-1}] \longrightarrow \mathbb{Z}[p^{-1}]$. This, in turn, induces an isomorphism $\underline{\mathrm{Ext}}^i\big(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]\big) \longrightarrow \mathrm{Ext}^i\big(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]\big)$. Since one can compute $\mathrm{Ext}^i\big(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]\big)$ using an injective resolution and morphisms of such resolutions are unique up to homotopy, we may assume without loss of generality that multiplication by p^r on the chosen injective resolution induces multiplication by p^r, which also applies to $\underline{\mathrm{Ext}}^i\big(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]\big)$. Since the argument for the second equation also applies to $\mathrm{Ext}^i\big(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]\big)$, the multiplication by p^r must be the zero map. ☐

Using an example, we will show that $\mathrm{R}^1 j_*T$ can be non-trivial even if the residue field is perfect. Let K be a strictly Henselian local field with perfect residue field k. Let L/K be a finite Galois extension and let T_N be the norm-one torus associated with this extension. Consider the corresponding long exact sequence in the smooth topology

$$0 \longrightarrow j_*T_N \longrightarrow j_*\mathfrak{R}_{L/K}(\mathbb{G}_{m,L}) \longrightarrow j_*\mathbb{G}_{m,K} \longrightarrow \mathrm{R}^1 j_*T_N.$$

Let $U := \mathbb{A}^1_{\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[T]$ be the affine line over $\mathcal{O}_K$. Then the restriction of the above sequence to a sequence in the étale topology over U is exact (cf. [M, III, Theorem 3.3]). Consider the stalks of this sequence at a geometric point over the generic point η_k of the special fiber U_k.

The generic point corresponds to the ring $\mathcal{O}_K[T]_{(\pi)}$, so it is an extension of $\mathcal{O}_K$ of ramification index one, but this ring is no longer complete. The residue field is $k(T)$, so it is no longer perfect.

This yields for $T = \mathbb{G}_{m,K}$, respectively $T = \mathfrak{R}_{L/K}(\mathbb{G}_{m,L})$, the following:

$$(j_*T)|_{(\text{ét})/U}\left(\mathcal{O}^{\text{sh}}_{U,\eta}\right) = (j_*T \times_{\mathcal{O}_K} \mathcal{O}_{U,\eta})\left(\mathcal{O}^{\text{sh}}_{U,\eta}\right).$$

Since Néron models are compatible with base changes of ramification index one, the following holds:

$$(j_*T) \times_{\mathcal{O}_K} \mathcal{O}_{U,\eta}\left(\mathcal{O}^{\text{sh}}_{U,\eta}\right) = j_*(T \times_{\mathcal{O}_K} \mathcal{O}_{U,\eta})\left(\mathcal{O}^{\text{sh}}_{U,\eta}\right).$$

For $T = \mathbb{G}_{m,K}$, this is equal to $(K(T)^{\text{nr}})^*$; for $T = \mathfrak{R}_{L/K}(\mathbb{G}_{m,L})$, this is equal to $(L(T)^{\text{nr}})^*$. So we obtain the exact sequence on the stalks

$$\cdots \longrightarrow (L(T)^{\text{nr}})^* \xrightarrow{N_{L/K}} (K(T)^{\text{nr}})^* \longrightarrow \left(\mathrm{R}^1 j_* T_N|_{(\text{ét})/U}\right)_{\bar{\eta}} \longrightarrow 0.$$

But if $[L\colon K]$ and $p := \text{char}(k)$ are not coprime, the last example from the previous section shows that the norm residue group may no longer be trivial. So $\mathrm{R}^1 j_* T_N$ as a smooth sheaf may not be trivial.

4.3 j_* and $\mathrm{R}^1 j_*$ for étale groups

The Néron model of an étale K-group F is an étale $\mathcal{O}_K$-group. This suffices to determine the Néron model in the étale topology. The étale sheaf represented by F corresponds to the continuous $\text{Gal}(k^{\text{sep}}/k)$-module $M_F := F(K^{\text{sep}})$.

By the decomposition theorem, j_*F is equal to the triple $\left(M_F, M_F^I, M_F^I \longrightarrow M_F^I\right)$ consisting of the module itself, its I-invariants and the identity on the invariants.

This can be understood as follows: as a scheme, F is a disjoint union of schemes $U_i := \operatorname{Spec} K_i$, where the K_i are finite separable extensions of K. Now the formation of the Néron model (as a scheme) is compatible with disjoint unions. A U_i as above for which K_i/K is an unramified extension has the Néron model $j_*U_i = \operatorname{Spec} \mathcal{O}_{K_i}$. Whereas a U_i as above that comes from a non-unramified extension has no K^{nr}-valued points, so it is its own Néron model.

By the decomposition theorem, it is clear that the Néron model of a constant group is again the same constant group (but regarded over $\mathcal{O}_K$). For the groups of roots of unity, we have $j_*\mu_{q,K} = \mu_{q,\mathcal{O}_K}$ if q is relatively prime to $p = \text{char}(k)$.

If q is relatively prime to the characteristic of K but is not prime to p, one must take into account the absolute ramification index of K. If this is one, then the p-th roots of unity are not in $\mathcal{O}_K^{\text{sh}}$ and then, for example, $j_*\mu_{p,K}$ is equal to $\mu_{p,K}$ glued to $\operatorname{Spec} \mathcal{O}_K$ along the identity section $\operatorname{Spec} K \longrightarrow \mu_{p,K}$. If the absolute ramification index is larger than 1, $\mu_{p,K}$ becomes isomorphic to the constant group $\mathbb{Z}/p\mathbb{Z}$ and $j_*\mu_{p,K}$ is a form of the constant group $\mathbb{Z}/p\mathbb{Z}$.

To analyze $\left(\mathrm{R}^1 j_* F\right)_{\bar{s}} = \mathrm{H}^1(\text{Gal}(K^{\text{sep}}/K^{\text{nr}}), F(K^{\text{sep}}))$, we must understand $\text{Gal}(K^{\text{sep}}/K^{\text{nr}})$. Define $\widehat{\mathbb{Z}}_{(p)} := \varprojlim_{n \in (\mathbb{N} \setminus p\mathbb{N})} \mathbb{Z}/n\mathbb{Z}$. Each tamely ramified extension L/K^{nr} can be normalized s that it has the form $L = K^{\text{nr}}[X]/(X^e - \pi_K)$ and these normalizations are compatible since, for every $x \in (\mathcal{O}_K^{\text{sh}})^*$, K^{nr} contains the e-th roots of x if $e \in \mathbb{N} \setminus p\mathbb{N}$.

This means that the maximal tamely ramified extension of K^{nr} in K^{sep} is Galois with Galois group $\hat{\mathbb{Z}}_{(p)}$. This gives us an exact sequence

$$0 \longrightarrow N \longrightarrow \text{Gal}(K^{\text{sep}}/K^{\text{nr}}) \longrightarrow \hat{\mathbb{Z}}_{(p)} \longrightarrow 0$$

where N is a closed normal subgroup whose finite quotients are all p-groups. Now, if M_F is a finitely generated continuous $\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})$-module with trivial action, then

$$\begin{aligned}\mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})\,,M_F) &= \mathrm{Hom}_{\mathrm{cont.}}(\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})\,,M_F)\\ &= \mathrm{Tors}_{(p^{-1})}(M_F) + p\text{-power torsion},\end{aligned}$$

where $\mathrm{Tors}_{(p^{-1})}(M_F)$ denotes the torsion part of M_F that is prime to p.

If M_F is not a trivial module, then there exists a finite Galois extension L/K^{nr} such that $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ acts trivially on M_F. From the inflation-restriction sequence we obtain the exact sequence

$$\mathrm{H}^1(\mathrm{Gal}(L/K^{\mathrm{nr}})\,,M_F) \hookrightarrow \mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})\,,M_F) \longrightarrow \mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/L)\,,M_F)\,,$$

so that the stalk of $\mathrm{R}^1 j_* F$ in $\bar{s}$ is an extension of a subgroup of $\mathrm{Hom}_{\mathrm{cont.}}(\mathrm{Gal}(K^{\mathrm{sep}}/L)\,,M_F)$ by $\mathrm{H}^1(\mathrm{Gal}(L/K^{\mathrm{nr}})\,,M_F)$.

As an example, consider the $\mu_{q,K}$ with q relatively prime to the characteristic of K. The short exact Kummer sequence

$$0 \longrightarrow \mu_{q,K} \longrightarrow \mathbb{G}_{m,K} \xrightarrow{(-)^q} \mathbb{G}_{m,K} \longrightarrow 0$$

induces a long exact sequence

$$0 \longrightarrow \mu_{q,\mathcal{O}_K} \longrightarrow j_*\mathbb{G}_{m,K} \longrightarrow j_*\mathbb{G}_{m,K} \longrightarrow \mathrm{R}^1 j_*\mu_{q,K} \longrightarrow 0.$$

The latter sequence induces, in turn, the following sequence on the group of components

$$i_*\mathbb{Z}/q\mathbb{Z} \xrightarrow{0-\mathrm{map}} i_*\mathbb{Z} \xrightarrow{\cdot q} i_*\mathbb{Z} \longrightarrow \Phi\big(\mathrm{R}^1 j_*\mu_{q,K}\big) \longrightarrow 0.$$

In the case that q is relatively prime to the characteristic of k, the sequence of identity components corresponds to the exact Kummer sequence

$$0 \longrightarrow \mu_{q,K} \longrightarrow \mathbb{G}_{m,K} \xrightarrow{(\cdot)^q} \mathbb{G}_{m,K} \longrightarrow 0$$

over $\mathcal{O}_K$. By an argument similar to that given in the proof of Theorem 5.3.4, the sequence of groups of components is exact, whence $\mathrm{R}^1 j_*\mu_{q,K} = i_*\mathbb{Z}/q\mathbb{Z}$. If q is not prime to $\mathrm{char}(k)$, then $\mathrm{R}^1 j_*\mu_{q,K}$ must be contained in the cokernel of $p^{\nu_p(q)}$-exponentiation on $(\mathcal{O}_K^{\mathrm{sh}})^*$. This is non-trivial if, and only if, the residue field is not perfect and in this case it is an infinite p-primary torsion group. If the residue field is perfect, then the Kummer sequence over $\mathcal{O}_K$ is exact since, as $\mathcal{O}_K^{\mathrm{sh}}$ is a complete discrete valuation ring and hence $\mathcal{O}_K^{\mathrm{sh}}$ contains a multiplicative representative system of k^{sep}, we obtain $\mathrm{R}^1 j_*\mu_{q,K} = \mathbb{Z}/q\mathbb{Z}$.

Summarizing, we have:

Proposition 4.3.1. *Let F be an étale K-group scheme and let L/K^{nr} be a finite Galois extension such that $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ acts trivially on $F(K^{\mathrm{sep}})$. Let $I \coloneqq \mathrm{Gal}(L/K^{\mathrm{nr}})$. Further, assume that $F(K^{\mathrm{sep}})$ is finitely generated as an abelian group. Then there exists a short exact sequence of abelian groups*

$$0 \longrightarrow H^1(I, F(K^{\mathrm{sep}})) \longrightarrow \big(R^1 j_* F\big)_{\bar{s}} \longrightarrow E(F) \longrightarrow 0.$$

The group $E(F)$ is zero if $F(K^{\mathrm{sep}})$ is torsion-free.

If $L = K^{\mathrm{nr}}$, i.e. $H^1(I, F(K^{\mathrm{sep}})) = 0$, then $\big(R^1 j_* F\big)_{\bar{s}} = E(F)$ is an extension of a p-group by the prime-to-p torsion part of $F(K^{\mathrm{sep}})$.

If the residue field is perfect and the characteristic of K is equal to that of k then, using the structure theory of complete discrete valuation rings, one can see that $\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}}) = \hat{\mathbb{Z}}$ (see [S, II, §4, Excercise]). For an étale K-group F with trivial $\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})$-action and a finitely generated M_F, we have $\big(\mathrm{R}^1 j_* F\big)_{\bar{s}} = \mathrm{Tors}(F(K^{\mathrm{sep}}))$.

Chapter 5

Groups of components via cohomological methods

In this chapter we develop methods for determining the group of components of the Néron model of an algebraic K-torus T. First, we consider an approach from [X]. This approach of Xarles consists in passing from short exact sequences of algebraic K-tori to the (short, in his case) exact sequences of the Néron models and from these to the long exact sequence for the functor $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$.

In order to obtain statements about the group of components, we have to do this in the smooth topology and take advantage of the fact that there exist canonical identifications for $i = 0, 1$

$$\begin{aligned} \mathrm{R}^i\underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z}) &\cong \mathrm{R}^i\underline{\mathrm{Hom}}(i_*\Phi(T), i_*\mathbb{Z}) \\ &\cong i_*\mathrm{R}^i\underline{\mathrm{Hom}}(\Phi(T), \mathbb{Z}) \cong i_*\mathrm{R}^i\,\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}). \end{aligned}$$

In the first three terms the smooth sheaves over $\mathcal{O}_K$, respectively k, are meant. In the last term, however, Φ and $\mathbb{Z}$ are the $\mathrm{Gal}\,(k^{\mathrm{sep}}/k)$-modules associated to the corresponding étale sheaves. We establish these identification in a more general version, so that later instead of $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ we may also use the functor $\underline{\mathrm{Hom}}\big(\cdot, i_*\mathbb{Z}\big[p^{-1}\big]\big)$.

Using Xarles' approach, we can then replace the free part of the group of components with an extension

$$0 \longrightarrow X(T)^I \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z}) \longrightarrow E(T) \longrightarrow 0,$$

where $E(T)$ is a finitely generated p-primary torsion module which is called the defect term. The existence of such a defect term is a direct consequence of the non-exactness of the Néron model.

We then describe the map that a homomorphism of algebraic tori induces on the free part of the group of components. We can describe this via a commutative diagram using the results for tori with multiplicative reduction.

We also analyze the case of a short exact sequence of algebraic tori. In this case we can produce a commutative diagram that is no longer exact and in which, in general, not all torsion parts of the group of components under consideration appear.

As a second approach, we take up the idea from proposition 2.3.1 and thereby generalize [LL, Proposition 4.2 a)]:

We consider the case of a sequence of algebraic tori

$$0 \longrightarrow T' \longrightarrow R \longrightarrow T \longrightarrow 0$$

such that the torsion parts of $\Phi(T')$ and $\Phi(R)$ have coprime orders. If $\mathrm{R}^1 j_* T' = 0$ in the smooth topology, then we obtain an exact sequence

$$0 \longrightarrow \Phi\left(T'\right)^{\vee\vee} \longrightarrow \Phi(R) \longrightarrow \Phi(T) \longrightarrow 0.$$

Otherwise we define $\mathcal{K} := \ker\left(\mathrm{R}^1 j_* T' \longrightarrow \mathrm{R}^1 j_* R\right)$ and obtain a sequence

$$0 \longrightarrow \Phi(T')^{\vee\vee} \longrightarrow \Phi(R) \longrightarrow \Phi(T) \longrightarrow \Phi(\mathcal{K}) \longrightarrow 0$$

which is exact except perhaps at $\Phi(R)$.

We specify conditions for exactness above and describe the morphism $\Phi\left(T'\right)^{\vee\vee} \longrightarrow \Phi(R)$. The latter morphism can be described via a map between free parts.

Finally, we use this and the canonical surjection

$$T(K^{\mathrm{nr}}) = j_* T\left(\mathcal{O}_K^{\mathrm{sh}}\right) \longrightarrow i_* \Phi(T)\left(\mathcal{O}_K^{\mathrm{sh}}\right)$$

to compute certain group of components. Since we can view the defect terms as groups of components, we obtain an estimate of their size. This estimate shows that the defect terms for tori that split after a residually unramified extension are trivial.

5.1 The free part of the group of components

We first consider the functors $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}^1$ on the smooth and étale sites over $\mathcal{O}_K$. To distinguish them, we use the indices "sm" and "ét" to indicate the site on which these functors are being considered.

Proposition 5.1.1. (cf. [X, 2.2 and 2.12]) *Let $\mathcal{T}$ be a smooth $\operatorname{Spec} \mathcal{O}_K$-group scheme with connected fibers and let C be a constant torsion-free abelian sheaf on the étale or smooth sites over $\operatorname{Spec} k$. In the étale setting, assume in addition that there exists an $l \in \mathbb{N}$ which is relatively prime to $p := \mathrm{char}(k)$ such that C contains no l-divisible part, i.e., for every $c \in C \setminus \{0\}$, there exists an $r \in \mathbb{N}$ such that $l^r x \neq c$ for every $x \in C$. Then*

$$\underline{\mathrm{Hom}}_{\text{ét}}(\mathcal{T}, i_* C) = 0 \qquad \underline{\mathrm{Hom}}_{\mathrm{sm}}(\mathcal{T}, i_* C) = 0$$
$$\underline{\mathrm{Ext}}^1_{\mathrm{sm}}(\mathcal{T}, i_* C) = 0,$$

whereas in general $\underline{\mathrm{Ext}}^1_{\text{ét}}(\mathcal{T}, i_* C) \neq 0$.

Proof. We begin with the statements about the étale topology. Since

$$\underline{\mathrm{Hom}}(\mathcal{T}, i_* C) \cong i_* \underline{\mathrm{Hom}}(i^* \mathcal{T}, C)\,,$$

it suffices to check that $\underline{\mathrm{Hom}}(i^*\mathcal{T}, C) = 0$. The étale site over $\operatorname{Spec} k$ is equivalent to the category of continuous $\mathrm{Gal}\,(k^{\mathrm{sep}}/k)$-modules and the sheaves $i^*\mathcal{T}$ and C correspond to the Galois modules $\mathcal{T}\left(\mathcal{O}_K^{\mathrm{sh}}\right)$ and C with trivial Galois action, respectively.

Now consider the given $l \in \mathbb{N}$. Since l is prime to the characteristic of k, multiplication by l on $\mathcal{T}$ is étale [BLR, 7.3.2]. This means (cf. [M, II, 2.19]) that the stalk $\mathcal{T}_{\bar{s}} = \mathcal{T}\left(\mathcal{O}_K^{\mathrm{sh}}\right)$ is an l-divisible group. Since a connected étale k-scheme U is the spectrum of a finite separable extension k' of k and

$$\mathrm{Hom}_U\left(i^*\mathcal{T}|_U\,, C|_U\right) \subset \mathrm{Hom}_{\mathbb{Z}}\left(\mathcal{T}\left(\mathcal{O}_K^{\mathrm{sh}}\right), C\right)^{\mathrm{Gal}\left(k^{\mathrm{sep}}/k'\right)},$$

we must have $\underline{\mathrm{Hom}}(i^*\mathcal{T}, C) = 0$ since a homomorphism from an l-divisible group to C must be trivial.

As an example regarding the last assertion of the proposition about $\underline{\mathrm{Ext}}^1$ in the étale topology, consider the sheaf $\underline{\mathrm{Ext}}^1_{\text{ét}}(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z})$, which in general does not vanish according to proposition 5.1.2.

Now consider the smooth sheaves. Let U be an arbitrary scheme and let G_1 and G_2 be two smooth U-group schemes. Via a Yoneda-type argument, we have

$$\mathrm{Hom}_{U-\mathrm{gr}}(G_1, G_2) \cong \mathrm{Hom}_{(\mathrm{sm})/U}(G_1, G_2)\,,$$

in other words, the homomorphisms of the smooth sheaves over U represented by the G_i coincide with the U-group scheme morphisms from G_1 to G_2.

We now have $\underline{\mathrm{Hom}}(\mathcal{T}, i_*C) \cong i_*\underline{\mathrm{Hom}}(i^*\mathcal{T}, C)$ since, by the smoothness of $\mathcal{T}$, the sheaf $i^*\mathcal{T}$ is represented by $\mathcal{T} \otimes_{\mathcal{O}_K} k$ on the smooth site over k. Now, for every smooth and connected k-scheme U, the group scheme $\mathcal{T}_k \times_k U$ is connected as well. Every group homomorphism from a connected k-group scheme into an étale k-group scheme factors through the identity component and is therefore trivial. This means that $\mathrm{Hom}_U(\mathcal{T}_k|_U, C|_U) = 0$ and therefore $\underline{\mathrm{Hom}}(\mathcal{T}_k, C) = 0$ as well.

By [SGA7, VIII 5.7], i_*C is represented by an étale $\mathcal{O}_K$-group scheme $Z := C_{\mathcal{O}_K,k}$ which is obtained by gluing copies S_i of $\mathrm{Spec}\,\mathcal{O}_K$ for each $i \in C$ along the generic fiber $\eta = \mathrm{Spec}\,K$. Note that this construction is compatible with base changes.

For a smooth $\mathrm{Spec}\,\mathcal{O}_K$-scheme U, an extension of $\mathcal{T}$ by Z as abelian fppf sheaves over U is an extension of U-groups in the sense of [SGA7, VIII]. Such an extension trivializes as a $\mathcal{T}_{Z_U}$-torsor, i.e., in general over an fppf cover, but in fact over a smooth cover because $\mathcal{T}$ is smooth over $\mathrm{Spec}\,\mathcal{O}_K$. Therefore the group of isomorphism classes of such extensions is equal to the group of isomorphism classes of extensions as smooth abelian sheaves over U.

By [SGA7, VIII, 5.9] the extensions of $\mathcal{T}_U$ by Z_U correspond to the extensions of $\mathcal{T}_{U_k}$ by C_{U_k} as U_k-groups in the fppf topology, i.e., as seen above, to the extensions of smooth abelian sheaves over U_k. Assume that U_k is irreducible and write η for the generic point of U_k. Since U_k is a smooth k-scheme, it is geometrically unibranched. Thus, by [SGA7, VIII, 5.2], the extensions of $\mathcal{T}_{U_k}$ by C_{U_k} correspond to the extensions of $\mathcal{T}_\eta$ by C_η. But these are extensions of groups over the field $k(\eta)$ and by [SGA7, VIII, 5.5] they are all trivial because $\mathcal{T}_\eta$ is connected. This yields $\mathrm{Ext}^1_U(\mathcal{T}_U, i_*C_U) = 0$.

Now $\underline{\mathrm{Ext}}^1(\mathcal{T}, i_*C)$ is the sheafification of the presheaf $U \mapsto \mathrm{Ext}^1_U(\mathcal{T}_U, Z_U)$, which completes the proof. □

Proposition 5.1.2. *Let K be a local field with perfect residue field. In the étale topology over $\mathrm{Spec}\,\mathcal{O}_K$, we have $\underline{\mathrm{Ext}}^1(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z}) \neq 0$.*

Proof. By definition, $\underline{\mathrm{Ext}}^1(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z})$ is the sheafification of the presheaf $U \mapsto \mathrm{Ext}^1_U(\mathbb{G}_{m,\mathcal{O}_U}, i_*\mathbb{Z}|_U)$. Since $\underline{\mathrm{Hom}}(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z}) = 0$, the local to global spectral sequence for Ext [M, III, Theorem 1.22] shows that it suffices to check that $\mathrm{Ext}^1_U(\mathbb{G}_{m,\mathcal{O}_U}, i_*\mathbb{Z}|_U) \neq 0$ for a suitable U.

Now let $n \in \mathbb{N}$ be relatively prime to $p = \mathrm{char}(k)$ and choose $U = \mathrm{Spec}\,\mathcal{O}_{K'}$ for an unramified extension K'/K containing the n-th roots of unity. Let the residue field of K' be denoted by k'. From the exact sequence of constant sheaves on the étale site over $\mathcal{O}_K$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

we obtain an exact sequence

$$\mathrm{Hom}_U\left(\mathbb{G}_{m,\mathcal{O}_U}, i_*\mathbb{Q}|_U\right) \longrightarrow \mathrm{Hom}_U\left(\mathbb{G}_{m,\mathcal{O}_U}, (i_*\mathbb{Q}/\mathbb{Z})|_U\right) \longrightarrow \mathrm{Ext}^1_U\left(\mathbb{G}_{m,\mathcal{O}_U}, i_*\mathbb{Z}|_U\right).$$

Now we have

$$\mathrm{Hom}_U(\mathbb{G}_{m,\mathcal{O}_U}, (i_*\mathbb{Q}/\mathbb{Z})|_U) = \mathrm{Hom}_{G_{k'}}((\mathcal{O}_K^{\mathrm{sh}})^*, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}((\mathcal{O}_K^{\mathrm{sh}})^*_{G_{k'}}, \mathbb{Q}/\mathbb{Z}),$$

where $G_{k'} := \mathrm{Gal}(k^{\mathrm{sep}}/k')$ and $(\cdot)_{G_{k'}}$ denotes the $G_{k'}$-coinvariants.

Since $\mathbb{Q}/\mathbb{Z}$ is divisible, i.e., $\mathbb{Z}$-injective, maps from $\mathbb{Z}/n\mathbb{Z} \cong \mu_n(\mathcal{O}_K^{\mathrm{sh}})$ to $\mathbb{Q}/\mathbb{Z}$ extend to maps $(\mathcal{O}_K^{\mathrm{sh}})^*_{G_{k'}} \longrightarrow \mathbb{Q}/\mathbb{Z}$. Now there exist exactly n different maps $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ and their extensions can restrict to a map $(\mathcal{O}_K^{\mathrm{sh}})^*_{G_{k'}} \longrightarrow \mathbb{Q}$ only if they are trivial on $\mu_n(\mathcal{O}_K^{\mathrm{sh}})$.

Therefore, by the long exact sequence, $\mathrm{Ext}^1_U(\mathbb{G}_{m,\mathcal{O}_U}, i_*\mathbb{Z}|_U)$ must contain a subgroup of the form $\mathbb{Z}/n\mathbb{Z}$. □

Now if K is a local field and T is an algebraic K-torus, the short exact sequence

$$0 \longrightarrow (j_*T)^0 \longrightarrow j_*T \longrightarrow i_*\Phi(T) \longrightarrow 0$$

induces an isomorphism in the smooth and also the étale topology

$$\underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z}) \cong \underline{\mathrm{Hom}}(i_*\Phi, i_*\mathbb{Z})\,. \tag{5.1.2.1}$$

Moreover, in the smooth topology we have

$$\underline{\mathrm{Ext}}^1(j_*T, i_*\mathbb{Z}) \cong \underline{\mathrm{Ext}}^1(i_*\Phi, i_*\mathbb{Z})\,. \tag{5.1.2.2}$$

We want to simplify this further. To do this, we first determine the functors $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}^1$ on the étale site in terms of Galois modules. Since the group of components Φ is an étale group scheme and the corresponding Galois module $\Phi(k^{\mathrm{sep}})$ is finitely generated, we can appeal to the following proposition.

Proposition 5.1.3. *Let F be a sheaf on the étale site over* $\mathrm{Spec}\, k$ *such that the associated continuous* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-module M_F is finitely generated. Then the equivalence of categories between étale sheaves over* $\mathrm{Spec}\, k$ *and continuous* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-modules induces an isomorphism of δ functors (in the second argument)*

$$R^i\underline{\mathrm{Hom}}(F, G) \rightsquigarrow R^i\,\mathrm{Hom}_{\mathbb{Z}}(M_F, M_G)\,.$$

Proof. Under these assumptions on F, the equivalence of categories maps $\underline{\mathrm{Hom}}(F, G)$ to the module $\mathrm{Hom}_{\mathbb{Z}}(M_F, M_G)$ [M, III, Example 1.7]. Further, under the equivalence of categories between étale sheaves and continuous Galois modules, injective sheaves correspond to injective Galois modules. Now $\mathrm{R}^i\underline{\mathrm{Hom}}(F, \cdot)$ is computed using injective resolutions and for every continuous Galois module there exists an injective resolution of Galois modules which are also injective as abelian groups. Since the forgetful functor is exact, we obtain an injective resolution of abelian groups and the proposition follows. □

Proposition 5.1.4. *Let k be a field, let Φ be a commutative étale k-group scheme and let C be a commutative constant k-group scheme. Assume that $\Phi(k^{\mathrm{sep}})$ is finitely generated as an abelian group.*

Then there exists an étale k-group scheme $\underline{\mathrm{Hom}}(\Phi, C)$ which represents $\underline{\mathrm{Hom}}(\Phi, C)$ as smooth and étale sheaf. Further, there exists an étale k-group scheme $\underline{\mathrm{Ext}}^1(\Phi, \mathbb{Z})$ that represents $\underline{\mathrm{Ext}}^1(\Phi, \mathbb{Z})$ as smooth and étale sheaf.

Proof. It is well-known that in the setting of the proposition there exists a k-group scheme $\underline{\mathrm{Hom}}(\Phi, C)$ that represents the group functor $T \mapsto \mathrm{Hom}_{T-\mathrm{grp}}(\Phi_T, C_T)$. This scheme must then also represent $\mathrm{Hom}(\Phi, C)$ in the smooth and the étale topologies.

Since Φ is étale and $\Phi(k^{\mathrm{sep}})$ is finitely generated, Φ is isomorphic to a constant group scheme after an étale base change. For constant group schemes C_1, C_2, the functor $\underline{\mathrm{Hom}}(C_1, C_2)$ corresponds to the constant group scheme for the group $\mathrm{Hom}_{\mathrm{grp}}(C_1, C_2)$. This means that $\underline{\mathrm{Hom}}(\Phi, C)$ becomes étale after an étale and surjective base change (since the base is a field), and is therefore already étale. For the sheaf $\underline{\mathrm{Ext}}^1(\Phi, \mathbb{Z})$ we consider the long exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(\Phi, \mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(\Phi, \mathbb{Q}) \longrightarrow \underline{\mathrm{Hom}}(\Phi, \mathbb{Q}/\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\Phi, \mathbb{Z}).$$

From Proposition 5.1.3 we see that $\underline{\mathrm{Ext}}^1(\Phi, \mathbb{Q}) = 0$ in the étale topology. Since the $\underline{\mathrm{Hom}}(\Phi, \cdot)$-sheaves are represented by étale group schemes, the map $\underline{\mathrm{Hom}}(\Phi, \mathbb{Q}) \longrightarrow \underline{\mathrm{Hom}}(\Phi, \mathbb{Q}/\mathbb{Z})$ is represented by a morphism of group schemes. By commutativity, the image of this morphism is a normal subgroup of the group scheme $\underline{\mathrm{Hom}}(\Phi, \mathbb{Q}/\mathbb{Z})$ and is also closed, because the topology on $\underline{\mathrm{Hom}}(\Phi, \mathbb{Q}/\mathbb{Z})$ is discrete. This means that the kernel of this homomorphism exists as an étale group scheme. The cokernel then represents the sheaf $\underline{\mathrm{Ext}}^1(\Phi, \mathbb{Z})$. Clearly, this also holds in the smooth topology. □

The above shows that the Galois modules $\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z})$ determine the smooth sheaves $\underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z})$ and $\underline{\mathrm{Ext}}^1(j_*T, i_*\mathbb{Z})$. However, this no longer applies to the higher Ext's, as the following example shows.

Proposition 5.1.5. *In the smooth topology over* $\mathrm{Spec}\, \mathcal{O}_K$ *we have* $\underline{\mathrm{Ext}}^2(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z}) \neq 0$.

Proof. Let $n \in \mathbb{N}$ be relatively prime to p. Then the Kummer sequence induces a long exact sequence for the functor $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$:

$$\cdots \longrightarrow 0 = \underline{\mathrm{Ext}}^1(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mu_n, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^2(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z}).$$

As we have seen, for the étale group scheme μ_n one can already compute the étale $\underline{\mathrm{Ext}}^1$ sheaf and this corresponds (in the stalk above the special fiber) to the non-trivial Galois module $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$. □

After this preliminary work, we want to analyze the free part of the group of components.

Theorem 5.1.6. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Let L/K be a finite Galois splitting extension for T with inertia group $I := \mathrm{Gal}(L/K^{\mathrm{nr}})$. Then the free part $\mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ of the group of components Φ of the Néron model of T is an extension of a $p = \mathrm{char}(k)$-primary torsion group by $X(T)^I$. More precisely, there exists an exact sequence of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules*

$$0 \longrightarrow X(T)^I \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z}) \longrightarrow E(T) \longrightarrow 0$$

where $E(T)$ represents the restriction of the smooth sheaf $\underline{\mathrm{Ext}}^1(\ker(R^1 j_* \widetilde{T} \longrightarrow R^1 j_* T), \mathbb{Z})$ to the étale site above $\mathrm{Spec}\, k$, where the torus $\widetilde{T}$ corresponds to the module $X(T)/X(T)^I$ via Cartier duality.

We call $E(T)$ the defect term *for T.*

Proof. From the canonical short exact sequence (6)

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0$$

we obtain, after forming the Néron models in the smooth topology, the exact sequence

$$0 \longrightarrow j_*\widetilde{T} \longrightarrow j_*T \longrightarrow j_*T^I \longrightarrow \mathcal{K} \longrightarrow 0,$$

where $\mathcal{K} := \ker\Big(\mathrm{R}^1 j_*\widetilde{T} \longrightarrow \mathrm{R}^1 j_*T\Big)$. We can split this sequence into two short exact sequences, namely:

$$0 \longrightarrow j_*\widetilde{T} \longrightarrow j_*T \longrightarrow \mathcal{N} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N} \longrightarrow j_*T^I \longrightarrow \mathcal{K} \longrightarrow 0$$

Applying $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$, we obtain the exact sequences

$$0 \longrightarrow \underline{\mathrm{Hom}}(\mathcal{N}, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(j_*\widetilde{T}, i_*\mathbb{Z}) = 0$$

$$0 = \underline{\mathrm{Hom}}(\mathcal{K}, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(j_*T^I, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{N}, i_*\mathbb{Z}) \twoheadrightarrow E,$$

where $E := \mathrm{Ext}^1(\ker(\mathrm{R}^1 j_*\widetilde{T} \longrightarrow \mathrm{R}^1 j_*T), i_*\mathbb{Z})$. The first sequence exploited the fact that the Néron model of $\widetilde{T}$ is of finite type, i.e., has a finite group of components. For the second sequence we used the facts that $\mathrm{R}^1 j_*\widetilde{T}$ is a torsion sheaf and j_*T^I has a torsion-free group of components, so that in the smooth topology we obtain $\mathrm{Ext}^1(j_*T^I, i_*\mathbb{Z}) = 0$.

By substituting the isomorphism from the first sequence we obtain an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(j_*T^I, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z}) \longrightarrow E \longrightarrow 0.$$

The restriction of this sequence to the étale site on $\mathrm{Spec}\, k$ remains exact and the restriction of E is also a p-primary torsion sheaf (just as E is). The étale sequence then yields the desired exact sequence of Galois modules via Proposition 5.1.3 and the description of the group of components in the case of tori with multiplicative reduction. □

By the above theorem, additional statements can be obtained that generalize Xarles' results to the case of an arbitrary residue field:

Corollary 5.1.7. *Let T be an algebraic K-torus as above. If T splits over a tamely ramified extension, then* $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) = X(T)^I$.

Proof. Consider again the sequence (6)

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0.$$

Since the character group of $\widetilde{T}$ is a quotient of $X(T)$, namely $X(\widetilde{T}) = X(T)/X(T)^I$ (see (7)), $\widetilde{T}$ also splits over a tamely ramified extension. This means that $\mathrm{R}^1 j_*\widetilde{T}$ is trivial, so that in the notation of the proof above $\mathcal{K} = 0$ and therefore $E = 0$. Consequently, $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}\big(\Phi\big(T^I\big), \mathbb{Z}\big) \cong X(T)^I$. □

5.2 The induced map on the free part

In order to use the results obtained so far for a further determination of the group of components, we need to identify the effect of the functor $\underline{\mathrm{Hom}}(j_*(\cdot), i_*\mathbb{Z})$ on morphisms between K-tori.

Proposition 5.2.1. *Let $\psi : T_1 \to T_2$ be a morphism of algebraic K-tori and let $D(\psi)\colon X(T_2) \to X(T_1)$ be the associated homomorphism between character groups. Then the descriptions from Theorems 1.1.3 and 5.1.6 induce a commutative diagram*

$$\begin{array}{ccccccc} X(T_2)^I & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2^I),\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2),\mathbb{Z}) & \twoheadrightarrow & E(T_2) \\ {\scriptstyle D(\psi)^I}\downarrow & & & & {\scriptstyle \bar\psi}\downarrow & & \\ X(T_1)^I & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_1^I),\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_1),\mathbb{Z}) & \twoheadrightarrow & E(T_1) \end{array}$$

where $\bar\psi := i_*\underline{\mathrm{Hom}}(j_*\psi, i_*\mathbb{Z})$ (on $i_*\underline{\mathrm{Hom}}(j_*T_2, i_*\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2), \mathbb{Z})$).

Proof. Since $D(\psi)$ is a Galois module homomorphism, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X(T_2)^I & \longrightarrow & X(T_2) & \longrightarrow & X(\widetilde{T_2}) & \longrightarrow & 0 \\ & & {\scriptstyle D(\psi)^I}\downarrow & & {\scriptstyle D(\psi)}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & X(T_1)^I & \longrightarrow & X(T_1) & \longrightarrow & X(\widetilde{T_1}) & \longrightarrow & 0 \end{array}$$

Using Cartier duality, we obtain a commutative diagram of algebraic tori with exact rows. Since j_* and $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ are functors, using the technique from the proof of Theorem 5.1.6 we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2^I),\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2),\mathbb{Z}) & \twoheadrightarrow & E(T_2) \\ \downarrow & & {\scriptstyle \bar\psi}\downarrow & & \\ \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_1^I),\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_1),\mathbb{Z}) & \twoheadrightarrow & E(T_1) \end{array}$$

By Theorem 1.1.3, the morphisms on the first row can be described using character groups. □

For later applications, we need to describe $\underline{\mathrm{Hom}}(j_*\cdot, i_*\mathbb{Z})$ when applied to a short exact sequence of algebraic tori.

Theorem 5.2.2. *Let K be a local field and let*

$$0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$$

be a short exact sequence of algebraic K-tori. Then we obtain a commutative diagram of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules with exact columns

$$\begin{array}{ccccccc} E(T_3) & & E(T_2) & & E(T_1) & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_3),\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2),\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T_1),\mathbb{Z}) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ X(T_3)^I & \hookrightarrow & X(T_2)^I & \longrightarrow & X(T_1)^I & \twoheadrightarrow & M, \end{array}$$

where we use the following notations:

$$\begin{aligned} M &:= \ker(H^1(I, X(T_3)) \longrightarrow H^1(I, X(T_2)) \\ \cdots &:= \underline{\mathrm{Ext}}^1(\mathcal{N}, i_*\mathbb{Z})|_{(\text{ét})/k} \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T_2), \mathbb{Z}), \\ \mathcal{N} &:= \ker(\mathcal{T}_3 \longrightarrow \mathcal{K}) \\ \mathcal{K} &:= \ker(R^1 j_* T_1 \to R^1 j_* T_2). \end{aligned}$$

The bottom row of the above diagram is exact and the middle row is a complex which is exact except perhaps at $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2), \mathbb{Z})$.

If $\underline{\mathrm{Ext}}^1(\mathcal{K}, i_*\mathbb{Z}) = 0$, *then the middle row is exact at* $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T_2), \mathbb{Z})$ *as well. In this case we have an inclusion* $\underline{\mathrm{Ext}}^1(\Phi(T_3), i_*\mathbb{Z}) \hookrightarrow \underline{\mathrm{Ext}}^1(\mathcal{N}, i_*\mathbb{Z})\big|_{(\text{ét})/k}$.

If, in addition, $\underline{\mathrm{Ext}}^2(\mathcal{K}, i_*\mathbb{Z}) = 0$, *then the preceding inclusion is an isomorphism.*

Proof. The given short exact sequence of tori corresponds to a short exact sequence of character groups and this induces an exact and commutative diagram with the above choice of M

$$\begin{array}{ccccccccccc}
 & & X(\widetilde{T_3}) & & X(\widetilde{T_2}) & & X(\widetilde{T_1}) & & & & \\
 & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & & & \\
0 & \longrightarrow & X(T_3) & \longrightarrow & X(T_2) & \longrightarrow & X(T_1) & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & X(T_3)^I & \longrightarrow & X(T_2)^I & \longrightarrow & X(T_1)^I & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

By Cartier duality, we obtain from the above an exact and commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & \widetilde{T_1} & & \widetilde{T_2} & & \widetilde{T_3} & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & T_3 & \longrightarrow & 0 \\
 & & & & \twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \\
0 & \longrightarrow & D(M) & \longrightarrow & T_1^I & \longrightarrow & T_2^I & \longrightarrow & T_3^I & \longrightarrow & 0
\end{array}$$

We then apply the functors j_* and $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ in the smooth topology. If we set $\mathcal{T}_i := j_*T_i$, then we obtain on the middle row

$$0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{T}_3 \longrightarrow \mathcal{K} \longrightarrow 0$$

where $\mathcal{K} := \ker(\mathrm{R}^1 j_* T_1 \to \mathrm{R}^1 j_* T_2)$. In order to apply $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ we split this sequence into the sequences

$$0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{N} \longrightarrow 0 \tag{5.2.2.1}$$

and

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{T}_3 \longrightarrow \mathcal{K} \longrightarrow 0. \tag{5.2.2.2}$$

We set $\mathcal{N} := \ker(\mathcal{T}_3 \to \mathcal{K})$. The long exact sequence induced by (5.2.2.1) yields

$$\underline{\mathrm{Hom}}(\mathcal{T}_3, i_*\mathbb{Z}) \hookrightarrow \underline{\mathrm{Hom}}(\mathcal{T}_2, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{T}_1, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{N}, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{T}_2, i_*\mathbb{Z})$$

where we take the inclusion $\underline{\mathrm{Hom}}(\mathcal{T}_3, i_*\mathbb{Z}) \hookrightarrow \underline{\mathrm{Hom}}(\mathcal{N}, i_*\mathbb{Z})$ from the long exact sequence induced by (5.2.2.2). By construction, the modified sequence is a complex and is exact except perhaps at $\underline{\mathrm{Hom}}(\mathcal{T}_2, i_*\mathbb{Z})$.

After restricting to the étale site over k and passing to the representing Galois modules (via (5.1.2.2) and (5.1.2.1)), we obtain the middle row of the diagram in the statement. This can be done using Proposition 5.2.1 to obtain the rows of a commutative diagram

$$\begin{array}{ccccccc}
E(T_3) & & E(T_2) & & E(T_1) & & \\
\uparrow & & \twoheaduparrow & & \twoheaduparrow & & \\
\operatorname{Hom}_{\mathbb{Z}}(\Phi(T_3),\mathbb{Z}) & \hookrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(T_2),\mathbb{Z}) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(T_1),\mathbb{Z}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
X(T_3)^I & \longrightarrow & X(T_2)^I & \longrightarrow & X(T_1)^I & &
\end{array}$$

where $\cdots$ stands for $\underline{\operatorname{Ext}}^1(\mathcal{N}, i_*\mathbb{Z})|_{(\text{ét})/k} \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(\Phi(T_2),\mathbb{Z})$ (as a morphism of the representing Galois modules).

By Proposition 5.2.1, the maps on the bottom row correspond to the canonical maps of the I-invariants of the character groups. Therefore the bottom row must be exact and $X(T_3)^I \longrightarrow X(T_2)^I$ is an inclusion. In addition, we must have $M = \operatorname{coker}\big(X(T_2)^I \longrightarrow X(T_1)^I\big)$.

The long exact sequence induced by the sequence (5.2.2.2) yields (in the smooth topology) the stated relationships between $\mathrm{R}^i\underline{\operatorname{Hom}}(\mathcal{N}, i_*\mathbb{Z})$ and

$$\mathrm{R}^i\underline{\operatorname{Hom}}(\mathcal{T}_3, i_*\mathbb{Z}) \cong \mathrm{R}^i\underline{\operatorname{Hom}}(i_*\Phi(T_3)\,, i_*\mathbb{Z}) \cong i_*\underline{\operatorname{Hom}}(\Phi(T_3)\,,\mathbb{Z})\,.$$

□

Theorem 5.2.3. *Let K be a local field and let*

$$0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$$

be a short exact sequence of K-tori. Further, assume either that the tori T_i split over a finite Galois tamely ramified extension L/K or that the residue field is perfect.

Then the description of the free part is functorial. In particular we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(T_3),\mathbb{Z}) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(T_2),\mathbb{Z}) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(T_1),\mathbb{Z}) & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(\Phi(T_3),\mathbb{Z}) \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & X(T_3)^I & \longrightarrow & X(T_2)^I & \longrightarrow & X(T_1)^I & &
\end{array}$$

of $\operatorname{Gal}(k^{\mathrm{sep}}/k)$-modules with exact rows, where the maps on the top row are induced by the maps on the Néron models and the maps on the bottom row are induced by the images of the character groups.

Proof. If the residue field is perfect or the tori split after tame ramification, all defect terms are trivial. Then the theorem above yields an isomorphism of the two bottom rows. In particular, the sequence of the free parts must then be exact. Since we also have $\mathrm{R}^1 j_* T_1 = 0$ under the stated hypotheses, we must have $\mathcal{K} = 0$, whence $\underline{\operatorname{Ext}}^1(\mathcal{N}, i_*\mathbb{Z}) \cong \underline{\operatorname{Ext}}^1(\mathcal{T}_3, i_*\mathbb{Z})$. □

5.3 Exact sequences of groups of components of tori

We now revisit the idea from Proposition 2.3.1 and consider the case of exact sequences of algebraic tori. This yields a second procedure for obtaining information about the group of components of the Néron model.

In this section, we seek to understand the torsion part $\mathrm{Tors}(\Phi(T))$ of a group of components $\Phi(T)$ as submodule and accordingly define the torsion-free part as

$$\Phi(T)^{\vee\vee} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}), \mathbb{Z}) \cong \Phi(T)/\,\mathrm{Tors}(\Phi(T)).$$

Proposition 5.3.1. [LL, 4.3 a)] *Let K be a local field and let*

$$0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$$

be a short exact sequence of algebraic K-tori with $R^1 j_* T_1 = 0$ in the smooth topology. Then the above sequence induces a short exact sequence

$$0 \longrightarrow j_*T_1 \xrightarrow{\iota} j_*T_2 \longrightarrow j_*T_3 \longrightarrow 0$$

which, in turn, induces a short exact sequence of groups of components

$$0 \longrightarrow j_*T_1/\iota^{-1}(j_*T_2^0)|_{(\text{ét})/k} \longrightarrow \Phi(T_2) \longrightarrow \Phi(T_3) \longrightarrow 0.$$

We have $j_*T_1^0 \subset \iota^{-1}(j_*T_2^0) \subset (j_*T_1)^{\mathrm{ft}}$. In particular, if the torsion parts of $\Phi(T_1)$ and $\Phi(T_2)$ have coprime orders, then

$$j_*T_1/\iota^{-1}(j_*T_2^0) = \Phi\left(T_1\right)^{\vee\vee}.$$

Proof. By Proposition 2.3.1, only the additional assertions require proof. The containment $j_*T_1^0 \subset \iota^{-1}(j_*T_2^0) \subset (j_*T_1)^{\mathrm{ft}}$ is clear since $\Phi(T_1) \longrightarrow \Phi(T_2)$ has a finite kernel (2.3.4) and the ft-Néron model corresponds exactly to the torsion part of the group of components (since this is the largest finite subgroup of the group of components).

In the étale topology we can identify the sheaf homomorphisms with the corresponding $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-module homomorphisms. If the orders of the torsion components of $\Phi(T_1)(k^{\mathrm{sep}})$ and $\Phi(T_2)(k^{\mathrm{sep}})$ are coprime, then the image of the torsion part of $\Phi(T_1)(k^{\mathrm{sep}})$ must be trivial, so that by the above containment the map $\Phi(T_1) \longrightarrow \Phi(T_2)$ is injective and can be factored through the quotient $\Phi(T_1)^{\vee\vee}$. □

In general, the sequence of Néron models is not expected to be right-exact. Therefore we want to consider the "group of components" $\Phi\big(\mathrm{R}^1 j_*T\big)$. This is just the (sheaf) quotient of $\mathrm{R}^1 j_*T$ by $\mathrm{R}^1 j_*T^0$ according to the definition above.

Proposition 5.3.2. *Let $\mathcal{G} \longrightarrow \mathcal{F}$ be an epimorphism of sheaves on the smooth site over* $\mathrm{Spec}\,\mathcal{O}_K$*, where $\mathcal{G}$ is represented by a smooth group scheme. Then*

$$\underline{\mathrm{Hom}}(\mathcal{F}^0, i_*\mathbb{Z}) = 0$$
$$\underline{\mathrm{Ext}}^1(\mathcal{F}, i_*\mathbb{Z}) \cong \underline{\mathrm{Ext}}^1(\Phi(\mathcal{F}), i_*\mathbb{Z}).$$

Proof. By 2.2.4 there exists a short exact sequence

$$0 \longrightarrow \kappa \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{F}^0 \longrightarrow 0.$$

Since $\underline{\mathrm{Hom}}\big(\mathcal{G}^0, i_*\mathbb{Z}\big) = 0$, the first assertion follows. Now $\mathcal{G}^0$ is an l-divisible sheaf for every natural number l that is not divisible by $p = \mathrm{char}(k)$. This means that the kernel κ is also an l-divisible sheaf, whence $\underline{\mathrm{Hom}}(\kappa, i_*\mathbb{Z}) = 0$.

Thus the long exact sequence for the functor $\underline{\mathrm{Hom}}(\cdot, i_*\mathbb{Z})$ in the smooth topology yields

$$0 = \underline{\mathrm{Hom}}(\kappa, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{F}^0, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{G}^0, i_*\mathbb{Z}) = 0.$$

Therefore, we obtain from the sequence

$$0 \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F} \longrightarrow \Phi(\mathcal{F}) \longrightarrow 0$$

an isomorphism $\underline{\mathrm{Ext}}^1(\Phi(\mathcal{F}), i_*\mathbb{Z}) \cong \underline{\mathrm{Ext}}^1(\mathcal{F}, i_*\mathbb{Z})$. □

Proposition 5.3.3. *Let $\mathcal{F}$ be a sheaf on the smooth site over* $\operatorname{Spec}\mathcal{O}_K$ *and let* $\psi\colon G_1 \longrightarrow G_2$ *be a morphism of smooth and commutative group schemes. If* $\mathcal{F} \cong \operatorname{coker}(G_1 \longrightarrow G_2)$ *as smooth sheaves, then* $\Phi(\mathcal{F})$ *is represented by an étale group scheme.*

If, in addition, the restriction of $\mathcal{F}$ *to the étale site is trivial, then* $\Phi(\mathcal{F})$ *is also trivial.*

Proof. Clearly $\Phi(\mathcal{F}) \cong \operatorname{coker}(\Phi(G_1) \longrightarrow \Phi(G_2))$ as smooth sheaves and via a Yoneda argument one sees that the morphism on the groups of components is induced by a homomorphism of the étale group schemes. The image of $\Phi(G_1)$ under this homomorphism must be a closed normal subgroup because the groups are abelian and the topology on the étale groups is discrete. Thus the cokernel exists as a group scheme and is étale; a fortiori, it represents the cokernel as a smooth sheaf.

The epimorphism $\mathcal{F} \longrightarrow \Phi(\mathcal{F})$ is surjective even after restriction to the étale site, so that $\Phi(\mathcal{F}) = 0$ as an étale sheaf. Since $\Phi(\mathcal{F})$ is an étale scheme, it follows from the triviality of $\Phi(k^{\mathrm{sep}})$ that $\Phi = 0$ as well. □

Theorem 5.3.4. *Let K be a local field and let*

$$0 \longrightarrow T' \longrightarrow R \longrightarrow T \longrightarrow 0$$

be a short exact sequence of algebraic K-tori. Assume that the torsion parts of the groups of components $\Phi(R)$ and $\Phi(T')$ have coprime orders. Then the corresponding long exact sequence of the Néron models

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{R} \longrightarrow \mathcal{T} \longrightarrow \mathcal{K} \longrightarrow 0,$$

where $\mathcal{K} := \ker(R^1 j_* T' \longrightarrow R^1 j_* R)$, *induces a sequence*

$$0 \longrightarrow \Phi(T')^{\vee\vee} \longrightarrow \Phi(\mathcal{R}) \longrightarrow \Phi(T) \longrightarrow \Phi(\mathcal{K}) \longrightarrow 0$$

which is exact except perhaps at $\Phi(\mathcal{R})$.

If the sequence $\mathcal{R}^0 \longrightarrow \mathcal{T}^0 \longrightarrow \mathcal{K}^0$ *is exact, then the last sequence is exact at* $\Phi(\mathcal{R})$ *too.*

Proof. We have a commutative diagram in the smooth topology

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \iota^{-1}(\mathcal{R}^0) & \longrightarrow & \mathcal{R}^0 & \longrightarrow & \mathcal{T}^0 & \longrightarrow & \mathcal{K}^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}' & \xrightarrow{\iota} & \mathcal{R} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{K} & \longrightarrow & 0
\end{array}$$

with an exact bottom row and a top row which is exact except perhaps at $\mathcal{R}^0$ and $\mathcal{T}^0$.

If we restrict this diagram to the étale site over $\operatorname{Spec} k$, then exactness is retained. In particular, we now consider the sheaves (without any distinction in the notation!) as continuous $\operatorname{Gal}(k^{\mathrm{sep}}/k)$-modules.

Thus the diagram induces a sequence of the group of components

$$0 \longrightarrow \Phi(T')^{\vee\vee} \longrightarrow \Phi(\mathcal{R}) \longrightarrow \Phi(T) \longrightarrow \Phi(\mathcal{K}) \to 0, \tag{5.1}$$

which is trivially exact at $\Phi(\mathcal{K})$. By 2.2.4, it is exact at $\Phi(T)$. The exactness at $\Phi(T')^{\vee\vee}$ is clear by the assumption on the orders of the torsion parts.

Now assume that the sequence $\mathcal{R}^0 \longrightarrow \mathcal{T}^0 \longrightarrow \mathcal{K}^0$ is exact and let us show that (5.1) is exact at $\Phi(\mathcal{R})$.

All that remains to show is that an element $\bar{x} \in \Phi(\mathcal{R})$ which is mapped to zero in $\Phi(T)$ already has a preimage in $\Phi\,(T')^{\vee\vee}$. To do this, let a preimage $x \in i^*\mathcal{R}$ be chosen: its image z under $i^*\mathcal{R} \longrightarrow i^*\mathcal{T}$ is obviously already in $i^*\mathcal{T}^0$. The image of z in $i^*\mathcal{K}$ is zero, so that by the exactness of the sequence $i^*\mathcal{R}^0 \longrightarrow i^*\mathcal{T}^0 \longrightarrow i^*\mathrm{R}^1 j_*(T')^0$ we can find a preimage x_0 of z in z in $i^*\mathcal{R}^0$. Summarizing, we obtain a section $x - x_0 \in i^*\mathcal{R}$ which is a preimage of $\bar{x}$ and has a preimage in $i^*\mathcal{T}'$. Its image in the group of components $\Phi(T')$ must obviously be a preimage of $\bar{x}$. □

Proposition 5.3.5. *In the setting of Theorem 5.3.4 we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi(T')^{\vee\vee} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X(T')^I, \mathbb{Z}) & \longrightarrow & E(T')^{\mathrm{pd}} & \longrightarrow & 0 \\ & & \alpha \big\downarrow & & \beta \big\downarrow & & & & \\ 0 & \longrightarrow & \Phi(R) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X(R)^I, \mathbb{Z}) & \longrightarrow & E(R)^{\mathrm{pd}} & \longrightarrow & 0 \end{array}$$

where the morphism β is induced by the maps on the Néron models and the morphism α is equal to the dual of the given map $X(R) \longrightarrow X(T')$ when restricted to the I-invariants. Above, $(\cdot)^{\mathrm{pd}}$ denotes Pontryagin dual.

Proof. The assertion corresponds to the assertion of Proposition 5.2.1 applied to the morphism $T' \longrightarrow R$ and dualized via $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$. Since, as abelian groups, the defect terms are finite groups, the functor $\mathrm{Ext}^1_{\mathbb{Z}}(\cdot, \mathbb{Z})$ corresponds to the Pontryagin dual. □

We now use the induced sequences to compute certain group of components .

Proposition 5.3.6. *Let L/K be a finite separable extension of local fields such that $L^{\mathrm{nr}}/K^{\mathrm{nr}}$ is Galois and let T_N be the associated norm-one torus. Then there exists an isomorphism of abelian groups*

$$\Phi\big(R^1 j_* T_N\big) \cong (\mathbb{Z}/p^s\mathbb{Z})^{[L_{\mathrm{nr}}:K]},$$

where $p^s := e_{L/K}/\nu_L(\pi_K) = [l:k]_{\mathrm{ins}}$ is the degree of inseparability of the associated extension of residue fields.

Proof. After changing the base to K^{nr}, the sequence of character groups that defines T_N is of the form

$$0 \longrightarrow \mathbb{Z} \stackrel{\alpha}{\longrightarrow} \bigoplus_{i=1}^{[K^{\mathrm{nr}}:K]} \mathrm{Ind}_{\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{nr}})}^{\mathrm{Gal}(K^{\mathrm{sep}}/L^{\mathrm{nr}})} \mathbb{Z} \longrightarrow X(T_N) \longrightarrow 0$$

Since α is the diagonal embedding, $X(T_N)$ as a cokernel is also a sum and each summand is isomorphic to the character group of the norm-one torus associated to the extension $L^{\mathrm{nr}}/K^{\mathrm{nr}}$. It suffices to consider the case $K = K^{\mathrm{nr}}$ and $L = L^{\mathrm{nr}}$:

Using Theorem 5.3.5, the defining sequence of the norm-one torus yields an exact sequence

$$\Phi(\mathcal{R}) \longrightarrow \Phi(\mathcal{G}) \longrightarrow \Phi\big(\mathrm{R}^1 j_* T_N\big) \longrightarrow 0,$$

where $\mathcal{R} := j_*\mathfrak{R}_{L/K}(\mathbb{G}_{m,L})$ and $\mathcal{G} \colon = j_*\mathbb{G}_{m,K}$. Since the group of components are étale groups, we can consider this sequence relative to the étale topology and consider the terms of the sequence as Galois modules. We obtain

$$\begin{array}{ccccccc} L^* = \mathcal{R}(\mathcal{O}_K) & \xrightarrow{N_{L/K}} & \mathcal{G}(\mathcal{O}_K) = K^* & & & & \\ \nu_L \big\downarrow & & \nu_K \big\downarrow & & & & \\ L^*/\mathcal{O}_L^* = \Phi(\mathcal{R}) & \longrightarrow & \Phi(\mathcal{G}) = K^*/\mathcal{O}_K^* & \longrightarrow & \Phi(\mathrm{R}_1 j_* T_N) & \longrightarrow & 0. \end{array}$$

Thus, the canonical map of the Néron models onto the group of components induces the map $1 = \nu_L(\pi_L) \longmapsto \nu_K\big(N_{L/K}(\pi_L)\big) = \nu_K\Big(\pi_K^{p^s}\Big) = p^s$. □

The above considerations yield another interpretation of the defect term that we found when we examined the free part. We also have the following simple alternative description.

Lemma 5.3.7. *Let T be an algebraic K-torus and recall the sequence (6)*

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0.$$

Let $\mathcal{K} := \ker(R^1 j_*\widetilde{T} \longrightarrow R^1 j_* T)$. Then the defect group can be expressed as $E(T) = \underline{\mathrm{Ext}}^1(\mathcal{K}, i_*\mathbb{Z}) \mid_{(\text{ét})/k} = \underline{\mathrm{Ext}}^1(\Phi(\mathcal{K}), \mathbb{Z})$. Further, the group of components $\Phi(\mathcal{K})$ can be determined from the diagram

$$\begin{array}{ccccccc} T(K^{\mathrm{nr}}) & \longrightarrow & T^I(K^{\mathrm{nr}}) = ((K^{\mathrm{nr}})^*)^{d_I} & & & & \\ \downarrow & & \nu\downarrow & & & & \\ \Phi(T) & \longrightarrow & \Phi(T^I) = \mathbb{Z}^{d_I} & \longrightarrow & \Phi(\mathcal{K}) & \longrightarrow & 0 \end{array}$$

Proof. We have already established the description of the $\underline{\mathrm{Ext}}^1$ terms above. Using the idea applied in the proof of Theorem 5.3.4, the sequence

$$0 \longrightarrow j_*\widetilde{T} \longrightarrow j_*T \longrightarrow j_*T^I \longrightarrow \mathcal{K} \longrightarrow 0$$

induces an exact sequence

$$\Phi(T) \longrightarrow \Phi(T^I) \longrightarrow \Phi(\mathcal{K}) \longrightarrow 0,$$

which can obviously be determined in the étale topology. The diagram in the statement arises by passing from the étale sheaves to the representing Galois modules (on the special fiber) and from the explicit description of the Néron models of algebraic tori with multiplicative reduction. □

The above yields an important statement:

Proposition 5.3.8. *Let K be a local field and let T be an algebraic K-torus which splits over a finite Galois extension L/K. Further, let $d := \mathrm{rank}\big(X(T)^I\big)$, where $I = \mathrm{Gal}(L/K^{\mathrm{nr}})$ is the inertia group of $\mathrm{Gal}(L/K)$. Then the defect term $E(T)$ is a quotient of $(\mathbb{Z}/p^s\mathbb{Z})^d$, where p^s is the degree of inseparability of the corresponding extension of residue fields. In particular, the defect term is trivial if T splits over a non-residually ramified extension.*

Proof. Using Lemma 5.3.7, it suffices to estimate the cokernel of the canonical map $T(K^{\mathrm{nr}}) \longrightarrow \Phi(T^I)$. We have $\Phi(T^I) = \mathbb{Z}^d$ and for an element $\vec{a} = (a_1, .., a_d) \in \Phi(T^I)$ one can construct in $T^I(K^{\mathrm{nr}})$ a pre-image $\vec{\alpha} = (\alpha_1, .., \alpha_{d_I})$ with elements $\alpha_i \in K^{\mathrm{nr}}$ such that $\nu_K(\alpha_i) = a_i$.

By choosing a trivialization, one can write $T^I(K^{\mathrm{nr}}) = \mathrm{Hom}_{\mathbb{Z}}(X(T^I), (K^{\mathrm{nr}})^*)$, so that by choosing a $\mathbb{Z}$-basis $(\chi_i)_{i=1,\dots,d_I}$ of $X(T^I)$ one can identify the point $\vec{\alpha}$ with the map induced by $\chi_i \longmapsto \alpha_i$.

Recall $L^{\mathrm{nr}} = LK^{\mathrm{nr}}$. Then, for a uniformizing element $\pi_{L^{\mathrm{nr}}}$ in $\mathcal{O}_{L^{\mathrm{nr}}}$, we have

$$\nu_K(N_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\pi_{L^{\mathrm{nr}}})) = p^r.$$

Now $T(K^{\mathrm{nr}}) = \mathrm{Hom}_{\mathbb{Z}}(X(T), (L^{\mathrm{nr}})^*)^{\mathrm{Gal}(L^{\mathrm{nr}}/K^{\mathrm{nr}})}$ and we can obtain a base (χ_i) of $X(T)^I$ and add elements (ξ_j) to form a $\mathbb{Z}$-basis of $X(T)$. For a tuple $\vec{b} = (b_1, .., b_d) \in \mathbb{Z}^d$, consider the induced homomorphism $\vec{\beta}$ in $\mathrm{Hom}_{\mathbb{Z}}(X(T),(L^{\mathrm{nr}})^*)$, which depends on the assignment $\chi_i \longmapsto \pi_{L^{\mathrm{nr}}}^{b_i}$ $\xi_j \longmapsto 1$. The norm $\overrightarrow{\beta'} := \sum_{\tau\in \mathrm{Gal}(L^{nr}/K^{\mathrm{nr}})} \tau(\vec{\beta})$ defines an element in $T(K^{\mathrm{nr}})$. This element clearly maps the element χ_i to $\beta_i' := N_{L^{nr}/K^{\mathrm{nr}}}(\pi_{L^{\mathrm{nr}}}^{b_i})$. The image of this element in $T^I(K^{\mathrm{nr}})$ is therefore a preimage of $p^s \cdot \vec{b}$, since $\nu_K(N_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\beta_i')) = p^s b_i$.

Thus, by the above construction in $\Phi(T^I) = \mathbb{Z}^d$, we find preimages for $p^s\mathbb{Z}^d$, so that $E(T)$ must be a quotient of $\mathbb{Z}^d/p^s\mathbb{Z}^d$. In the case of a non-residually ramified splitting extension we have $p^s = 1$, which means that $E(T)$ must be trivial. □

Chapter 6

Main results

In the previous chapter we summarized our results on the group of components in the general case. First we generalized the description from [X, Theorem 3.1] on algebraic tori that split over a tamely ramified extension. Since the defect terms are trivial for these tori, the description is even compatible with homomorphisms. From the generalization for these tori it follows immediately that the results of Xarles also apply to Weil restrictions of such tori, but they need not be compatible with homomorphisms anymore.

Next we consider algebraic tori T which split over a non-residually ramified extension. For these tori we can show that the description of the free part remains valid. We can describe the torsion part only as an extension of the group of components $\Phi\big(R^1 j_* T'\big)$ (for a suitable torus T') by $\mathrm{H}^1(I, X(T))$.

Since the defect terms and $\mathrm{R}^1 j_* T$ are always p-primary torsion sheaves, we give a description of the group of components as a $\mathbb{Z}\big[p^{-1}\big][G_k]$-module. In this setting, Xarles' description can be extended to the general case. However, the description as $\mathbb{Z}\big[p^{-1}\big][G_k]$-modules only covers the prime-to-p part and the isomorphism classes of modules become larger.

We then determine the group of components of the Néron model of norm-one tori T_N with respect to cyclic, Galois and totally ramified extensions L/K of local fields. For these, the torsion part of the group of components is always a quotient of $\mathrm{H}^1(I, X(T_N))$ and $\mathbb{Z}/p^s\mathbb{Z}$, where p^s is the degree of inseparability of the residue field extension associated to L/K. From such a counterexample we construct a torus T for which the free part of the group of components is not isomorphic to $X(T)^I$.

Finally, we look at the p-primary torsion part. We show that the p-primary torsion part is annihilated by multiplication by p^s if there exists a splitting extension such that p^s is the highest power of p in the order of the inertia group of this extension. A general description of the p-primary component has not yet been obtained, but we suspect that the p-primary component will remain bounded. This conjecture would imply that the results of Xarles apply to algebraic tori that split over a non-residually ramified extension.

6.1 Néron models and tame ramification

Now we want to generalize the results of Xarles to algebraic K-tori that split over a tamely ramified extension. We have already seen that the result for the free part holds for algebraic tori that split over a tamely ramified extension.

Theorem 6.1.1. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$ that splits over a finite Galois tamely ramified extension L/K. Further, let $G_K := \mathrm{Gal}(L/K)$ and let I be the inertia group of G_K. Now let*

$$0 \longrightarrow X(T) \longrightarrow J_0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \cdots$$

be a resolution of $X(T)$ by torsion-free, I-acyclic G_K-modules. Then, as a $G_k := G_K/I$-module,

$$\Phi(T) \cong \mathrm{coker}\big(\mathrm{Hom}_{\mathbb{Z}}\big((X')^I, \mathbb{Z}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(J_0^I, \mathbb{Z}\big)\big),$$

where $X' := \ker(J_1 \longrightarrow J_2)$.

Proof. Following Xarles, we consider the short exact sequence of algebraic K-tori

$$0 \longrightarrow T' \longrightarrow R \longrightarrow T \longrightarrow 0$$

induced via Cartier duality by the exact sequence

$$0 \longrightarrow X(T) \longrightarrow J_0 \longrightarrow X' \longrightarrow 0.$$

We may assume without loss of generality that J_0 is an induced $\mathrm{Gal}(L/K)$-module, i.e., the torus R is a Weil restriction of a product of multiplicative groups. This means that the group of components of the Néron model of R is torsion-free. Further, since with this choice of T the tori R and T' also split over the tamely ramified extension L, we obtain a short exact sequence of Néron models

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{R} \longrightarrow \mathcal{T} \longrightarrow 0.$$

Since the defect terms are trivial due to tame ramification, Propositions 5.3.1 and 5.3.5 induce the following commutative diagram of G_k-modules

$$\begin{array}{ccccccc} 0 \longrightarrow & \Phi(T')^{\vee\vee} & \longrightarrow & \Phi(R) & \longrightarrow & \Phi(T) & \longrightarrow 0 \\ & \| & & \| & & & \\ & \mathrm{Hom}_{\mathbb{Z}}((X')^I, \mathbb{Z}) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}}(J_0^I, \mathbb{Z}) & & & \end{array}$$

where α is the linear dual of the restriction of the morphism $J_0 \longrightarrow X'$ to I-invariant subgroups. Since the top row is exact, the claim follows. □

Corollary 6.1.2. *Let T be an algebraic K-torus that splits over a tamely ramified extension L/K. Then $H^1(I, X(T)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z})$ as $G_k = G_K/I$-modules.*

Proof. By Theorem 6.1.1, we have a short exact sequence of G_k-modules

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big((X')^I, \mathbb{Z}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(J_0^I, \mathbb{Z}\big) \longrightarrow \Phi(T) \longrightarrow 0.$$

By applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$, we obtain the exact sequence of G_k-modules

$$\cdots \longrightarrow J_0^I \longrightarrow (X')^I \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \longrightarrow 0.$$

Above we used the fact that X' and J_0 are torsion-free. Now, by the left-exactness of the functor $(\cdot)^I$, it follows that $(X')^I = \ker\big(J_1^I \longrightarrow J_2^I\big)$, whence

$$\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \cong \mathrm{coker}\big(J_0^I \longrightarrow \ker\big(J_1^I \longrightarrow J_2^I\big)\big)$$

as G_k-modules. Since the J_i formed an I-acyclic resolution of $X(T)$, this must be $\mathrm{H}^1(I, X(T))$. □

From Theorem 1.2.1 we also immediately obtain the statement:

Proposition 6.1.3. *Let L/K be a finite separable extension of local fields and let T' be an algebraic L-torus which splits over a tamely ramified extension of L. Then the description from* [X, Theorem 3.1] *applies to the group of components of the Néron model of the K-torus $T := \mathfrak{R}_{L/K}(T')$.*

6.2 Néron models and non-residual ramification

By Proposition 5.3.8 and Theorem 5.1.6, the following holds:

Proposition 6.2.1. *Let K be a local field and let T be an algebraic K-torus which splits over a finite Galois and non-residually ramified extension L/K. Then $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \cong X(T)^I$ as $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules.*

However, since in this case we cannot yet describe the group of components of $\mathrm{R}^1 j_* T$, we can only describe the torsion part of the group of components as an extension:

Proposition 6.2.2. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$ which splits over a finite, Galois and non-residually ramified extension L/K. Let I be the inertia group of $G_K := \mathrm{Gal}(L/K)$. Then there exists an exact sequence of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules*

$$0 \longrightarrow H^1(I, X(T)) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(R^1 j_* Q'), \mathbb{Z}) \longrightarrow 0,$$

where Q' is a suitable torus.

Proof. We first consider the case where $\mathrm{H}^1(I, X(T)) = 0$. By Theorem 5.2.2, the sequence (3)

$$0 \longrightarrow T' \longrightarrow R := \mathfrak{R}_{L/K}(T_L) \longrightarrow T \longrightarrow 0$$

induces a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(R), \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(T'), \mathbb{Z}) & \twoheadrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(\mathcal{N}, \mathbb{Z})|_{(\text{ét})/k} \\
\| & & \| & & \| & & \\
X(T)^I & \hookrightarrow & X(R)^I & \longrightarrow & X(T')^I & \longrightarrow & 0
\end{array}$$

where, according to the notation of the theorem, $\mathcal{N} = \ker\big(\mathcal{T} \longrightarrow \mathrm{R}^1 j_* T'\big)$. The last map on the top row is surjective because $\Phi(\mathcal{R})$ is torsion-free. Since the defect terms vanish by Proposition 5.3.8, the vertical maps are isomorphisms. Now, by the exactness of the bottom row, the top row must be exact at $\underline{\mathrm{Hom}}(\mathcal{R}, i_* \mathbb{Z})$ also. In the long exact sequence

$$\begin{aligned}
0 \longrightarrow \underline{\mathrm{Hom}}(\mathcal{T}, i_*\mathbb{Z}) \xrightarrow{\alpha} \underline{\mathrm{Hom}}(\mathcal{N}, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(K, i_*\mathbb{Z}) \\
\longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{T}, i_*\mathbb{Z}) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{N}, i_*\mathbb{Z})
\end{aligned}$$

the morphism α is an isomorphism after restriction to the étale site over k. From the diagram we see that $\underline{\mathrm{Ext}}^1(\mathcal{N}, i_*\mathbb{Z})\big|_{(\text{ét})/k} = 0$, which yields

$$\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(\mathrm{R}^1 j_* T'), \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}).$$

Since we assumed that $\mathrm{H}^1(I, X(T)) = 0$, we obtain the sequence of the proposition by setting $Q' = T'$.

Now let T be an arbitrary torus that splits after a non-residually ramified extension L/K. Recall the sequence (8)

$$0 \longrightarrow M \longrightarrow Q \longrightarrow T \longrightarrow 0,$$

where M has multiplicative reduction and Q is such that $H^1(I, X(Q)) = 0$. Then Theorem 5.2.2 yields an exact and commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathbb{Z}}(\Phi(T),\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(Q),\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\Phi(M),\mathbb{Z}) & \twoheadrightarrow & N \\ \| & & \| & & \| & & \\ X(T)^I & \hookrightarrow & X(Q)^I & \longrightarrow & X(M) & \twoheadrightarrow & H^1(I, X(T)) \end{array}$$

where $N = \ker[\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T),\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(Q),\mathbb{Z})]$. The diagram shows that $N = \mathrm{H}^1(I, X(T))$, whence there exists a short exact sequence

$$0 \longrightarrow \mathrm{H}^1(I, X(T)) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T),\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\Phi(Q),\mathbb{Z}) \longrightarrow 0.$$

Now, by the first part of the proof, we have $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(Q),\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\Phi(R^1 j_* Q'),\mathbb{Z})$, which yields the proposition. □

6.3 The prime-to-p part

The notation $[p^{-1}]$ always means the localization with respect to the multiplicative system $\{1, p, p^2, ..\}$. Further, G_k denotes the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of the residue field.

We now wish to determine the group of components as $\mathbb{Z}[p^{-1}][G_k]$-module by replacing Φ with its localization $\Phi[p^{-1}] \cong \Phi \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$. The action of the Galois group is the canonical induced action. The p-primary torsion component of the torsion part is annihilated by the localization. But the isomorphism classes also change, since Galois module homomorphisms with coefficients in $\mathbb{Z}[p^{-1}]$ are now permitted instead of just from $\mathbb{Z}$. As we will see, in this coarser setting the results from [X] can be generalized without difficulty.

We first examine how the localization behaves under the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot,\mathbb{Z})$.

Lemma 6.3.1. *Let Φ be a finitely generated continuous G_k-module. Then we have the following isomorphisms of $\mathbb{Z}[p^{-1}][G_k]$-modules*

$$\mathrm{Hom}_{\mathbb{Z}[p^{-1}]}(\Phi[p^{-1}],\mathbb{Z}[p^{-1}]) \cong \mathrm{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]) \cong \mathrm{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z})[p^{-1}]$$
$$\mathrm{Ext}^1_{\mathbb{Z}[p^{-1}]}(\Phi[p^{-1}],\mathbb{Z}[p^{-1}]) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z})[p^{-1}].$$

Proof. It is shown in [Wei, Proposition 3.3.10 and Lemma 3.3.8] that, for a noetherian ring R and a finitely generated R-module A, $\mathrm{Hom}_R(A,B)$ and $\mathrm{Ext}^n_R(A,B)$ are compatible with localization.

This means that in the statement the first and third terms of the two lines are isomorphic as abelian groups. These isomorphisms are compatible with the action of G_k since the action canonically depends on the induced action on Φ (for the trivial action on $\mathbb{Z}$ or $\mathbb{Z}[p^{-1}]$). The isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]) \cong \mathrm{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z})[p^{-1}]$ is clear. For the isomorphism $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z})[p^{-1}]$, note that the canonical $\mathbb{Z}$-injective resolution of $\mathbb{Z}$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

yields, upon tensoring with the flat $\mathbb{Z}$-module $\mathbb{Z}[p^{-1}]$, a $\mathbb{Z}$-injective resolution

$$0 \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] \longrightarrow 0,$$

because quotients of divisible abelian groups are divisible. Thus we obtain

$$\begin{aligned}\operatorname{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z})[p^{-1}] &= (\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Q}/\mathbb{Z})/\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Q}))\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}]\\ &= \operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Q}/\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}]/\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}]\\ &= \operatorname{Hom}_{\mathbb{Z}}(\Phi,(\mathbb{Q}/\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}])/\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}])\\ &= \operatorname{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]).\end{aligned}$$

□

Thus we can determine Φ as $\mathbb{Z}[p^{-1}][G_k]$-module using the functors $\mathrm{R}^i\operatorname{Hom}_{\mathbb{Z}}(\cdot,\mathbb{Z}[p^{-1}])$ for $i=0,1$.

Now let T be an algebraic K-torus. Using Proposition 5.1.1, the short exact sequence

$$0\longrightarrow j_*T^0\longrightarrow j_*T\longrightarrow i_*\Phi(T)\longrightarrow 0$$

induces an isomorphism in the smooth and étale topologies

$$\underline{\operatorname{Hom}}(j_*T,i_*\mathbb{Z}[p^{-1}])\cong\underline{\operatorname{Hom}}(i_*\Phi(T),i_*\mathbb{Z}[p^{-1}]).$$

In the smooth topology, we also have an isomorphism

$$\underline{\operatorname{Ext}}^1(j_*T,i_*\mathbb{Z}[p^{-1}])\cong\underline{\operatorname{Ext}}^1(i_*\Phi(T),i_*\mathbb{Z}[p^{-1}]).$$

Using Propositions 5.1.3 and 5.1.4, we obtain the following equalities

$$\begin{aligned}\underline{\operatorname{Hom}}(i_*\Phi(T),i_*\mathbb{Z}[p^{-1}]) &= i_*\operatorname{Hom}_{\mathbb{Z}}(\Phi(T),\mathbb{Z}[p^{-1}])\\ \underline{\operatorname{Ext}}^1(i_*\Phi(T),i_*\mathbb{Z}[p^{-1}]) &= i_*\operatorname{Ext}^1_{\mathbb{Z}}(\Phi(T),\mathbb{Z}[p^{-1}]),\end{aligned}$$

where on the right-hand side the abelian sheaves are represented by the étale group scheme associated to the respective Galois module.

For a finitely generated continuous G_k-module Φ, one has a canonical short exact sequence

$$\operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z}),\mathbb{Z})\hookrightarrow\Phi\twoheadrightarrow\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z}),\mathbb{Z}),$$

which corresponds to the decomposition of Φ into a torsion part (as a submodule) and a free part (as a quotient modulo the torsion part). As seen above, after tensoring with $\mathbb{Z}[p^{-1}]$, the above sequence becomes isomorphic to the sequence

$$\operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{Ext}^1_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]),\mathbb{Z}[p^{-1}])\hookrightarrow\Phi[p^{-1}]\twoheadrightarrow\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z}[p^{-1}]),\mathbb{Z}[p^{-1}]),$$

and this obviously corresponds to the decomposition of $\Phi[p^{-1}]$ into a torsion part and a free part in the category of $\mathbb{Z}[p^{-1}][G_k]$-modules.

Proposition 6.3.2. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Let L/K be a finite Galois splitting extension for T and let I be the inertia group of* Gal(L/K)*. Then*

$$\operatorname{Hom}_{\mathbb{Z}}(\Phi(T),\mathbb{Z}[p^{-1}])\cong X(T)^I\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}]$$

in the category of continuous $\mathbb{Z}[p^{-1}][G_k]$-modules. This description is compatible with homomorphisms of algebraic tori.

Proof. Recall the sequence (6)

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0.$$

This induces a long exact sequence of the Néron models from which we obtain an exact sequence

$$0 \longrightarrow j_*\widetilde{T} \longrightarrow j_*T \longrightarrow j_*T^I \longrightarrow \mathcal{K} \longrightarrow 0,$$

where $\mathcal{K} := \ker(\mathrm{R}^1 j_*\widetilde{T} \longrightarrow \mathrm{R}^1 j_*T)$. If we split the latter sequence into two short exact sequences as in 5.1.6 and apply the functor $\mathrm{Hom}(\cdot, i_*\mathbb{Z}[p^{-1}])$, we obtain

$$0 \longrightarrow \underline{\mathrm{Hom}}(\mathcal{N}, i_*\mathbb{Z}[p^{-1}]) \longrightarrow \underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z}[p^{-1}]) \longrightarrow \underline{\mathrm{Hom}}(j_*\widetilde{T}, i_*\mathbb{Z}[p^{-1}]) = 0$$

and

$$0 = \underline{\mathrm{Hom}}(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]) \longrightarrow \underline{\mathrm{Hom}}(j_*T^I, i_*\mathbb{Z}[p^{-1}]) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{N}, i_*\mathbb{Z}[p^{-1}]) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]).$$

By Proposition 4.2.7, we have $\underline{\mathrm{Ext}}^1(\mathcal{K}, i_*\mathbb{Z}[p^{-1}]) = 0$. This means that

$$\underline{\mathrm{Hom}}(j_*T^I, i_*\mathbb{Z}[p^{-1}]) \cong \underline{\mathrm{Hom}}(j_*T, i_*\mathbb{Z}[p^{-1}]) = \underline{\mathrm{Hom}}(i_*\Phi(T), i_*\mathbb{Z}[p^{-1}]).$$

Considering this in the étale topology over k, this induces an isomorphism $X(T)^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] \cong \mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}[p^{-1}])$ of the representing Galois modules.

Now let $\psi\colon T_1 \longrightarrow T_2$ be a homomorphism of algebraic tori. This corresponds to a homomorphism $D(\psi)\colon X(T_2) \longrightarrow X(T_1)$ of the character groups. We now consider the exact and commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{T_1} & \longrightarrow & T_1 & \longrightarrow & T_1^I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{T_2} & \longrightarrow & T_2 & \longrightarrow & T_2^I & \longrightarrow & 0. \end{array}$$

The above diagram induces a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(j_*T_1^I, i_*\mathbb{Z}[p^{-1}]) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}(j_*T_1, i_*\mathbb{Z}[p^{-1}]) \\ \uparrow & & \uparrow \\ \underline{\mathrm{Hom}}(j_*T_2^I, i_*\mathbb{Z}[p^{-1}]) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}(j_*T_2, i_*\mathbb{Z}[p^{-1}]). \end{array}$$

Using the description of the group of components in the case of multiplicative reduction, restricting to the étale site over k and passing to the representing Galois modules, we see that the vertical maps above correspond exactly to the morphism $D(\psi)^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]\colon X(T_2)^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] \longrightarrow X(T_1)^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$. □

We now show that the description of the torsion part in the category of continuous $\mathbb{Z}[p^{-1}][G_k]$-modules also remains valid:

Proposition 6.3.3. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Let L/K be a finite Galois splitting extension for T and let I be the inertia group of* $\mathrm{Gal}(L/K)$. *Then*

$$\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}[p^{-1}]) \cong H^1(I, X(T)) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$$

in the category of continuous $\mathbb{Z}[p^{-1}][G_k]$-modules.

Proof. Recall the sequence (3)

$$0 \longrightarrow T' \longrightarrow R := \mathfrak{R}_{L/K}(T_L) \longrightarrow T \longrightarrow 0,$$

where L/K is a finite Galois splitting extension for T. From this sequence we obtain an exact sequence by passing to the Néron models

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{R} \longrightarrow \mathcal{T} \longrightarrow \mathrm{R}^1 j_* T' \longrightarrow 0$$

Note that, by Corollary 4.2.7, we have $\underline{\mathrm{Ext}}^i(\mathrm{R}^1 j_* T', i_*\mathbb{Z}[p^{-1}]) = 0$ for $i = 1, 2$. Thus, if we split the latter sequence into two short exact sequences and apply $\mathrm{Hom}(\cdot, i_*\mathbb{Z}[p^{-1}])$ to these sequences, we obtain an exact sequence (cf. proof of Theorem 5.2.2)

$$\underline{\mathrm{Hom}}(\mathcal{T}, i_*\mathbb{Z}[p^{-1}]) \hookrightarrow \underline{\mathrm{Hom}}(\mathcal{R}, i_*\mathbb{Z}[p^{-1}]) \to \underline{\mathrm{Hom}}(\mathcal{T}', i_*\mathbb{Z}[p^{-1}]) \to \underline{\mathrm{Ext}}^1(\mathcal{T}, i_*\mathbb{Z}[p^{-1}]).$$

Now, if we restrict the above sequence to the étale site over k and pass from the Néron models to their groups of components, we obtain by Proposition 6.3.2 an exact and commutative diagram of G_k-modules

$$\begin{array}{ccccccc}
\underline{\mathrm{Hom}}(\Phi(T), i_*\mathbb{Z}[p^{-1}]) & \hookrightarrow & \underline{\mathrm{Hom}}(\Phi(R), i_*\mathbb{Z}[p^{-1}]) & \longrightarrow & \underline{\mathrm{Hom}}(\Phi(T'), i_*\mathbb{Z}[p^{-1}]) & \twoheadrightarrow & E \\
\| & & \| & & \| & & \\
X(T)^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & \hookrightarrow & X(R)^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & \longrightarrow & X(T')^I \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & \twoheadrightarrow & H,
\end{array}$$

where E and H denote, respectively, $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}[p^{-1}])$ and $\mathrm{H}^1(I, X(T)) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$. The proposition follows immediately from this diagram. □

We now establish the corresponding generalization of [X, 3.1].

Theorem 6.3.4. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Let L/K be a finite Galois extension such that T splits over L. Further, let I be the inertia group of $G_K := \mathrm{Gal}(L/K)$ and let*

$$0 \longrightarrow X(T) \longrightarrow J_0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \cdots$$

be a resolution of $X(T)$ via torsion-free and I-acyclic G_K-modules. Then

$$\Phi(T) \otimes_{\mathbb{Z}} \mathbb{Z}\left[p^{-1}\right] \cong \mathrm{coker}\left(\mathrm{Hom}_{\mathbb{Z}}\left((X')^I, \mathbb{Z}\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(J_0^I, \mathbb{Z}\right)\right) \otimes_{\mathbb{Z}} \mathbb{Z}\left[p^{-1}\right],$$

where $X' := \ker(J_1 \longrightarrow J_2)$.

Proof. We consider the short exact sequence

$$0 \longrightarrow T' \longrightarrow R \longrightarrow T \longrightarrow 0$$

of algebraic K-tori, which arises via Cartier duality from the sequence

$$0 \longrightarrow X(T) \longrightarrow J_0 \longrightarrow X' \longrightarrow 0.$$

We can assume without loss of generality that J_0 is an induced $\mathrm{Gal}(L/K)$-module, i.e., the torus R is a Weil restriction of a product of multiplicative groups. This means that the group of components of the Néron model of R is torsion-free and we have $\mathrm{R}^1 j_* R = 0$. So we obtain a long exact sequence

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{R} \longrightarrow \mathcal{T} \longrightarrow \mathrm{R}^1 j_* T' \longrightarrow 0.$$

This induces a sequence via Proposition 5.3.4

$$0 \longrightarrow \Phi(T')^{\vee\vee} \longrightarrow \Phi(R) \longrightarrow \Phi(T) \longrightarrow \Phi(\mathrm{R}^1 j_* T') \longrightarrow 0$$

of group of components, which is exact at $\Phi(T)$. After tensoring with the flat $\mathbb{Z}$-module $\mathbb{Z}[p^{-1}]$, we obtain $\mathrm{R}^1 j_* T' \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] = 0$ and thus, by Proposition 5.3.5, a commutative diagram

$$\begin{array}{ccccc}
\Phi(T')^{\vee\vee} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & \overset{\alpha}{\hookrightarrow} & \Phi(R) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & \overset{\beta}{\twoheadrightarrow} & \Phi(T) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] \\
\| & & \| & & \\
\mathrm{Hom}_{\mathbb{Z}}((X')^I, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & \overset{\alpha}{\longrightarrow} & \mathrm{Hom}_{\mathbb{Z}}(X(R)^I, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] & &
\end{array}$$

Since the p-primary torsion defect terms become trivial after localization, the vertical maps in Proposition 5.3.5 become isomorphisms in this case. Since, a priori, the upper sequence may not be exact at $\Phi(R) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$, we consider the isomorphism

$$\begin{aligned}
\Phi(T) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] &\cong \Phi(R)/\ker(\beta) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] \\
&\cong \big(\Phi(R)/\operatorname{im}(\alpha)/\ker(\beta)/\operatorname{im}(\alpha)\big) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}].
\end{aligned}$$

By dimension reasons, $\ker(\beta)/\operatorname{im}(\alpha)$ must be a torsion group. Now the torsion components of $\Phi(T) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ and $\Phi(R)/\operatorname{im}(\alpha)$ are isomorphic via the map α constructed in Proposition 6.3.3. Since the torsion parts are finitely generated, i.e., finite, we must have $(\ker(\beta)/\operatorname{im}(\alpha)) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}] = 0$ and thus the theorem follows. □

6.4 The group of components for norm-one tori

In this section we describe the group of components of the special fiber of the Néron model of a norm-one torus using exact sequences of the groups of components.

Proposition 6.4.1. *Let T_N be a norm-one torus with respect to a finite separable extension L/K of local fields such that $L^{\mathrm{nr}}/K^{\mathrm{nr}}$ is a cyclic Galois extension. Let $e_{L/K} := \nu_L(\pi_K)$ be the ramification index of L/K, let $I := \mathrm{Gal}(L/K)$ be the inertia group and let $f := [K^{\mathrm{nr}} \colon K]$ be the separable degree of the extension of the residue fields. Further, let p^s be the degree of inseparability of the extension of the residue fields, so that $[L^{\mathrm{nr}} \colon K^{\mathrm{nr}}] = p^s e_{L/K}$.*

Then $\Phi(T_N) = \big(\mathbb{Z}/e_{L/K}\mathbb{Z}\big)^f$ and $H^1(I, X(T_N)) = \big(\mathbb{Z}/p^s e_{L/K}\mathbb{Z}\big)^f$. Further, there exists an exact sequence of abelian groups

$$0 \longrightarrow \Phi(T_N) \longrightarrow H^1(I, X(T_N)) \longrightarrow \Phi(R^1 j_* T_N) \longrightarrow 0.$$

Proof. As shown in Proposition 5.3.6, after changing the base to K^{nr}, the torus T_N becomes isomorphic to the f-fold product of the norm-one torus T_N^{nr} associated to $L^{\mathrm{nr}}/K^{\mathrm{nr}}$. Since group of components and cohomology are compatible with fiber products, respectively, sums, it suffices to consider the case $K = K^{\mathrm{nr}}$. Since the extension $L^{\mathrm{nr}}/K^{\mathrm{nr}}$ is cyclic, the norm-one torus T_N is isomorphic to the torus S with character group $\mathrm{Hom}_{\mathbb{Z}}(X(T_N), \mathbb{Z})$ by [LL, Lemma 4.1]. Consequently, T_N admits the resolution

$$0 \longrightarrow \mathbb{G}_{m,K^{\mathrm{nr}}} \longrightarrow \mathfrak{R}_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\mathbb{G}_{m,L^{\mathrm{nr}}}) \longrightarrow T_N \longrightarrow 0. \tag{6.1}$$

The above sequence induces an exact sequence of Néron models which, by Proposition 5.3.1, induces, in turn, an exact sequence of the associated groups of components

$$0 \longrightarrow \Phi(\mathbb{G}_{m,K^{\mathrm{nr}}}) \longrightarrow \Phi(\mathfrak{R}_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\mathbb{G}_{m,L^{\mathrm{nr}}})) \longrightarrow \Phi(T_N) \longrightarrow 0.$$

The latter sequence is isomorphic to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \Phi(T_N) \longrightarrow 0.$$

The K^{nr}-valued points of the map $\mathbb{G}_{m,K^{\mathrm{nr}}} \to \mathfrak{R}_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\mathbb{G}_{m,L^{\mathrm{nr}}})$ is the inclusion $(K^{\mathrm{nr}})^* \longrightarrow (L^{\mathrm{nr}})^*$. However, as already seen, the groups $\Phi(\mathbb{G}_{m,K^{\mathrm{nr}}})$ and $\Phi(\mathfrak{R}_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\mathbb{G}_{m,L^{\mathrm{nr}}}))$ are generated by the images of the uniformizing elements, so that we must have $\Phi(T_N) = \mathbb{Z}/e_{L/K}\mathbb{Z}$.

The exact sequence (6.1) corresponds to the exact sequence of character groups

$$0 \longrightarrow X(T_N) \longrightarrow \mathbb{Z}[I] \xrightarrow{\text{aug.}} \mathbb{Z} \longrightarrow 0,$$

that is, the group of characters of T_N is the kernel of the augmentation map. The corresponding long exact sequence in I-cohomology yields the exact sequence

$$\cdots \longrightarrow \mathbb{Z}[I]^I \xrightarrow{\text{aug.}} \mathbb{Z} \longrightarrow \mathrm{H}^1(I, X(T_N)) \longrightarrow \mathrm{H}^1(I, \mathbb{Z}[I]) = 0.$$

Now the I-invariant elements of $\mathbb{Z}[I]$ are of the form $\sum_{\sigma \in I} k e_\sigma$ with $k \in \mathbb{Z}$, whence $\mathrm{H}^1(I, X(T_N)) = \mathbb{Z}/p^s e_{L/K}\mathbb{Z}$.

Now there exists a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & j_*T_N & & & & \\
 & & & & \big\downarrow & & & & \\
0 & \longrightarrow & j_*\mathbb{G}_{m,K^{\mathrm{nr}}} & \longrightarrow & j_*\mathfrak{R}_{L^{\mathrm{nr}}/K^{\mathrm{nr}}}(\mathbb{G}_{m,L^{\mathrm{nr}}}) & \longrightarrow & j_*T_N & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow & & & & \\
 & & \mathbb{G}_{m,K^{\mathrm{nr}}} & \xrightarrow{(\cdot)^{p^s e_{L/K}}} & \mathbb{G}_{m,K^{\mathrm{nr}}} & & & & \\
 & & & & \big\downarrow & & & & \\
 & & & & \mathrm{R}^1 j_* T_N & & & &
\end{array}$$

The middle column corresponds to the long exact sequence of the Néron models associated to the canonical sequence of the norm-one torus. The top row is the sequence of the Néron models associated to (6.1). Since the norm map on $(K^{\mathrm{nr}})^*$ corresponds to the exponentiation with $p^s e_{L/K}$, the diagram is commutative. After passing to the group of components, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot e_{L/K}} & \mathbb{Z} & \longrightarrow & \Phi(T_N) & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow{\scriptstyle \cdot p^s} & & \vdots & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot p^s e_{L/K}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^s e_{L/K}\mathbb{Z} & \longrightarrow & 0 \\
 & & & & \big\downarrow & & & & \\
 & & & & \Phi(\mathrm{R}^1 j_* T_N). & & & &
\end{array}$$

The dashed arrow on the right-hand column exists by commutativity of the diagram and the isomorphism on the left-hand column. The claim now follows via the snake lemma if one notes that the map

$$\Phi(T_N) \longrightarrow \mathbb{Z}/p^s e_{L/K}\mathbb{Z} \cong \mathrm{H}^1(I, X(T_N))$$

must be injective. □

In general, however, the groups $\mathrm{H}^1(I, X(T))$ and $\Phi\big(\mathrm{R}^1 j_* T\big)$ are not related. To show this, we consider the field $K := \mathbb{F}_3(X)((\pi))^{\mathrm{nr}}$ and the extension $L = K[Y]/\big(Y^3 + 2\pi Y + X\big)$. This extension is clearly separable and induces an inseparable extension of the residue fields. If $\bar{y}_0 \in L$ is a solution of the equation, then $\bar{y}_1 := \bar{y}_0 + \sqrt{\pi}$ and $\bar{y}_2 := \bar{y}_0 + 2\sqrt{\pi}$ are solutions, since $\sqrt{\pi}\pi + 2\pi\sqrt{\pi} = 3\sqrt{\pi}\pi = 0$. Thus the extension is not Galois and the normal closure L^{nor} arises from the adjunction of a root of the unifomizing element. In particular, L^{nor}/K is totally ramified. Since permuting the two roots of π also induces a permutation of the roots of $Y^3 + 2\pi Y X$, we have $\mathrm{Gal}(L^{\mathrm{nor}}/K) = \mathbb{S}_3$. By [LL, Proposition 4.17(c)], we conclude that $\mathrm{H}^1(\mathbb{S}_3, X(T_N)) = 0$ for the norm-one torus T_N associated to the extension L/K. On the other hand, it follows from Proposition 5.3.6 that $\Phi\big(\mathrm{R}^1 j_* T_N\big) = \mathbb{Z}/3\mathbb{Z}$. Thus the group of components of $\mathrm{R}^1 j_* T_N$ can be nontrivial even if $\mathrm{H}^1(I, X(T_N)) = 0$.

6.5 An example of the free part

Using our computations for norm-one tori with respect to totally ramified Galois extensions of degree p, we will now construct a family of examples which show that, in general, the free part is no longer isomorphic to $X(T)^I$. So let $p = \mathrm{char}(k)$ be a prime number and consider a $(p+1)$-dimensional torus T which splits over an extension L/K with Galois group $\mathrm{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. More precisely, let σ be a generator of the first factor and let τ be a generator of the second factor. Without loss of generality, we assume that the second factor $\langle\tau\rangle$ corresponds to the inertia group of L/K.

We now let $\mathrm{Gal}(L/K)$ act on $X(T) = \mathbb{Z}^{p+1}$ via the matrices

$$M_\sigma := \begin{pmatrix} & & & 0 \\ & I_p & & \vdots \\ & & & 0 \\ 1 & \dots & 1 & -1 \end{pmatrix} \quad M_\tau := \begin{pmatrix} & & & 0 \\ & Z & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

where I_p is the $(p \times p)$-identity matrix and Z is the $(p \times p)$-matrix that permutes cyclically the basis vectors $e_1, .., e_p$. One computes that M_τ^p and M_σ^2 are each the identity matrix and that $M_\tau M_\sigma = M_\sigma M_\tau$. So such an action of the Galois group is defined on $X(T)$. Now consider the sequence

$$0 \longrightarrow (\tau - \mathrm{Id})X(T) \longrightarrow X(T) \xrightarrow{p} X(T)_I \longrightarrow 0. \tag{6.5}$$

The image of $\tau - \mathrm{Id}$ is a submodule because τ and σ commute. This submodule has rank $p-1$, which is easy to read off the explicit shape of the associated matrix. We define the linear map

$$\psi\colon X(T) \longrightarrow \mathbb{Z}^2 \quad (a_1, \dots, a_p, a_{p+1})^t \longmapsto \left(\sum_{i=1}^{p} a_i, a_{p+1}\right)^t$$

Since τ permutes only the first p components, we have $\psi \circ \tau = \psi$, whence $(\tau - \mathrm{Id})(X(T)) \subset \ker(\psi)$. Conversely, let $\vec{a} := (a_1, \dots, a_{p+1})^t \in \ker(\psi)$ be given. Then $a_{p+1} = 0$ and $\sum_{i=1}^{p} a_i = 0$. For the vector

$$\vec{b} = (b_1, \dots, b_{p+1})^t := \left(-\sum_{i=1}^{1} a_1, -\sum_{i=1}^{2} a_i, \dots, -\sum_{i=1}^{p-1} a_i, -\sum_{i=1}^{p} a_i, 0\right)^t$$

we have $\tau(\vec{b}) - \vec{b} = (b_p - b_1, b_1 - b_2, \dots, b_{p-1} - b_p, 0) = (a_1, .., a_{p-1}, a_p, 0)^t = \vec{a}$. Thus, ψ corresponds to the projection $X(T) \longrightarrow X(T)_I$. The action of σ on $X(T)$ induces the action on

$X(T)_I$ such that

$$\begin{array}{ccc}
(a_1, a_2, \ldots, a_p, a_{p+1})^t & \xmapsto{\sigma_{X(T)}} & (a_1, a_2, \ldots, a_p, \sum_{i=1}^{p} a_i - a_{p+1})^t \\
\psi \downarrow & & \psi \downarrow \\
(\sum_{i=1}^{p} a_i, a_{p+1})^t & \xmapsto{\sigma_{X(T)_I}} & (\sum_{i=1}^{p} a_i, \sum_{i=1}^{p} a_i - a_{p+1})^t
\end{array}$$

The τ-invariants of $X(T)$ are also isomorphic to $\mathbb{Z}^2$ and correspond to the span $\langle (1, \ldots, 1, 0)^t, (0, \ldots, 0, 1)^t \rangle$. We take these vectors as a basis. The action of σ on $X(T)$ induces the following action on $X(T)^I$

$$\begin{aligned}
(a, b)^t &= a(1, \ldots, 1, 0)^t + b(0, \ldots, 0, 1)^t \\
&\longmapsto a(1, \ldots, 1, 0)^t + (pa - b)(0, \ldots, 0, 1)^t = (a, pa - b)^t.
\end{aligned}$$

In line with this, the long I-cohomology sequence induced by (6.5) is the sequence

$$0 \longrightarrow 0 \longrightarrow X(T)_I \xrightarrow{\psi^I} X(T)^I \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where, in the coordinates just chosen, the map ψ^I is given by

$$(a, b)^t \longmapsto \psi\big((a, \ldots, a, b)^t\big) = (pa, b).$$

Now the actions of σ correspond to the matrices

$$M_{\sigma_{X(T)^I}} = \begin{pmatrix} 1 & 0 \\ p & -1 \end{pmatrix} \quad M_{\sigma_{(X(T)_I)}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

and these matrices are not conjugate over $\mathbb{GL}_2(\mathbb{Z})$ since a conjugate matrix would have to be of the form

$$\begin{pmatrix} a & 0 \\ c & p(a + c) \end{pmatrix},$$

i.e., with a non-invertible determinant in $\mathbb{Z}$. This means that $X(T)^I$ and $X(T)_I$ are not isomorphic as $\mathbb{Z}[\langle\sigma\rangle]$-modules.

We now consider the short exact sequence of algebraic tori

$$0 \longrightarrow T_I \longrightarrow T \longrightarrow T' \longrightarrow 0$$

that corresponds to the sequence (6.5) under Cartier duality. By Proposition 5.3.1, this yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X(T)_I, \mathbb{Z}) \longrightarrow \Phi(T) \longrightarrow \Phi(T') \longrightarrow 0.$$

After changing base to K^{nr}, the torus T' is isomorphic to the norm-one torus of the extension L/K^{nr}, which can be seen from the character groups: the norm-one torus of a cyclic extension of degree p has character group $X(T_N) = \mathbb{Z}^p/\mathbb{Z}(\delta_1 + \cdots + \delta_p)$, where $\delta_1, \ldots, \delta_p$ is a basis of $\mathbb{Z}^p$. The Galois group $\mathbb{Z}/p\mathbb{Z} = \langle\tau\rangle$ acts through $\tau(\delta_i) = \delta_{i+1}$ for $1 \leq i \leq p - 1$ and $\tau(\delta_p) = \delta_1$ on the basis of $\mathbb{Z}^p$ and this action induces the action on $X(T_N)$.

Now we have for $(\tau - \mathrm{Id})(X(T)) \subset X(T) = \mathbb{Z}^{p+1}$ a basis $(\tilde{e}_i)_{i=1,\dots,p-1}$ with $\tilde{e}_i := e_i - e_p$. The assignment

$$\begin{aligned}
\delta_1 &\longmapsto \tilde{e}_1\\
\delta_2 = \tau(\delta_1) &\longmapsto \tau(\tilde{e}_1) = \tau(e_1 - e_p) = e_2 - e_1 = \tilde{e}_2 - \tilde{e}_1\\
\delta_3 = \tau(\delta_2) &\longmapsto \tau(\tilde{e}_2 - \tilde{e}_1) = \tilde{e}_3 - \tilde{e}_1 - \tilde{e}_2 + \tilde{e}_1 = \tilde{e}_3 - \tilde{e}_2\\
&\vdots\\
\delta_{p-1} = \tau(\delta_{p-2}) &\longmapsto \tilde{e}_{p-1} - \tilde{e}_{p-2}\\
\delta_p = \tau(\delta_{p-1}) &\longmapsto \tau(e_{p-1} - e_p - e_{p-2} + e_p) = e_p - e_{p-1} = -\tilde{e}_{p-1}
\end{aligned}$$

induces a *surjective* linear map $\mathbb{Z}^p \longrightarrow (\tau - \mathrm{Id})(X(T))$, which can be seen from the matrix that represents it. Further, a vector $\sum_{i=1}^p a_i\delta_i$ is mapped to zero if and only if all a_i agree. By construction, this map is compatible with the action of $\mathrm{Gal}\,(L/K^{\mathrm{nr}})$, so that altogether we have an isomorphism of Galois modules

$$X(T_N) \cong (\tau - \mathrm{Id})(X(T)) = X(T').$$

However, as we saw in the explicit calculation of the Néron model of norm-one tori with respect to Galois totally ramified extensions of degree p, T' has a trivial group of components if L/K^{nr} induces a degree p purely inseparable extension of the residue field. In this case we must have $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \cong X(T)_I$, which is therefore not isomorphic to $X\big(T^I\big)$.

6.6 The *p*-primary torsion part and open questions

Unfortunately, with the results obtained so far it is not yet possible to provide a comprehensive description of the group of components of the Néron model in the general case, as we still know too little about the p-primary torsion part. However, we can estimate the order of the elements of the p-primary part.

If the group of components $\Phi(T)$ of an algebraic torus T can be described using the results of Xarles, then $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi, \mathbb{Z}) = \mathrm{H}^1(I, X(T))$ for the inertia group I of a splitting extension of T. This means that every element of the torsion part of $\Phi(T)$ is annihilated by multiplication by the order of I [S, VIII, §2, Corollary 2]. In particular, the p-primary torsion part is annihilated by the highest power of p that divides the order of I. We can also establish this estimate independently of the validity of the description found in [X]:

Proposition 6.6.1. *Let T be an algebraic K-torus and let L/K^{nr} be a finite Galois splitting extension of $T_{K^{\mathrm{nr}}}$. Let n be the order of $I := \mathrm{Gal}(L/K^{\mathrm{nr}})$. Then the torsion part of $\Phi(T)$ is annihilated by n. In particular, the p-primary torsion component is annihilated by the order of the p-primary part of I.*

Proof. It suffices to consider the torus $T_{K^{\mathrm{nr}}}$. Thus let $K = K^{\mathrm{nr}}$. We define $R := \mathfrak{R}_{L/K}(T_L)$ and consider the canonical immersion $T \hookrightarrow R$. We set $G := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $G_L := \mathrm{Gal}(K^{\mathrm{sep}}/L)$ and denote by $\sigma_1, \dots, \sigma_n$ a representative system for the G_L-cosets of G.

The map $T \hookrightarrow R$ has the following form on the character groups

$$\mathrm{Ind}_G^{G_L} X(T_L) \longrightarrow X(T), \quad (x_1, .., x_n) \longmapsto \sum_{i=1}^n \sigma_i(X).$$

Conversely, we can define a canonical map $R \longrightarrow T$ which, on character groups, has the form (Proposition 0.4.4)

$$X(T) \longrightarrow \operatorname{Ind}_G^{G_L} X(T_L)\,, \quad x \longmapsto \left(\sigma_1^{-1}(X), .., \sigma_n^{-1}(X)\right).$$

The composition $T \hookrightarrow R \longrightarrow T$ induces the multiplication by n on the character groups, so that the corresponding map of the tori is also the multiplication by n (in additive notation of the group law on T).

By the Néron mapping property, we obtain morphisms of the corresponding Néron models and these induce morphisms between groups of components

$$\Phi(T) \longrightarrow \Phi(R) \longrightarrow \Phi(T).$$

Now one can see from the proof of Proposition 1.2.1 that $\Phi(R) = \operatorname{Ind}_G^{G_L} \Phi(T_L)$. Since T_L is split, $\Phi(T_L)$ cannot have torsion. So $\Phi(R)$ has no torsion either.

This means that, a fortiori, the torsion part of $\Phi(T)$ is in the kernel of the map

$$\Phi(T) \longrightarrow \Phi(R) \longrightarrow \Phi(T)$$

and this is the multiplication by n map.

The statement about the p-primary torsion part is elementary. □

The above result is an analogue of [ELL, Theorem 1], which, to our knowledge, is the only general result on the p-primary torsion part of groups of components of Néron models of abelian varieties.

Finally, we would like to formulate the solutions we assumed as hypotheses for some obvious open questions and show cross-relationships and consequences of these hypotheses.

The first important unsolved problem is the question of whether the order of the torsion part of the group of components, as is the case in the examples with the norm-one tori, is bounded. One approach to test this is provided by the following conjecture:

Conjecture 6.6.2 (generalization of [X, 2.7]). Let K be a local field and let T be an algebraic K-torus whose character group satisfies $H^1(I, X(T)) = 0$. Then $\Phi(T)$ is torsion-free.

This conjecture would enable a description of the torsion part of the group of components as in [X, 2.14]:

Proposition 6.6.3. *Let K be a local field and let T be an algebraic K-torus. Suppose further that Conjecture 6.6.2 holds. Then the torsion part of the group of components of the Néron model of T can be written as a quotient*

$$\operatorname{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) = H^1(I, X(T))/E,$$

where E is an appropriate p-group. Further, E is trivial if T splits over a non-residually ramified extension.

Proof. Recall again the sequence (8)

$$0 \longrightarrow M \longrightarrow Q \longrightarrow T \longrightarrow 0,$$

where M is a torus with multiplicative reduction and Q is a torus such that $\mathrm{H}^1(I, X(Q)) = 0$. Since $\mathrm{R}^1 j_* M = 0$, we obtain from Theorem 5.2.2 a commutative diagram of G_k-modules

$$\begin{array}{ccccccccc}
 & & E(T) & & E(Q) & & 0 & & \\
 & & \uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(Q), \mathbb{Z}) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\Phi(M), \mathbb{Z}) & \twoheadrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) \\
 & & \uparrow & & \uparrow & & \cong\uparrow & & \\
0 & \longrightarrow & X(T)^I & \longrightarrow & X(Q)^I & \longrightarrow & X(M)^I & \twoheadrightarrow & H^1(I, X(T))
\end{array}$$

The surjectivity at the end of the middle row follows from the conjecture. The surjectivity at the end of the bottom row follows from the vanishing $\mathrm{H}^1(I, X(Q)) = 0$. By the commutativity, the diagram can clearly be prolonged without losing commutativity with a map $E(Q) \longrightarrow 0$ on the first row.

Now apply the snake lemma to the two middle columns. This produces an exact sequence

$$E(Q) \longrightarrow \mathrm{H}^1(I, X(T)) \longrightarrow \underline{\mathrm{Ext}}^1(\Phi(T), i_*\mathbb{Z}) \longrightarrow 0.$$

Since $E(Q)$ is a p-group as a defect term for Q, the statement follows. The last assertion in the statement is clear since in this case Q also splits over a non-residually ramified extension, giving $E(Q) = 0$. □

A counterexample to the conjecture 6.6.2 naturally also yields a counterexample to the conclusion from this theorem. Perhaps an answer to this conjecture could be found through further study of the explicit construction of Néron models.

From the conjecture above, it is easy to derive the following weaker conjecture:

Conjecture 6.6.4. Let K be a local field and let T be an algebraic K-torus which splits over a finite Galois and non-residually ramified extension L/K. Then $R^1 j_* T$ is connected.

The validity of this assumption would imply that deviations from Xarles' description can only occur if the torus under consideration splits over a residually ramified field extension.

Proposition 6.6.5. *Assume that conjecture 6.6.4 holds. Let K be a local field and let T be an algebraic K-torus which splits over a finite Galois and non-residually ramified extension L/K. Then the results of Xarles apply to T.*

Proof. In Proposition 6.2.1 we showed that $\mathrm{Hom}_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) = X(T)^I$. By the conjecture it follows from Proposition 6.2.2 that $\mathrm{Ext}^1_{\mathbb{Z}}(\Phi(T), \mathbb{Z}) = \mathrm{H}^1(I, X(T))$, so the description of the torsion part remains valid. For the generalization of [X, Theorem 3.1], we consider the sequence (3)

$$0 \longrightarrow T' \longrightarrow \mathcal{R} := \mathfrak{R}_{L/K}(T_L) \longrightarrow T \longrightarrow 0,$$

which arises from the resolution $X(T) \longrightarrow \mathrm{Ind}_{\mathrm{Gal(L/K)}} \mathbb{Z} \longrightarrow \cdots$. Theorem 5.3.4 and Proposition 5.3.5 yield a commutative diagram

$$\begin{array}{ccccccc} \Phi(T')^{\vee\vee} & \overset{\alpha}{\hookrightarrow} & \Phi(\mathcal{R}) & \longrightarrow & \Phi(T) & \longrightarrow & \Phi(\mathrm{R}^1 j_* T') = 0 \\ \| & & \| & & & & \\ \mathrm{Hom}_{\mathbb{Z}}(X(T')^I, \mathbb{Z}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}}(X(\mathcal{R})^I, \mathbb{Z}) & & & & \end{array}$$

Since the defect terms vanish, the vertical maps are isomorphisms. A priori, the top row is exact except perhaps at $\Phi(\mathcal{R})$, i.e., we only have an isomorphism

$$\Phi(T) \cong \Phi(\mathcal{R})/\ker(\beta) \cong [\Phi(\mathcal{R})/\operatorname{im}(\alpha)]/[\ker(\beta)/\operatorname{im}(\alpha)].$$

Now, as seen above, the free and finite parts of $\Phi(T)$ agree with the free and finite parts of $\Phi(\mathcal{R})/\operatorname{im}(\alpha)$ (correspondingly). For reasons of rank, $\ker(\beta)/\operatorname{im}(\alpha)$ must be a torsion group. Since all groups considered are finitely generated, the torsion parts must be finite and by the equality of the torsion parts as abelian groups, the quotient $\ker(\beta)/\operatorname{im}(\alpha)$ must be trivial. So we have an isomorphism

$$\Phi(T) \cong \operatorname{coker}\big(\mathrm{Hom}_{\mathbb{Z}}\big(X(T')^I, \mathbb{Z}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(X(\mathcal{R})^I, \mathbb{Z}\big)\big)$$

as in the representation from [X, Theorem 3.1]. □

Appendix A

Wound unipotent groups

Let K be a field with characteristic $p > 0$. Then a connected unipotent K-group G is called a K-*wound* group if every homomorphism $\mathbb{G}_{a,K} \longrightarrow G$ is trivial.

In the case that G is also smooth and commutative and the multiplication by $p = \text{char}(K)$ is the zero map, one can see G as a closed subgroup of the $\mathbb{G}_{a,K}^{n+1}$ with $n = \dim(G)$ (see [BLR, Proposition 10.2.10] and [T, III, §3]). More precisely, G has the form $\operatorname{Spec} K[T_0, \dots, T_n] \,/\, (F(T_0, \dots, T_n))$ with $F(T_0, \dots, T_n) = \sum_{i=0}^{n} \sum_{j=0}^{m_i} c_{i,j} T^{p^j}$ and the principal part $\sum_{i=0}^{n} c_{i,m_i} T^{p^{m_i}}$ has no nontrivial K-rational zero. Here the specific form of F is necessary so that F is compatible with the group law of $\mathbb{G}_{a,K}^{n+1}$.

We need the converse of this statement:

Proposition A.1 (cf. [T, III, 3.3.5]). *Let $G := K[T_0, \dots, T_n] \,/ (F(T_0, \dots, T_n))$ be a subgroup of $\mathbb{G}_{a,k}^{n+1}$, where $F = \sum_{i=0}^{n} \sum_{j=0}^{m_i} c_{i,j} T^{p^j}$ is a p-polynomial. Then G is smooth if $c_{i_0,0} \neq 0$ for some $i_0 \in \{0, .., n\}$. Further, G is connected if for some $i_0 \in \{1, .., n\}$ exactly one of the $c_{i_0,j} \neq 0$. In this case, G is K-wound if the principal part of F has no nontrivial rational zero in $\mathbb{A}_k^{n+1}$.*

Proof. By definition, it is clear that G is unipotent as a subgroup of a unipotent group. We will show smoothness using the Jacobi criterion [BLR, Proposition 2.2.7]: G is viewed as a closed subscheme of the smooth scheme $\mathbb{G}_{a,K}^{n+1}$ generated everywhere by the ideal sheaf (F) and $dF = \sum_{i=0}^{n} c_{i,0} dT_i$. Let $x \in G$ be a point and let $z \in \mathbb{G}_{a,K}^{n+1}$ be its image. The stalk of $\Omega^1_{\mathbb{G}_a^{n+1}/K}$ in z has the basis $(dT_i)_{i=0,\dots,n}$. If there is an $i_0 \in \{1, .., n\}$ with $c_{i_0,0} \neq 0$, then $\left((dT_i)_{i=0,\dots,\widehat{i_0},\dots,n}, dF\right)$ is a basis.

Now for an $i_0 \in \{1, .., n\}$ let exactly one $c_{i_0,j} \neq 0$ and let $f \in K[T_0, \dots, T_n] \,/ (F)$ be an idempotent element. We will show that f is 0 or 1, i.e., that G is connected. We take f as the residue class of a polynomial $\tilde{f} \in K[T_0, \dots, T_n]$. By the idempotence, $\tilde{f}^{p^j} \equiv \tilde{f} \bmod F$. Now we can use f^{p^j} as an element of the subring $K[T_0, \dots, T_{i_0}^{p^j}, \dots, T_n]/(F) \subset K[T_0, \dots, T_n] \,/ (F)$. However, By the requirement on F, this subring is isomorphic to $K[T_0, \dots, \widehat{T_{i_0}}, \dots, T_n]$, so it contains only 0 and 1 as idempotents. So f is 0 or 1.

Now suppose that there is a non-constant map $\mathbb{G}_{a,k} \longrightarrow G$. This corresponds to an algebra homomorphism

$$k[T_0, \dots, T_n] \,/ (F) \longrightarrow k[T] \quad T_i \longmapsto \phi_i(T) := a_{i,s_i} T^{s_i} + \ldots + a_{i,0} \in k[T],$$

where at least one of the polynomials ϕ_i must be nontrivial. Since it is well-defined, we must have

$$F(\phi_0(T), \ldots, \phi_n(T)) = \sum_{i=0}^{n} \sum_{j=0}^{m_i} c_{i,j} \phi_i(T)^{p^j} = 0 \in k[T].$$

Let $N := \max_{i=0,\ldots,n} s_i p^{m_i}$. This is clearly the highest power of T which appears in $F(\phi_0(T), \ldots, \phi_n(T))$. Set $b_i = a_{i,s_i}$ if $s_i p^{m_i} = N$ and zero otherwise. Then the coefficient of T^N equals

$$\sum_{i=0}^{n} b_i^{p^{n_i}} c_{i,m_i}.$$

But this is the principal part of F evaluated at the position $(T_i = b_i)_{i=0,\ldots n}$. Since the principal part only has the trivial k-rational zero, all b_i would have to be zero, contradicting our choice. □

Appendix B

Right-exactness of the ft-Néron model

In this section $\mathcal{G}$ denotes the lft-Néron model of $\mathbb{G}_{m,K}$. From the definition of the ft-Néron model and our description of the corresponding étale sheaf, one obtains a long exact sequence for an algebraic K-torus T:

$$0 \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), \mathcal{G}) \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T), i_*\mathbb{Z})$$
$$\longrightarrow \underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathcal{G}) \longrightarrow \underline{\mathrm{Ext}}^1(j_*\underline{X}(T), i_*\mathbb{Z}) \longrightarrow \cdots,$$

the beginning of which can be identified canonically with the sequence

$$0 \longrightarrow (j_*T)^{ft} \longrightarrow j_*T,$$

so that we obtain an inclusion

$$\mathrm{coker}\Big((j_*T)^{\mathrm{ft}} \longrightarrow j_*T\Big) = i_*\Phi(T)^{\vee\vee} \subset \underline{\mathrm{Hom}}\big(\underline{X}(T)^I, i_*\mathbb{Z}\big) \cong i_* \mathrm{Hom}_{\mathbb{Z}}\big(X(T)^I, \mathbb{Z}\big).$$

We now want to examine the $\underline{\mathrm{Ext}}^1$ terms that appear above in more detail. The sheaf $j_*\underline{X}(T)$ is represented by an étale group scheme. By Proposition 5.1.3, it follows that

$$\underline{\mathrm{Ext}}^1(j_*\underline{X}(T), i_*\mathbb{Z}) = i_* \mathrm{Ext}^1_{\mathbb{Z}}\big(X(T)^I, \mathbb{Z}\big) = 0,$$

because $X(T)^I$ is torsion-free. Next we want to examine $\underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathcal{G})$.

Proposition B.1. *Let K be a local field and let T be an algebraic K-torus. Then, in the étale topology,* $\mathrm{Ext}^1(j_*\underline{X}(T), \mathcal{G}) \cong R^1 j_*T$.

Proof. Since the torus T splits over a finite Galois extension L/K, $\underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathcal{G})$ is a skyscraper sheaf, because for every étale morphism $U \longrightarrow \mathrm{Spec}\, L$ we have

$$\mathrm{Ext}^1_U(j_*\underline{X}(T)|_U, \mathcal{G}|_U) = \mathrm{Ext}^1_U\big(\mathbb{Z}^d, \mathbb{G}_{m,U}\big) = \mathrm{H}^1(U, \mathbb{G}_{m,U})^d = 0,$$

where d is the dimension of T.

Let $U = \mathrm{Spec}\, \mathcal{O}_{K'} \longrightarrow \mathrm{Spec}\, \mathcal{O}_K$ be an étale morphism, where K'/K is a finite unramified extension. Then $\mathrm{Ext}^1_U(j_*\underline{X}(T)|_U, \mathcal{G}|_U)$ is the group of isomorphism classes of extensions of sheaves

$$0 \longrightarrow \mathcal{G}|_U \longrightarrow E \longrightarrow j_*\underline{X}(T)|_U \longrightarrow 0$$

on the étale site over U. By [M, II, 3.12], we can write this as an extension of triples

$$0 \longrightarrow \begin{pmatrix} (K^{\mathrm{nr}})^* \\ (K^{\mathrm{sep}})^* \\ (K^{\mathrm{nr}})^* \xrightarrow{\mathrm{id}} (K^{\mathrm{nr}})^* \end{pmatrix} \to \begin{pmatrix} E_{\bar{s}} \\ E_{\bar{\eta}} \\ \psi\colon E_{\bar{s}} \to E_{\bar{\eta}}^I \end{pmatrix} \to \begin{pmatrix} X(T)^I \\ X(T) \\ X(T)^I \xrightarrow{\mathrm{id}} X(T)^I \end{pmatrix} \longrightarrow 0.$$

This means that the top and middle rows are exact sequences of continuous $\mathrm{Gal}(K^{\mathrm{nr}}/K')$-modules and $\mathrm{Gal}(K^{\,\mathrm{sep}}/K')$-modules and that the top row forms a commutative diagram (with the morphisms from the bottom row) with the sequence of I-invariants of the middle row. By the local-to-global spectral sequence for $\underline{\mathrm{Ext}}^1$ on the étale site over $U_K := \operatorname{Spec} K'$, one obtains an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^1(U_K, \underline{\mathrm{Hom}}(j_*\underline{X}(T)|_{U_K}, \mathcal{G}|_{U_K})) \longrightarrow \mathrm{Ext}^1_{U_K}(j_*\underline{X}(T)|_{U_K}, \mathcal{G}|_{U_K}) \\ &\longrightarrow \mathrm{H}^0\big(U_K, \underline{\mathrm{Ext}}^1(j_*\underline{X}(T)|_{U_K}, \mathcal{G}|_{U_K})\big)\,. \end{aligned}$$

As seen above, the last term must be trivial, so by Cartier duality we have a functorial isomorphism

$$\mathrm{H}^1(U_K, T) = \mathrm{Ext}^1_{U_K}(j_*\underline{X}(T)|_{U_K}, \mathcal{G}|_{U_K}) \cong \mathrm{Ext}^1_{\mathrm{Gal}(K^{\mathrm{sep}}/K')}(X(T), (K^{\mathrm{sep}})^*).$$

By [M, III, 1.13], the sheafification of the presheaf $U \longmapsto \mathrm{H}^1(U_K, T)$ is equal to $\mathrm{R}^1 j_* T$ and the sheafification of the presheaf $U \longmapsto \mathrm{Ext}^1_U(\,j_*\underline{X}(T)|_U\,,\,\mathcal{G}|_U)$ is equal to $\mathrm{Ext}^1(j_*\underline{X}(T), \mathcal{G})$. So we have to show, in a functorial way, that

$$E_U := \mathrm{Ext}^1_U(\,j_*\underline{X}(T)|_U\,, \mathcal{G}|_U) = \mathrm{Ext}^1_{U_K}(j_*\underline{X}(T)|_{U_K}, \mathcal{G}|_{U_K}) =: E_{U_K},$$

so that the isomorphism classes of extensions of triples are the isomorphism classes of extensions of $X(T)$ by $(K^{\mathrm{sep}})^*$ as $\mathrm{Gal}(K^{\mathrm{sep}}/K')$-modules.

To do this, we define a map F that maps an extension of triples to their middle row as an extension of $X(T)$ by $(K^{\mathrm{sep}})^*$. This assignment is clearly compatible with isomorphisms, so that we obtain a map $F\colon E_U \longrightarrow E_{U_K}$ on the Ext groups.

An extension of $X(T)$ by $(K^{\mathrm{sep}})^*$ (as $\mathrm{Gal}(K^{\mathrm{sep}}/K')$-modules) induces a commutative diagram with exact rows via Hilbert's Theorem 90

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (K^{\mathrm{nr}})^* & \longrightarrow & E_{\bar{\eta}}^I & \longrightarrow & X(T)^I & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & (K^{\mathrm{sep}})^* & \longrightarrow & E_{\bar{\eta}} & \longrightarrow & X(T) & \longrightarrow & 0 \end{array}$$

This gives us a map $G\colon E_{U_K} \longrightarrow E_U$ since the construction is compatible with isomorphisms of extensions.

One immediately sees that $F \circ G = \mathrm{id}$. Conversely, an extension E of triples is uniquely determined by the middle row and the isomorphism $E_{\bar{s}} \cong E_{\bar{\eta}}^I$, and this isomorphism induces an isomorphism between the extension E and its image under $G \circ F$.

An étale and surjective $\mathcal{O}_K$-morphism $U' \longrightarrow U$ with connected U' corresponds to an unramified field extension $K' \subset K''$ such that $U' \cong \operatorname{Spec} \mathcal{O}_{K''}$.

The restriction of a sheaf with respect to $U' \longrightarrow U$ corresponds, at the level of the Galois module, to the restriction of the inclusion $\mathrm{Gal}(K^{\mathrm{sep}}/K'') \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K')$, so $E_U \cong E_{U_K}$ (functorially in U). □

Lemma B.2. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Then* $\operatorname{coker}[\underline{\operatorname{Hom}}(j_*\underline{X}(T),\mathcal{G}) \longrightarrow \underline{\operatorname{Hom}}(j_*\underline{X}(T),i_*\mathbb{Z})] \cong E(T)^{\mathrm{pd}}$, *where* $E(T)^{\mathrm{pd}}$ *means the étale sheaf induced by the Pontryagin dual of the defect term.*

Proof. We consider the canonical exact sequence (6) in the smooth topology

$$0 \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow T^I \longrightarrow 0.$$

This induces an exact sequence

$$0 \longrightarrow j_*\widetilde{T} \longrightarrow j_*T \longrightarrow j_*T^I \longrightarrow \mathcal{K}.$$

By Proposition 3.3.1, we can complete this sequence and transform it into an exact and commutative diagram using the corresponding ft-Néron models

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j_*\tilde{T} & \longrightarrow & j_*T^{\mathrm{ft}} & \xrightarrow{\beta} & \mathbb{G}^d_{m,\mathcal{O}_K} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_*\tilde{T} & \longrightarrow & j_*T & \longrightarrow & j_*T^I & \twoheadrightarrow & \mathcal{K} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \Phi(T) & \xrightarrow{\alpha} & \underline{\operatorname{Hom}}(j_*\underline{X}(T)^I, i_*\mathbb{Z}) & &
\end{array}$$

Let $d = \operatorname{rank} X(T)^I$. By the commutativity of the diagram, we obtain a map

$$\alpha\colon \Phi(T)^{\vee\vee} \longrightarrow \underline{\operatorname{Hom}}\big(j_*\underline{X}(T)^I, i_*\mathbb{Z}\big)\,.$$

This is a map of sheaves represented by étale group schemes. Thus we can determine α after restricting to the étale topology. In that topology, we have a commutative diagram

$$\begin{array}{ccc}
j_*T = \underline{\operatorname{Hom}}(j_*\underline{X}(T),\mathcal{G}) & \longrightarrow & j_*T^I = \underline{\operatorname{Hom}}(j_*\underline{X}(T^I),\mathcal{G}) \\
\downarrow & & \downarrow \\
\underline{\operatorname{Hom}}(j_*\underline{X}(T),i_*\mathbb{Z}) & \xrightarrow{\cong} & \underline{\operatorname{Hom}}(j_*\underline{X}(T^I),i_*\mathbb{Z})
\end{array}$$

since the inclusion $\underline{X}\big(T^I\big) = \underline{X}(T)^I \hookrightarrow \underline{X}(T)$ becomes an isomorphism after applying $i^* \circ j_*$. Thus α corresponds to an inclusion

$$\operatorname{im}(\underline{\operatorname{Hom}}(j_*\underline{X}(T),\mathcal{G}) \longrightarrow \underline{\operatorname{Hom}}(j_*\underline{X}(T),i_*\mathbb{Z})) \subseteq \underline{\operatorname{Hom}}(j_*\underline{X}(T),i_*\mathbb{Z})\,.$$

Applying the snake lemma we obtain a short exact sequence

$$0 \longrightarrow \operatorname{coker}(\beta) \longrightarrow \mathcal{K} \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0.$$

If we then apply the functor $\underline{\operatorname{Hom}}(\cdot, i_*\mathbb{Z})$ in the smooth topology, we obtain a long exact sequence

$$\underline{\operatorname{Hom}}(\operatorname{coker}(\beta), i_*\mathbb{Z}) \longrightarrow \underline{\operatorname{Ext}}^1(\operatorname{coker}(\alpha), i_*\mathbb{Z}) \longrightarrow \underline{\operatorname{Ext}}^1(\mathcal{K}, i_*\mathbb{Z}) \longrightarrow \underline{\operatorname{Ext}}^1(\operatorname{coker}(\beta), i_*\mathbb{Z})\,.$$

Now $\underline{\operatorname{Hom}}(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z})$, $\underline{\operatorname{Ext}}^1(\mathbb{G}_{m,\mathcal{O}_K}, i_*\mathbb{Z})$ and $\underline{\operatorname{Hom}}\big(j_*T^{\mathrm{ft}}, i_*\mathbb{Z}\big)$ are all trivial, whence $\underline{\operatorname{Hom}}(\operatorname{coker}(\beta), i_*\mathbb{Z}) = 0$ and $\underline{\operatorname{Ext}}^1(\operatorname{coker}(\beta), i_*\mathbb{Z}) = 0$. By definition, the restriction of $\underline{\operatorname{Ext}}^1(\mathcal{K}, i_*\mathbb{Z})$ to the étale site is represented by the defect term $E(T)$. Thus, by restricting to the étale site and dualizing again using Proposition 5.1.3, the assertion follows, because $\operatorname{coker}(\alpha)$ is represented by an étale group scheme. □

The above immediately gives us the statement

Proposition B.3. *Let K be a local field and let T be an algebraic K-torus with character group $X(T)$. Then there exists an exact sequence*

$$0 \longrightarrow E(T) \longrightarrow \underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \mathrm{R}^1 j_* T \longrightarrow 0.$$

In particular, $\underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K})$ is trivial if the residue field is perfect or if T splits over a tamely ramified extension. In general, the above sheaf is a p-primary torsion sheaf.

With these considerations we can see that in principle one could also determine the free part via the inclusion $j_*T^{\mathrm{ft}} \hookrightarrow j_*T$. Then the defect term would be defined as $E(T) := \ker\big(\underline{\mathrm{Ext}}^1(j_*\underline{X}(T), \mathbb{G}_{m,\mathcal{O}_K}) \twoheadrightarrow \mathrm{R}^1 j_* T\big)$. However, this definition is more difficult to compute than our definition.

Applying the functor $\underline{\mathrm{Hom}}(j_*\cdot, \mathbb{G}_{m,\mathcal{O}_K})$ to an exact sequence of character groups yields a sequence that contains the associated sequence of the ft-Néron models, but it is generally difficult to describe what the next term in this sequence should be. If j_* is exact on the character groups, a simple solution exists.

Proposition B.4. *Let K be a local field and let*

$$0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$$

be a short exact sequence of K-tori. If

$$\ker\big(H^1(I, X(T_3)) \longrightarrow H^1(I, X(T_2))\big) = 0,$$

then the long exact sequence of the ft-*Néron models is isomorphic to the sequence*

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T_1), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T_2), \mathbb{G}_{m,\mathcal{O}_K}) \\ \longrightarrow \underline{\mathrm{Hom}}(j_*\underline{X}(T_3), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \underline{\mathrm{Ext}}^1(j_*\underline{X}(T_1), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \cdots \end{aligned}$$

Proof. Under the stated conditions, the associated short exact sequence of the character groups induces an exact sequence

$$0 \longrightarrow j_*\underline{X}(T_3) \longrightarrow j_*\underline{X}(T_2) \longrightarrow j_*\underline{X}(T_1) \longrightarrow 0.$$

By applying the functor $\underline{\mathrm{Hom}}(\cdot, \mathbb{G}_{m,\mathcal{O}_K})$, we obtain a long exact sequence, the beginning of which corresponds exactly to the sequence of the ft-Néron models. More precisely, by Proposition 3.3.3, the $\mathrm{Hom}(j_*\underline{X}(T_i), \mathbb{G}_{m,\mathcal{O}_K})$-terms are isomorphic to the corresponding ft-Néron models and a homomorphism between these as sheaves is already clearly determined on the generic fiber. There the isomorphism is clear after defining the sequences. □

With the notations as in the last sentence, the following holds. For an exact sequence of algebraic K-tori with $\ker\big(\mathrm{H}^1(I, X(T_3)) \longrightarrow \mathrm{H}^1(I, X(T_2))\big) = 0$, the sequence of ft-Néron models is exact precisely when

$$\mathcal{E} := \ker\big(\underline{\mathrm{Ext}}^1(j_*\underline{X}(T_1), \mathbb{G}_{m,\mathcal{O}_K}) \longrightarrow \underline{\mathrm{Ext}}^1(j_*\underline{X}(T_2), \mathbb{G}_{m,\mathcal{O}_K})\big) = 0.$$

If, additionally, $\Phi(T_2)$ has no p-primary torsion and $\mathrm{H}^1(I, X(T_3)) = 0$, then $\Phi(T_3) = \Phi(\mathcal{E})$. This can be seen as a generalization of Proposition 6.2.2 in the case where $\mathrm{H}^1(I, X(T)) = 0$.

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